Exact formulas for random growth off a flat interface

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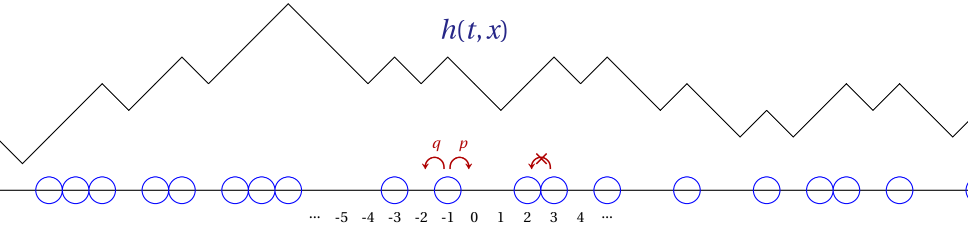
Asymmetric Simple Exclusion Process (ASEP)

local min ↦ local max at rate q
local max ↦ local min at rate p



$\eta_t(x) \in \{0, 1\}$ is the occupation variable.

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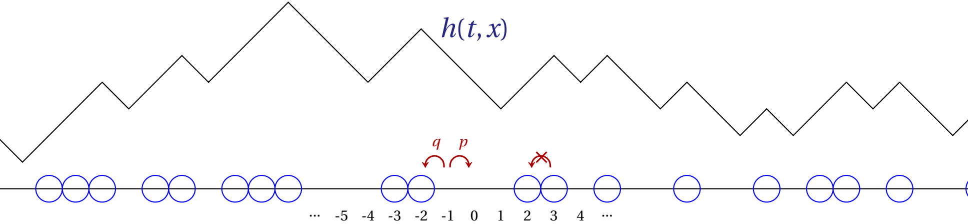

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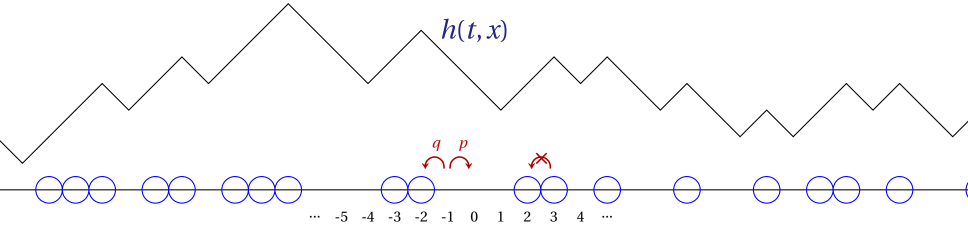

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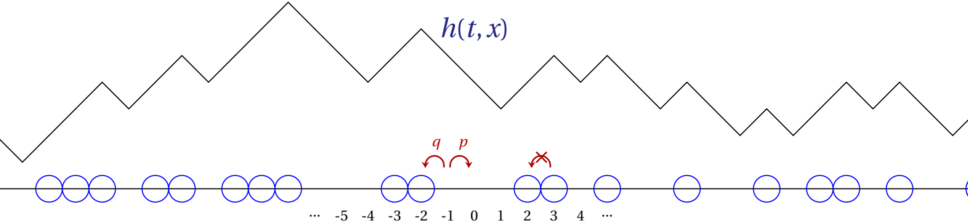

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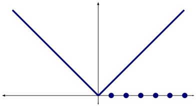
The case $p=0$, $q=1$ (TASEP) is equivalent to last passage percolation.

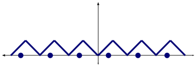
Three special classes of initial data

(scale invariance)

- Step / curved: $\eta_0(x) = \mathbf{1}_{x\geq 0} \quad \rightsquigarrow \quad h(0,x) = |x|.$



- Flat / periodic: $\eta_0(x) = \mathbf{1}_{x\in 2\mathbb{Z}} \quad \rightsquigarrow \quad h(0,x) = \frac{1}{2}(1+(-1)^x).$



- Stationary / Bernoulli: $\eta_0 =$ product of Bernoullis

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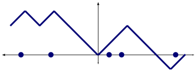

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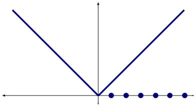

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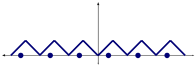

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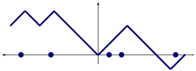

There are also three "mixed" cases.

TASEP asymptotics

For TASEP/LPP these six cases can be fully analyzed. Exact computations based on determinantal structure give exact limiting distributions.

[2000-07: Baik, Deift, Johansson, Rains, Borodin, Ferrari, Prähofer, Spohn, Sasamoto,...]

