Quantization in modular setting, and its applications

Roman Travkin

Institute of Advanced Study
School of Mathematics
travkin@alum.mit.edu

September 30, 2015

Acknowledgements

I want to thank the IAS faculty and staff for their constant support and attention since the first days (and even before) of my stay at the Institute. Special thanks to Prof. MacPherson for sending me in advance a personal invitation to give a talk.

I am also indebted to Roman Bezrukavnikov for his support over the years. I am grateful to him and Prof. Goresky for useful comments and suggestions during my preparation to this talk.

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What do we mean by quantization?

In mathematics, *quantization* refers to a class of algebraic and analytic problems motivated by trying to construct quantum-mechanical systems from classical ones ("quantization" in the physics sense).

In algebra, one is usually interested in deforming a commutative algebra A_0 to non-commutative A over some base, which we usually take to be either the power series algebra $\mathbb{k}[\![\hbar]\!]$ (the standard setting for the so-called *deformation quantization*), or the polynomial algebra $\mathbb{k}[\hbar]$ over a field $\mathbb{k}$. The input for the quantization problem is the $\mathbb{k}$-algebra A_0, and the Poisson structure on A_0 determining the "1st order deformation" of A_0.

In this talk, the Poisson structure is always *symplectic*—i.e., $\operatorname{Spec} A_0$ is smooth and the Poisson bracket is non-degenerate in an appropriate sense.

Let us illustrate the notion of quantization as described above, by the example of quantization of a symplectic affine space.

Basic example: the Weyl algebra

Given a symplectic vector space V over $\Bbbk$ of dimension $2n$, the corresponding algebraic variety is $\operatorname{Spec} \mathsf{S}(V^*)$ where $\mathsf{S}(-)$ denotes the symmetric algebra (basically, a polynomial algebra in $2n$ variables).

One constructs a deformation of $\mathsf{S}(V^*)$ called the *Weyl algebra* $W_\hbar(V^*)$ of V^* as the quotient of the tensor algebra of V^* by the (ideal generated by) relations

$$xy - yx = \hbar\langle x, y\rangle \quad \text{for } x, y \in V^*,$$

where $\langle \cdot, \cdot \rangle$ is the dual symplectic form on V^*.

If we choose a decomposition of V into a direct sum or Lagrangian subspaces: $V = L \oplus L^*$, then the above algebra $W_\hbar(V^*)$ specialized at $\hbar = 1$ is $\mathcal{D}(L)$—a version of the algebra of differential operators on the affine space L (L^* corresponds to multiplication, L—to differentiation).

This is the local model for all quantizations considered in the talk.

Differential operators, and locality of quantization

More generally, for any algebraic variety Y one can consider its cotangent bundle T^*Y. The algebra of functions on it has a natural deformation to a family whose general member is the algebra of differential operators $\mathcal{D}(Y)$ on Y. (Unless Y is affine, have to talk about *sheaves* here.)

Deformation quantization (i.e., over $\Bbbk[\![\hbar]\!]$) of a symplectic variety X is local in X (on affine charts, the product on the quantized algebra is given by a series of bi-differential operators—e.g., Moyal product for the Weyl algebra). This is useful for quantizing non-affine varieties, and for local-to-global techniques à la.Gelfand's formal geometry / Fedosov quantization. Over $\Bbbk[\hbar]$, this is not true: can localize only in half the directions, e.g., on Y instead of T^*Y (cf. "uncertainty principle" in quantum mechanics).

However, in positive characteristic the locality is regained due to the big center of the Weyl algebra. (E.g., the center of the Weyl algebra $\mathcal{D}(\mathbb{A}^1)$ in 2 variables x, ∂ is generated by x^p, ∂^p.) This gives rise to the notions of *p-curvature* of a flat connection and *p-support* of a $\mathcal{D}$-module (a non-conical version of singular support).

Classification of quantizations

The question of classification of quantizations of a given symplectic variety was studied by De Wilde–Lecomte, Fedosov in the C^∞-context, and by Kontsevich, Bezrukavnikov-Kaledin, Yekutieli. . . in the algebraic context.

For the C^∞-case, quantizations are classified, up to a *non-canonical* isomorphism by certain series in $\hbar$ with coefficients in de Rham cohomology of the manifold. In the algebraic case, certain cohomological vanishing restrictions are needed for the quantization to exist. Once those are imposed, have a similar result to the C^∞ case.

References for quantization in the algebraic context are:

- Yekutieli, *Twisted deformation quantization of algebraic varieties* (survey)
- Bezrukavnikov–Kaledin, *Fedosov quantization in algebraic context*
- Bezrukavnikov–Kaledin, *Fedosov quantization in positive characteristic*

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Categorical quantization

In applications, one is often interested in *categories of modules* over the quantized algebras, rather than the algebras themselves, so it is natural to consider a variant on the notion of quantization which deals with the deformation of the category: a sheaf of categories *locally* equivalent to the category of modules over a quantized algebra.