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For curved i.c., $h(0,x) = |x|$, we have

$$\mathbb{P}^{\text{TASEP}}\left(\frac{h(t,0)-\frac{1}{2}t}{t^{1/3}} \geq -r\right) \xrightarrow[t\to\infty]{} F_{\text{GUE}}(2^{1/3}r).$$

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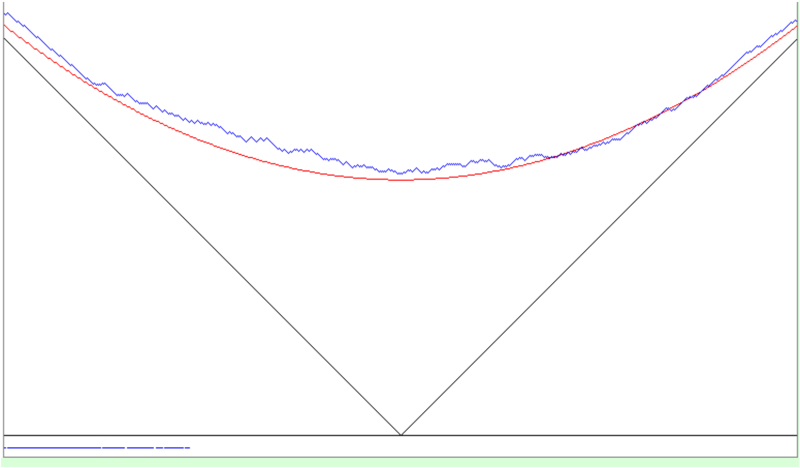
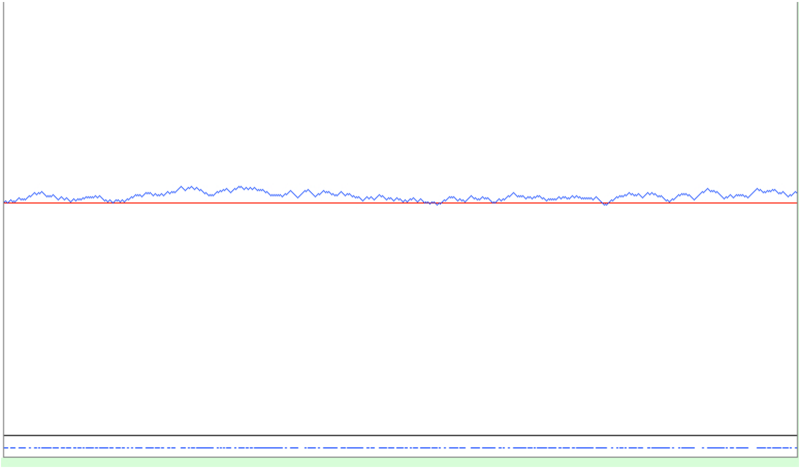
Limits given in terms of Fredholm determinants are not so surprising in view of the determinantal structure.

There are also multipoint results, leading to the Airy processes.



Difference between the blue and red curves is an approximate Airy$_2$ process. At each x the distribution is F_{GUE}.



Difference between the blue and red curves is an approximate Airy$_1$ process. At each x the distribution is F_{GOE}.

Exact formulas for step ASEP [Tracy-Widom '08]

$q \geq p \quad \rightsquigarrow \quad$ non-determinantal/"positive temperature" model

By *KPZ universality* one expects the same limiting distributions in the partially asymmetric (ASEP) case.

Most well understood case is the step initial condition:

$$\mathbb{P}^{\text{step}}\big(h(t/(q-p),x) \geq y\big) = \frac{1}{2\pi i}\oint \frac{d\mu}{\mu}\prod_{k=0}^{\infty}\Big(1-\mu\big(\tfrac{p}{q}\big)^k\Big)\det(I-J),$$

$$J(\eta,\eta') = \oint d\zeta\, \frac{e^{\Psi(\zeta)-\Psi(\eta')}}{\eta'(\zeta-\eta)} \sum_{k=-\infty}^{\infty} \frac{\big(\frac{p}{q}\big)^k\big(\frac{\zeta}{\eta'}\big)^k}{\mu-\big(\frac{p}{q}\big)^k}, \qquad \Psi(\zeta) = -x\log(1-\zeta) + \tfrac{t\zeta}{(1-\zeta)} + (y+x)\log\zeta.$$

Obtained by solving the forward equation for the n-particle transition functions (Bethe ansatz). Combinatorial miracles (Cauchy determinant, Ramanujan summation formula, etc...) lead to the Fredholm determinant.

Significant "post-processing" yields a formula suitable for asymptotic analysis and the conjectured GUE asymptotics [Tracy-Widom '08].

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The same method yields the conjectured asymptotics for the step-Bernoulli case [Tracy-Widom '09].

The flat cases remain unclear.

- Exact formulas for the ASEP distribution function [Lee '10]. Not suitable for asymptotics, not even at a formal level.
- Physics derivations for flat and half-flat KPZ [Le Doussal-Calabrese '12, Le Doussal '14]. Highly non-rigorous (replica method + ...).

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Let $Z(t,x) = e^{h(t,x)}$ with $\partial_t h = \frac{1}{2}(\partial_x h)^2 + \frac{1}{2}\partial_x^2 h + \xi$ (KPZ equation).