In characteristic 0 the picture for the categorical quantization is much cleaner than for the usual formulation. Namely, there is a canonical categorical quantization over $\mathbb{k}[\![\hbar]\!]$ of arbitrary symplectic variety, as well as a classification of all categorical quantizations in terms of de Rham classes as before, but with no restriction on the variety. (Bezrukavnikov–Kaledin, Yekutieli)

(rough explanation: non-canonicity in the algebra case comes from automorphisms of the Weyl algebra that are identity mod $\hbar$, but they are inner, at least on the level of formal neighborhoods, and act trivially on the category of modules)

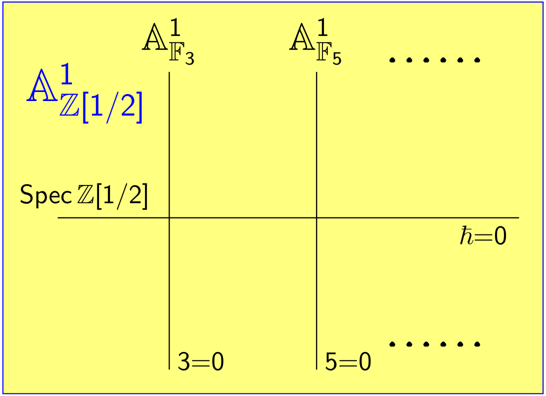
The case of positive characteristic

Due to the big center of the quantized algebras in characteristic p for invertible $\hbar$, the corresponding categories are certain twists of the undeformed categories of coherent sheaves. For differential operators it is known that these twists (given by $\mathbb{G}_m$-gerbes $\approx$ classes in Brauer group $H^2(X, \mathcal{O}_X^\times)$) are obtained by a natural construction from the canonical 1-form on the cotangent bundle. Thus for general symplectic variety in char p, we get canonical quantization category for $\hbar \neq 0$ (i.e., over $\Bbbk[\hbar, \hbar^{-1}]$) in presence of an anti-derivative of the symplectic form.

In a work-in-progress, we show that (for $p > 2$) the category above extends to all $\hbar$ (i.e., over $\Bbbk[\hbar]$). (Cf. Bezrukavnikov–Kaledin, whose setting is a bit different.) Moreover we extend this construction to the case where the base of the quantization is a flat $\mathbb{Z}/p^n$-scheme. Further questions related to this are: classification of *all* quantizations (locally isomorphic to this one), and generalization of the Azumaya property (this is related to p-adic Hodge theory and results by Vologodsky et al.).

Picture of the universal base of quantization

So we get a theory of quantization for scheme over the formal (infintesimal) neighborhood of the following subscheme in $\mathbb{A}^1_{\mathbb{Z}[1/2]} = \operatorname{Spec} \mathbb{Z}[1/2][\hbar]$:



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Applications of quantization in positive characteristic

We finish with a discussion of two applications of quantization in positive characteristic. One of them is concerned with geometric Langlands program, and the other has to do with representation theory of semisimple Lie algebras.

Geometric Langlands

Fix a reductive algebraic group G and a smooth complete algebraic curve C. The geometric Langlands correspondence is (roughly) a conjectural equivalence of derived categories of $\mathcal{D}$-modules on the moduli space Bun_G of G-bundles on C and of quasi-coherent sheaves on the moduli space of de Rham ${}^LG$–local systems on C. Here ${}^LG$ is the so-called *Langlands dual group*.

Originally geometric Langlands correspondence (in the étale sheaf version) arose from non-abelian generalizations of Artin's reciprocity law (Langlands program) in number theory.

Geometric Langlands (in the $\mathcal{D}$-module version) can be considered as a quantization of a Fourier–Mukai duality between *Hitchin integrable systems* for G and ${}^LG$. The Hitchin system is an algebraic integrable system (Lagrangian fibration) whose generic fibers are abelian varieties (for $G = \mathrm{GL}_N$—a family of Jacobians of proper curves—*spectral curves*—in T^*C).

Geometric Langlands via quantization

In order to apply the above theory of quantization to geometric Langlands, we have to work either in the deformation quantization setting, or in positive characteristic.

The former setting was studied by Arinkin; the characteristic p one—by Bezrukavnikov-Braverman, Gröchenig for GL_N, T.-H. Chen and X. Zhu for general reductive group.

My thesis deals with (the GL_N case of) *quantum geometric Langlands* in characteristic p, which is a deformation of the classical one that has twisted $\mathcal{D}$-modules on both sides of the equivalence.

Also, in a joint paper with Bezrukavnikov, we reprove the Beilinson–Drinfeld theorem on the algebra of global twisted differential operators on Bun_G for $G = \mathrm{GL}_N$ using reduction to characteristic p.

Note that all the projects listed above except Gröchenig's work deal with the "generic locus" of the Hitchin base that for GL_N parametrizes *smooth* spectral curves.

Symplectic resolutions

A *symplectic resolution* is a proper birational map $X \to Y$ from a (smooth) symplectic variety X to a singular Poisson variety Y (often with contracting $\mathbb{G}_m$-action), having certain good properties. An example is the Springer resolution of the nilpotent cone of a semisimple Lie algebra $\mathfrak{g}$. Its quantization is the quotient of the enveloping algebra $U(\mathfrak{g})$ by a central character.

Many questions in representation theory are reduced to the study of modules over quantized symplectic singularity (Y above), which using a derived equivalence result can be translated to the quantization of X.

Jointly with D. Kubrak, we show that in characteristic p the Brauer group class coming from the quantization of X descends from X to Y. This allows one to pass between X, Y and their quantizations and is quite useful for connecting representation theory to geometry by realizing representations of the quantized algebra of functions in positive characteristic as complexes of coherent sheaves.

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Other projects

I want to conclude by saying that there are other projects I work on but couldn't mention in the talk, and describing one of them briefly.

Namely, in a joint work with V. Ginzburg and G. Dobrovolska, we generalize the geometric interpretation of the Kac polynomial (counting quiver representations over finite fields) used as a key step in the recent proof of Kac conjecture by Hausel, Letellier and Rodriguez-Villegas, and give it a more geometric proof.

Thank you for coming!