Then $u(t,\vec{x}) = \mathbb{E}[Z(t,x_1)\cdots Z(t,x_n)]$ satisfies

$$\partial_t u(t,\vec{x}) = \Big(\tfrac{1}{2}\sum_{i=1}^n \partial_{x_i}^2 + \tfrac{1}{2}\sum_{i\neq j=1}^n \delta(x_i - x_j)\Big) u(t,\vec{x}) \quad \text{attractive } \delta\text{-Bose gas}.$$

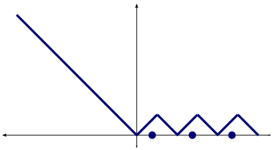
- Explicit (rigorous) eigenfunctions/eigenvalues (Bethe ansatz) leads to formulas for $\mathbb{E}[Z(t,x)^n]$.
- Sum the moments to obtain $\mathbb{E}[e^{-\zeta Z}] = \sum_{n=0}^\infty \frac{(-\zeta)^n}{n!}\mathbb{E}[Z^n]$.
  But $\mathbb{E}[Z^n] \approx e^{\frac{1}{24}n(n^2-1)}$!
- Use the Airy trick $\int dx\, \mathrm{Ai}(x) e^{nx} = e^{n^3}$ to manipulate the divergent series. For example,

  $$\sum_{n=0}^\infty (-1)^n e^{n^3} \text{ "=" } \int dx\, \mathrm{Ai}(x) \sum_{n=0}^\infty (-1)^n e^{nx} dx = \int dx\, \mathrm{Ai}(x) \frac{1}{1+e^x}.$$
- Resulting formulas "formally" lead to conjectured asymptotics.

Our results: [Ortmann-Quastel-R '13+]

- Exact formulas for half-flat and flat ASEP.
- Fredholm Pfaffian in the flat case.
- Suitable **in principle** for asymptotics, but technical problems remain.
- Formal critical point analysis leads to the conjectured asymptotics

I will focus first on the half-flat case $\eta_0(x) = \mathbf{1}_{x \in 2\mathbb{Z}_{>0}}$.



ASEP duality [Sasamoto-Imamura '10, Borodin-Corwin-Sasamoto '13]

Suppose there is a *left-most particle*. Let $N_x(t)$ be the net number of particles which have crossed to the right of site x up to time t. Then

$$u(t;\vec{x}) := \mathbb{E}\left[\tau^{N_{x_1-1}(t)}\eta_{x_1}(t)\cdots\tau^{N_{x_k-1}(t)}\eta_{x_k}(t)\right]$$

is the unique solution of

(1) $\partial_t u(t,\vec{x}) = \sum_{a=1}^{k}\left[pu(t,\vec{x}_a^-) + qu(t,\vec{x}_a^+) - u(t,\vec{x})\right]$

$\vec{x}_a^{\pm} = (x_1,\dots,x_a \pm 1,\dots,x_k)$

(2) When $x_1 \le \dots \le x_k$, $x_{a+1} = x_a + 1$,

$$pu(t,\vec{x}_{a+1}^-) + qu(t,\vec{x}_a^+) = u(t,\vec{x}).$$

(3) For $x_1 < x_2 < \cdots < x_k$

$$u(0,\vec{x}) = u_0(\vec{x}).$$

- Borodin and Corwin found contour integral formulas for moments of the q-Whittaker process, arising from their study of Macdonald processes.
- Suitable scaling limit yields formulas for the moments of the delta Bose gas with δ_0 initial condition [Heckman, Opdam '97]:

  (1)' $\partial_t u(t,\vec{x}) = \Delta u(t,\vec{x})$,

  (2)' $\left[\frac{\partial}{\partial_{x_a}} - \frac{\partial}{\partial_{x_{a+1}}} - 1\right] u(t,\vec{x}) = 0$ when $x_1 \le \cdots \le x_k$ and $x_a = x_{a+1}$,

  (3)' $k! \int_{x_1 < \cdots < x_k} d\vec{x}\, u(0,\vec{x}) f(\vec{x}) = f(0)$ for suitable f.

- By analogy, this allowed them to find contour integral formulas for ASEP with step (and step-Bernoulli) initial conditions:

$$u(t,\vec{x}) = \frac{1}{(2\pi \mathrm{i})^k} \int_{C^k} d\vec{z} \prod_{1\le a<b\le k} \frac{z_a - z_b}{z_a - \tau z_b} \prod_{a=1}^{k} \frac{\left(\frac{1-\tau z_a}{1-z_a}\right)^{x_a-1} e^{t(p\frac{1-z_a}{1-\tau z_a} + q\frac{1-\tau z_a}{1-z_a} - 1)}}{1 - z_a}$$

C is a small circle around 1

- Nevertheless, ASEP does not fit into theory of Macdonald processes.

ASEP with half-flat initial condition translates into the initial condition

$$u_0(\vec{x}) = \tau^{-k} \prod_{a=1}^{k} 1_{x_a \in 2\mathbb{Z}_{>0}} \tau^{\frac{1}{2} x_a}.$$

Our solution of the equations in this case is based on a careful study of E. Lee's half-flat ASEP formula

$$\begin{aligned} \mathbb{P}\big(N_x(t) \geq m\big) = (-1)^m \sum_{k \geq m} & \frac{\tau^{(k-m)(k-m+1)/2}}{(1+\tau)^{k(k-1)} k!} \begin{bmatrix} k-1 \\ k-m \end{bmatrix}_\tau \\ & \times \int_{C_R^k} d\vec{\xi} \prod_i \frac{\xi_i^x e^{t(p\xi_i^{-1} + q\xi_i - 1)}}{(1-\xi_i)(\xi_i^2 - \tau)} \\ & \times \prod_{i \neq j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \prod_{i<j} \frac{1 + \tau - (\xi_i + \xi_j)}{\tau - \xi_i\xi_j}. \end{aligned}$$

Contour integral ansatz for half-flat ASEP

(1) $\partial_t u(t,\vec{x}) = \sum_{a=1}^k \left[p u(t,\vec{x}_a^-) + q u(t,\vec{x}_a^+) - u(t,\vec{x}) \right]$,

(2) $p u(t,\vec{x}_{a+1}^-) + q u(t,\vec{x}_a^+) - u(t,\vec{x}) = 0$ when $x_{a+1} = x_a + 1$,

(3) $u(0,\vec{x}) = \tau^{-k} \prod_{a=1}^{k} 1_{x_a \in 2\mathbb{Z}_{>0}} \tau^{\frac{1}{2}x_a}$ for $x_1 < x_2 < \cdots < x_k$,

is solved by

$$u(t,\vec{x}) = \frac{\tau^{\frac{1}{2}k(k-1)}}{(2\pi \mathrm{i})^k} \int_{C^k} d\vec{z} \prod_{1\le a<b\le k} \frac{z_a - z_b}{z_a - \tau z_b} \frac{1 - z_a z_b}{1 - \tau z_a z_b} \times \prod_{a=1}^{k} \frac{\left(\frac{1-\tau z_a}{1-z_a}\right)^{x_a - 1} e^{t\left(p\frac{1-z_a}{1-\tau z_a} + q\frac{1-\tau z_a}{1-z_a} - 1\right)}}{\tau z_a^2 - 1}.$$

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Half-flat KPZ moments

We can similarly solve the delta Bose gas with initial condition $u(0,\vec{x}) = \prod_a e^{-\theta x}\mathbf{1}_{x\geq 0}$:

$$u(t,\vec{x}) := \mathbb{E}\left[\prod_{i=1}^{k} Z(t,x_i)\right] = \int_{\vec{\delta}+(\mathrm{i}\mathbb{R})^k} \frac{d\vec{z}}{(2\pi\mathrm{i})^k} \prod_{a<b}\left(\frac{z_a - z_b}{z_a - z_b - 1}\frac{z_a + z_b - 1}{z_a + z_b}\right) \times \prod_{a=1}^{k}\frac{1}{z_a} e^{\frac{t}{2}\sum_{a=1}^{k}(z_a-\theta)^2 + \sum_{a=1}^{k}(z_a-\theta)x_a}$$

where $\delta_1 > \delta_2 + 1 > \cdots > \delta_k + k - 1 > k-1$ and $x_1 < \cdots < x_k$.

After some post-processing we recover Le Doussal-Calabrese's formulas:

$$\mathbb{E}\left[e^{-\zeta Z(t,x)}\right] = \sum_{k=0}^{\infty}\frac{1}{k!}\int_{(\mathrm{i}\mathbb{R})^k} d\vec{u} \sum_{n_1,\dots,n_k=1}^{\infty} \prod_a \frac{2^{n_a}}{n_a}\frac{\Gamma(1-2u_a-2\gamma-n_a)}{\Gamma(1-2u_a-2\gamma)} e^{(n_a^3-n_a)\frac{t}{12}+n_a t u_a^2 - \gamma n_a s + x n_a u_a}$$

$$\times \prod_{a<b}\frac{\Gamma(1-u_a-u_b-2\gamma-\frac{n_a+n_b}{2})\Gamma(1-u_a-u_b-2\gamma+\frac{n_a+n_b}{2})}{\Gamma(1-u_a-u_b-2\gamma+\frac{n_a-n_b}{2})\Gamma(1-u_a-u_b-2\gamma-\frac{n_a-n_b}{2})}\frac{(n_a-n_b)^2-4(u_a-u_b)^2}{(n_a+n_b)^2-4(u_a-u_b)^2}$$

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$$\times \prod_{a<b} \frac{\Gamma(1-u_a-u_b-2\gamma-\frac{n_a+n_b}{2})\Gamma(1-u_a-u_b-2\gamma+\frac{n_a+n_b}{2})}{\Gamma(1-u_a-u_b-2\gamma+\frac{n_a-n_b}{2})\Gamma(1-u_a-u_b-2\gamma-\frac{n_a-n_b}{2})} \frac{(n_a-n_b)^2-4(u_a-u_b)^2}{(n_a+n_b)^2-4(u_a-u_b)^2}$$

In the case of half-flat q-TASEP and half-flat semi-discrete polymers, it appears that formulas are much more complicated.

Half-flat ASEP moments

Recall that we obtained a formula for

$$u(t;\vec{x}) := \mathbb{E}\left[\tau^{N_{x_1-1}(t)}\eta_{x_1}(t)\cdots\tau^{N_{x_k-1}(t)}\eta_{x_k}(t)\right] \qquad \text{with} \qquad x_1 < \cdots < x_k.$$

Additional combinatorics + enlarging contours yields

(Imamura-Sasamoto '11, Borodin-Corwin-Sasamoto '13)

$$\mathbb{E}\left[\tau^{mN_x(t)}\right] = m_\tau! \sum_{k=0}^{m} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k\geq 1\\ n_1+\cdots+n_k=m}} \int_{C^k} \frac{d\vec{w}}{(2\pi i)^k} \det\left[\frac{-1}{w_a\tau^{n_a}-w_b}\right]_{a,b=1}^{k}$$
$$\times \prod_a f(w_a;n_a)g(w_a;n_a) \prod_{a<b} \alpha(w_a,w_b;n_a,n_b),$$

$\zeta \in \mathbb{C}\setminus\mathbb{R}$, C contour containing $-1, 0$ with $-\tau^{-1}, \tau^{-1}$ on outside

with

$$f(w;n) = (1-\tau)^{n_a} e^{t\left[\frac{1}{1+w_a} - \frac{1}{1+\tau^{n_a}w_a}\right]} \left(\frac{1+\tau^{n_a}w_a}{1+w_a}\right)^{x-1},$$

$$g(w;n) = \frac{(-w;\tau)_\infty}{(-\tau^n w;\tau)_\infty} \frac{\left(\tau^{2n}w^2;\tau\right)_\infty}{\left(\tau^n w^2;\tau\right)_\infty}, \quad \alpha(w_1,w_2;n_1,n_2) = \frac{(w_1w_2;\tau)_\infty \left(\tau^{n_1+n_2}w_1w_2;\tau\right)_\infty}{(\tau^{n_1}w_1w_2;\tau)_\infty (\tau^{n_2}w_1w_2;\tau)_\infty}.$$

Here the q-Pochhammer symbol is $\quad (a;q)_\infty = \prod_{\ell=0}^{\infty}(1-q^\ell a).$

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$$\underbrace{\mathbb{E}\left[\tau^{mN_x(t)}\right]}_{\leq 1} = m_\tau! \sum_{k=0}^{m} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k\geq 1\\ n_1+\cdots+n_k=m}} \int_{C^k} \frac{d\vec{w}}{(2\pi i)^k} \det\left[\frac{-1}{w_a\tau^{n_a}-w_b}\right]_{a,b=1}^{k}$$

$$\times \prod_a f(w_a;n_a)g(w_a;n_a) \prod_{a<b} \alpha(w_a,w_b;n_a,n_b),$$

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e_τ-Laplace transform and formal asymptotics

Standard q-exponential fn.: $$e_q(x) = \frac{1}{((1-q)x;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{k_q!}.$$

(series only valid for $|x| < 1$)

Here $k_q! = [1]_q \cdot [2]_q \cdots [k]_q$, with $[k]_q = \frac{1-q^k}{1-q}$.

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Summing the moment formula + analytic continuation yields

$$\mathbb{E}^{\mathrm{hf}}\left[e_\tau\left(\zeta\tau^{N_x(t)}\right)\right] = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{(\delta+\mathrm{i}\mathbb{R})^k} \frac{d\vec{s}}{(2\pi\mathrm{i})^k} \int_{C^k} \frac{d\vec{w}}{(2\pi\mathrm{i})^k} \det\left[\frac{-1}{w_a\tau^{s_a} - w_b}\right]_{a,b=1}^{k}$$

$$\times \prod_a \frac{\pi}{\sin(-\pi s_a)} \zeta^{s_a} f(w_a;s_a) g(w_a;s_a) \prod_{a<b} \alpha(w_a,w_b;s_a,s_b),$$

Mellin-Barnes:

$$\sum_{n\geq 1} \zeta^n f(n) = \frac{1}{2\pi\mathrm{i}} \int_{\delta+\mathrm{i}\mathbb{R}} ds \frac{\pi}{\sin(-\pi s)} \zeta^s f(s).$$

e_τ-Laplace transform and formal asymptotics

$$\mathbb{E}^{\mathrm{hf}}\left[e_\tau\left(\zeta\tau^{N_x(t)}\right)\right] = \sum_{k=0}^{\infty}\frac{1}{k!}\int_{(\delta+\mathrm{i}\mathbb{R})^k}\frac{d\vec{s}}{(2\pi i)^k}\int_{C^k}\frac{d\vec{w}}{(2\pi i)^k}\det\left[\frac{-1}{w_a\tau^{s_a}-w_b}\right]_{a,b=1}^{k}$$
$$\times\prod_a\frac{\pi}{\sin(-\pi s_a)}\zeta^{s_a}f(w_a;s_a)g(w_a;s_a)\prod_{a<b}\alpha(w_a,w_b;s_a,s_b),$$

To compute the asymptotics the idea is to use the basic trick that for $\alpha>0$ and a sequence of random variables X_m,

$$\lim_{m\to\infty}\mathbb{E}\left[\exp\left(-e^{\alpha X_m}\right)\right] = \lim_{m\to\infty}\mathbb{P}(X_m>0).$$

e_τ-Laplace transform and formal asymptotics

$$\mathbb{E}^{\mathrm{hf}}\left[e_\tau\left(\zeta\tau^{N_x(t)}\right)\right] = \sum_{k=0}^{\infty}\frac{1}{k!}\int_{(\delta+\mathrm{i}\mathbb{R})^k}\frac{d\vec{s}}{(2\pi\mathrm{i})^k}\int_{C^k}\frac{d\vec{w}}{(2\pi\mathrm{i})^k}\,\det\left[\frac{-1}{w_a\tau^{s_a}-w_b}\right]_{a,b=1}^{k}$$
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$e_\tau(x)$ behaves sufficiently like $\exp(x)$ for $x\in(-\infty,0)$ so that

$$\lim_{t\to\infty}\mathbb{E}^{\mathrm{hf}}\left[e_\tau\left(-\tau^{N_{t^{2/3}x}(\frac{t}{q-p})-\frac{1}{2}t^{2/3}x-\frac{1}{2}t-\frac{1}{2}t^{1/3}x^2\mathbf{1}_{x\geq 0}+t^{1/3}r}\right)\right]$$
$$=\lim_{t\to\infty}\mathbb{P}^{\mathrm{hf}}\left(\frac{h(\frac{t}{q-p},t^{2/3}x)-\frac{1}{2}t-\frac{1}{2}t^{1/3}x^2\mathbf{1}_{x\geq 0}}{t^{1/3}}>-r\right).$$

(this uses $h(t,x)=N_x(t)-\frac{1}{2}x$)

e_τ-Laplace transform and formal asymptotics

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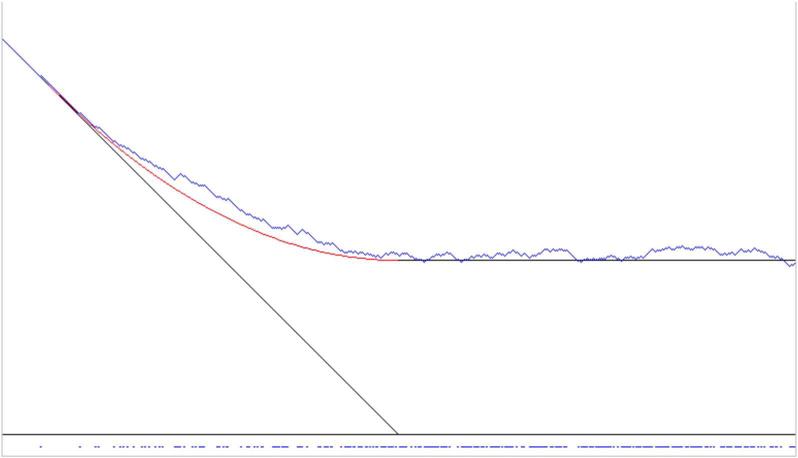
This transform determines the distribution of $N_x(t)$.

Formal critical point analysis shows that the asymptotic fluctuations are the conjectured ones:

$$\lim_{t\to\infty}\mathbb{P}^{\mathrm{hf}}\left(\frac{h\left(\frac{t}{q-p},t^{2/3}x\right)-\frac{1}{2}t-\frac{1}{2}t^{1/3}x^2\mathbf{1}_{x\geq 0}}{t^{1/3}}>-r\right)$$
$$=\mathbb{P}\left(\mathscr{A}_{2\to 1}(2^{-1/3}x)\leq 2^{1/3}r\right).$$

Unfortunately, $\alpha(w_a,w_b;s_a,s_b)$ is not controlled away from the critical point.



Difference between the blue and red curves is an approximate Airy$_{2\to 1}$ process. The distribution at each x interpolates between F_{GOE} and F_{GUE}.

From half-flat to flat

Recall we have

$$\mathbb{E}^{\mathrm{hf}}\left[\tau^{mN_x(t)}\right] = m_\tau! \sum_{k=0}^{m} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k\geq 1\\ n_1+\dots+n_k=m}} \int_{C^k} \frac{d\vec{w}}{(2\pi i)^k} \det\left[\frac{-1}{w_a\tau^{n_a}-w_b}\right]_{a,b=1}^{k}$$

$$\times \prod_a \left(\frac{1-\tau z_a}{1-z_a}\right)^{x-1} \tilde{f}(w_a;n_a) g(w_a;n_a) \prod_{a<b} \alpha(w_a,w_b;n_a,n_b),$$

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$$\mathbb{E}^{\mathrm{hf}}\left[\tau^{m(N_x(t)-\frac{1}{2}x)}\right] = m_\tau! \sum_{k=0}^{m} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k\geq 1 \\ n_1+\cdots+n_k=m}} \int_{C^k} \frac{d\vec{w}}{(2\pi i)^k} \det\left[\frac{-1}{w_a\tau^{n_a}-w_b}\right]_{a,b=1}^{k}$$

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We want to take the wedge to $-\infty$:

$$\mathbb{E}^{\mathrm{flat}}\left[\tau^{\frac{1}{2}mh(t,0)}\right] = \lim_{x\to\infty} \mathbb{E}^{\mathrm{hf}}\left[\tau^{m(N_{2x}(t)-x)}\right].$$

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It turns out that

$$\left|\frac{1-\tau z_a}{1-z_a}\tau^{-\frac{1}{2}n_a}\right| < 1 \qquad \Longleftrightarrow \qquad |z_a| > \tau^{1-\frac{1}{2}n_a}.$$

So we have to deform z_a to a circle of radius $R_a > \tau^{1-\frac{1}{2}n_a}$.

Flat formula

- Flat formula is monster sum of residues collected as we deform each contour to a circle of radius R_a
- There are two types of poles:
  1. $\prod_{a<b} \alpha(w_a, w_b; n_a, n_b)$ gives rise to *"paired variables"*
  2. $\prod_a g(w_a; n_a)$ gives rise to *"unpaired variables"*

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- Miracle 1: All residues lose dependence on x.

We end up with a formula like

$$\mathbb{E}^{\text{flat}}(\tau^{\frac{1}{2}mh(t,0)}) = m!_\tau \sum_{k=0}^{m} \sum_{\substack{k_\text{u},k_\text{p}\geq 0 \\ k_\text{u}+2k_\text{p}=k}} \frac{1}{k_\text{u}!2^{k_\text{p}}k_\text{p}!} \sum_{(\vec{\varsigma},\vec{n}^\text{u},\vec{n}^\text{p})\in\Lambda^m_{k_\text{u},k_\text{p}}} \\ \times \frac{1}{(2\pi \text{i})^{k_\text{p}}} \int_{C^{k_\text{p}}} d\vec{z}^\text{p}\, I(\vec{n}^\text{u},\vec{n}^\text{p};\vec{z}^\text{u},\vec{z}^\text{p},\vec{z}^{-\text{p}})$$

with C a circle of radius slightly > 1, $\vec{z}^\text{u}_a = \varsigma_a$ and $z^{-\text{p}}_a = 1/z^\text{p}_a$.

- Miracle 2: The integrand contains product of $\alpha(z_a, z_b; n_a, n_b)$ involving all pairs among the variables z_a^{u}, z_a^{p}, $z_a^{-\mathrm{p}}$.
  After massive cancellations these products turn into things like

$$\prod_{1\le a<b\le 2k} \frac{y_b - y_a}{y_a y_b - 1} = \mathrm{Pf}\left[\frac{y_b - y_a}{y_a y_b - 1}\right]_{a,b=1}^{2k}$$

(non-linear Schur Pfaffian identity) and

$$\prod_{1\le a<b\le k} (-\varsigma_a \varsigma_b)^{m_a \wedge m_b + 1} \mathrm{sgn}(\varsigma_a m_a - \varsigma_b m_b)$$
$$= \mathrm{Pf}\left[(-\varsigma_a \varsigma_b)^{m_a \wedge m_b} \mathrm{sgn}(\varsigma_b m_b - \varsigma_a m_a)\right]_{a,b=1}^{k}$$

for positive integers $m_1, \ldots, m_k$, and $\varsigma_1, \ldots \varsigma_k \in \{-1, 1\}$.

Here

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} A_{\sigma(2i-1),\sigma(2i)}, \quad \mathrm{Pf}(A)^2 = \sqrt{\det(A)}.$$

(A antisymmetric matrix)

- At this point the formulas still involve two Pfaffians, in different variables, and works only for k_{u} even.
- Step 3: The result can (almost) be written as a single Pfaffian while getting rid of the distinction between paired and unpaired variables.

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Half-flat moment formula:

$$\mathbb{E}^{\mathrm{flat}}\left[\tau^{\frac{1}{2}mh(t,0)}\right] = m!_\tau \sum_{k=0}^{m} \frac{(-1)^{\frac{1}{2}k(k+1)}}{2^k k!} \sum_{\substack{m_1,\dots,m_k=1,\\ m_1+\cdots+m_k=m}}^{\infty} \int_{C_{0,1}^k} d\vec{y}$$

$$\times \prod_{1\le a<b\le k} \tau^{-\frac{1}{2}m_a m_b} \prod_{a=1}^{k} u(y_a,m_a)\,\mathrm{Pf}\big[K(\vec{y};\vec{m})\big]$$

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with

$$K_{1,1}(y_a,y_b;m_a,m_b) = 4u_{\mathrm{p}}(y_a,m_a)\mathbf{1}_{m_a=m_b}\delta_{y_a-\frac{1}{y_b}} + (-y_a y_b)^{m_a\wedge m_b}\mathrm{sgn}(y_b m_b - y_a m_a)$$

$$\times u_{\mathrm{u}}(y_a,m_a)u_{\mathrm{u}}(y_b,m_b)(\delta_{y_a-1}+\delta_{y_a+1})(\delta_{y_b-1}+\delta_{y_b+1}),$$

$$K_{1,2}(y_a,y_b;m_a,m_b) = u_{\mathrm{u}}(-1,m_a)\delta_{y_a+1} - u_{\mathrm{u}}(1,m_a)\delta_{y_a-1},$$

$$K_{2,2}(y_a,y_b;m_a,m_b) = \frac{\tau^{\frac{1}{2}m_a}y_a - \tau^{\frac{1}{2}m_b}y_b}{\tau^{\frac{1}{2}(m_a+m_b)}y_a y_b - 1},$$

$$u(z,n) = \exp\left[t\left(\frac{1}{1+\tau^{-n/2}z} - \frac{1}{1+\tau^{-n/2}z}\right)\right]\tau^n(1-\tau)^n \frac{\left(-\tau^{-n/2}z;\tau\right)_\infty\left(\tau^{1+n}z^2;\tau\right)_\infty}{\left(-\tau^{n/2}z;\tau\right)_\infty\left(\tau z^2;\tau\right)_\infty},$$

$$u_{\mathrm{u}}(z,n) = \tau^{-\frac{1}{2}n}z\frac{1-\tau^n z^2}{1-\tau^n}, \qquad u_{\mathrm{p}}(z,n) = (-1)^n\tau^{-n}\frac{1+z^2}{z^2-1}.$$

Half-flat moment formula:

$$\underbrace{\mathbb{E}^{\text{flat}}\left[\tau^{\frac{1}{2}mh(t,0)}\right]}_{\sim\tau^{-\frac{1}{4}m^2}} = m!_\tau \sum_{k=0}^{m} \frac{(-1)^{\frac{1}{2}k(k+1)}}{2^k k!} \sum_{\substack{m_1,\dots,m_k=1,\\ m_1+\cdots+m_k=m}}^{\infty} \int_{C_{0,1}^k} d\vec{y}$$

$$\times \prod_{1\le a<b\le k} \tau^{-\frac{1}{2}m_a m_b} \prod_{a=1}^{k} u(y_a, m_a)\,\mathrm{Pf}\big[K(\vec{y};\vec{m})\big]$$

Thus, if we want to sum the moments,

we are forced to premultiply by $\tau^{\frac{1}{4}m^2}$!

Note that premultiplying by $\tau^{\frac{1}{4}m^2}$ gets rid of $\prod_{1\le a<b\le k}\tau^{-\frac{1}{2}m_a m_b}$

Flat ASEP moment formula

Symmetric q-exponential fn.:

$$\exp_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{k_{\tilde{q}}!} = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{4}} \frac{x^k}{k_q!},$$

where $[k]_{\tilde{q}} = \frac{q^{k/2}-q^{-k/2}}{q^{1/2}-q^{-1/2}}$, satisfies

$$\exp_q(x) = \exp_{q^{-1}}(x).$$

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Theorem (Ortmann-Quastel-R '13+)

For $\zeta \in \mathbb{C}$, $|\zeta| < \tau^{1/4}$, $\quad \mathbb{E}^{\text{flat}}\left[\exp_\tau\left(-\zeta\tau^{\frac{1}{2}h(t,0)}\right)\right] = \operatorname{Pf}(J-K)_{L^2[0,\infty)}$.

The Fredholm Pfaffian is defined as

$$\operatorname{Pf}(J-K)_{L^2[0,\infty)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{[0,\infty)^n} \operatorname{Pf}\left[K(x_i,x_j)\right]_{i,j=1}^{n} d\vec{x}$$
$$= \sqrt{\det(I+JK)_{L^2([0,\infty))\otimes L^2([0,\infty))}} \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$K^{\text{ASEP}} = \begin{pmatrix} K_{1,1} & K_{1,2} \\ -K_{1,2} & K_{2,2} \end{pmatrix}$$

$$\begin{aligned}
K_{1,1}(\lambda_1,\lambda_2) &= \sum_{m=1}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_{C_{0,1}} dy\, \tau^{\frac{1}{2}m^2} \zeta^{2m} v(\lambda_1, y, m) v(\lambda_2, 1/y, m) u_1(y,m) \\
&\quad + \sum_{m_1,m_2=1}^{\infty} \sum_{\varsigma_1,\varsigma_2 \in \{-1,1\}} (-\varsigma_1\varsigma_2)^{m_1 \wedge m_2} \operatorname{sgn}(\varsigma_2 m_2 - \varsigma_1 m_1) u_2(\varsigma_1, m_1) u_2(\varsigma_2, m_2) \\
K_{1,2}(\lambda_1,\lambda_2) &= -\sum_{m=1}^{\infty} \sum_{\varsigma \in \{-1,1\}} \varsigma \tau^{\frac{1}{4}m^2} \zeta^m v(\lambda_1, \varsigma, m) u_2(\varsigma, m) \\
K_{2,2}(\lambda_1,\lambda_2) &= \tfrac{1}{2} \operatorname{sgn}(\lambda_2 - \lambda_1)
\end{aligned}$$

with

$$\begin{aligned}
v(\lambda, y, m) = \tfrac{1-\tau^{m/2}y}{1+\tau^{m/2}y} \exp\Bigl(-\lambda \tfrac{1-\tau^{m/2}y}{1+\tau^{m/2}y} + t\Bigl[\tfrac{1}{1+\tau^{-m/2}y} - \tfrac{1}{1+\tau^{-m/2}y}\Bigr]\Bigr) \\
\times \tau^m (1-\tau)^m \frac{\left(-\tau^{-n/2}y;\tau\right)_\infty \left(\tau^{1+n}y^2;\tau\right)_\infty}{\left(-\tau^{n/2}y;\tau\right)_\infty \left(\tau y^2;\tau\right)_\infty},
\end{aligned}$$

$$u_1(y,m) = (-1)^m \tau^{-m} \tfrac{1+y^2}{y^2-1}, \qquad u_2(y,m) = \tau^{-\frac{1}{2}m} y \frac{1-\tau^m y^2}{1-\tau^m}.$$

By formal critical point analysis we can verify that, under the correct scaling, $\mathrm{Pf}(J-K)_{L^2[0,\infty)}$ converges to

$$\mathrm{Pf}\left[J-\begin{pmatrix}\partial_{\lambda_2}K_{\mathrm{Ai}}(\lambda_1,\lambda_2)+\frac{1}{2}\mathrm{Ai}(\lambda_1)\mathrm{Ai}(\lambda_2) & \frac{1}{2}\mathrm{Ai}(\lambda_1)\\ -\frac{1}{2}\mathrm{Ai}(\lambda_2) & \frac{1}{2}\mathrm{sgn}(\lambda_1-\lambda_2)\end{pmatrix}\right]=F_{\mathrm{GOE}}(r),$$

as conjectured.

- The asymptotic analysis of $\mathrm{Pf}(J-K)_{L^2[0,\infty)}$ presents technical difficulties similar to those in the half-flat case.
- Additionally $\exp_\tau$ behaves very badly on $(-\infty,0)$ so convergence of $\mathbb{E}^{\mathrm{flat}}\left[\exp_\tau\left(-\zeta\tau^{\frac{1}{2}h(t,0)}\right)\right]$ does not hold

(moment problem)