SPECTRA OF METRIC GRAPHS AND CRYSTALLINE MEASURES

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JOINT WORK WITH P. KURASOV

X A COMPACT RIEMANNIAN MANIFOLD

Δ THE LAPLACIAN OF FUNCTIONS

SPECTRUM: $\quad \Delta\phi + k^2\phi = 0$

$\mathrm{SPEC}(X) = \{k\}$ DISCRETE IN $\mathbb{R}$ SATISFYING WEYL ASYMP.

· CHAZARAIN; DUISTERMAAT /GUILLEMIN DETERMINE THE SINGULAR SUPPORT OF THE TEMPERED DISTRIBUTION; THE "WAVE TRACE"

$$\hat{\mu}_X(t) = \mathrm{TRACE}\left(2\cos(\sqrt{\Delta}\,t)\right); \quad \mu_X := \sum_{k \in \mathrm{SPEC}(X)} \delta_k ,$$

IN TERMS OF THE CLOSED ORBITS OF THE GEODESIC FLOW ON $T_1^*(X)$. HERE δ_s IS THE DIRAC POINT MASS AT s.

· IF X HAS BOUNDARY OR IS SINGULAR THE ANALYSIS OF THE PROPOGATION OF SINGULARITIES IS MUCH MORE DELICATE (MELROSE, ...)

EXAMPLE $X = S^1 = \mathbb{R}/\mathbb{Z}$ WITH ARC LENGTH

$\mathrm{Spec}(X) = \mathbb{Z}$; $\phi_m(x) = e^{2\pi i m x}$.

SUMMATION FORMULA IS THE CLASSICAL POISSON SUM

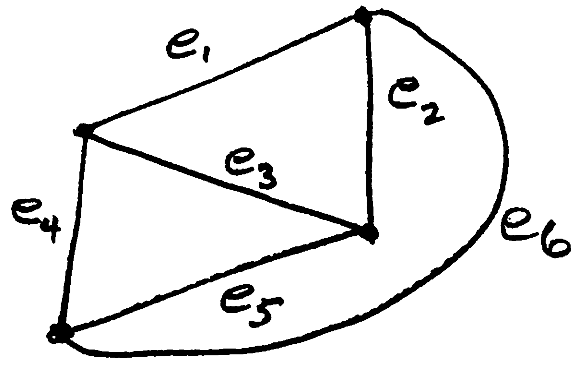
$$\widehat{\sum_{k \in \mathbb{Z}} \delta_k} = \sum_{m \in \mathbb{Z}} \delta_m \; ; \quad \text{ARITHMETIC PROGRESSIONS.}$$

IN GENERAL IT IS RARE THAT $\widehat{\mu_X}$ IS A SUM OF A DISCRETE SET OF POINT MASSES ; WHAT IS CALLED A "CRYTALLINE MEASURE" (MEYER).

• SELBERG'S TRACE FORMULA FOR LOCALLY SYMMETRIC X's GIVES THE FULL DISTRIBUTION $\widehat{\mu_X}$ EXPLICITLY ; THE RIEMANN - GUINAND EXPLICIT FORMULA IN THE THEORY OF ZETA FUNCTIONS GIVES SUCH A CRYTALLINE LIKE STRUCTURE IF "RH" HOLDS.

• WE STICK TO X ONE DIMENSIONAL AND ALLOW IT TO HAVE A FINITE NUMBER OF POINT SINGULARITIES.

METRIC OR QUANTUM GRAPHS:



G COMBINATORIAL CONNECTED GRAPH
N EDGES e_j
M VERTICES v_k

EQUIP THE EDGES WITH LENGTHS l_j, $j = 1, 2, \ldots N$

TO GET A METRIC GRAPH X WHICH IS SMOOTH ON THE EDGES (INTERIOR) SINGULAR AT THE VERTICES.

$$\Delta = \frac{d^2}{dx_j^2}$$ ON FUNCTIONS ϕ ON THE EDGES W.R.T x_j

FOR THE BOUNDARY CONDITIONS AT THE VERTICES WE CHOOSE THE NEUMANN OR KIRCHOFF CONDITION:

- ϕ IS CONTINUOUS AT THE v's
- $\sum_{e} \partial_e \phi(v) = 0$ FOR EACH VERTEX v AND e IS INWARD EDGE TO v.

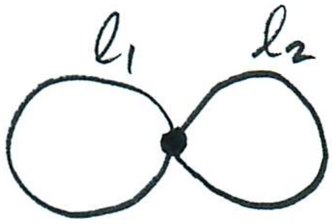
WITH THIS A DEGREE ONE VERTEX  CORRESPONDS TO THE USUAL NEUMANN CONDITION.

A DEGREE TWO VERTEX  HAS A REMOVABLE SINGULARITY; SO ASSUME THERE ARE NO DEGREE TWO VERTICES.

Δ IS SELF ADJOINT AND HAS DISCRETE k SPECTRUM IN $\mathbb{R}$.

- IT IS CONVENIENT TO DEFINE <u>SPEC(X)</u> TO TO BE THE NON-ZERO 'k-SPECTRUM OF Δ' AND TO INCLUDE 0 WITH MULTIPLICITY $2+N-M$.

<u>EXAMPLE:</u>

$X =$ 

; FIGURE EIGHT , $N=2, M=1$

$$\mathrm{Spec}(X) = \left\{ \frac{2\pi k_1}{\ell_1}, \frac{2\pi k_2}{\ell_2}, \frac{2\pi k_3}{\ell_1+\ell_2} : k_1, k_2, k_3 \in \mathbb{Z} \right\}$$

<u>WEYL LAW:</u> FOR ANY X AS ABOVE

$$\#\left\{ \mathrm{Spec}(X) \cap [-T, T] \right\} = \frac{2(\ell_1+\ell_2+\cdots+\ell_N)}{\pi} T + O(1)$$

AS $T \to \infty$.

SO SPEC(X) HAS A DENSITY IN $\mathbb{R}$ WHICH IS THAT OF AN ARITHMETIC PROGRESSION AND μ_X IS LOCALLY UNIFORMLY BOUNDED — THE NUMBER OF ATOMS IN AN INTERVAL OF FIXED LENGTH IS BOUNDED FROM ABOVE.

COMPUTING SPEC(X):

ON THE EDGES AN EIGENCTION TAKES THE FORM $\phi(x_j) = a e^{k x_j} + b e^{-k x_j}$; OUR BOUNDARY CONDITIONS LEAD TO THE SECULAR DETERMINANT (KOTTOS/ SMILANSKY)

. GIVEN THE UNDERLYING GRAPH G DEFINE THE 2N BY 2N MATRICES INDEXED BY THE ORIENTED EDGES $e_1, \hat{e}_1, e_2, \hat{e}_2, \ldots, e_N, \hat{e}_N$

$$U(z_1, \ldots, z_N) = (u_{fg}) \quad ; \quad u_{fg} = z_f \delta_{fg}$$

AND THE SCATTERING MATRIX

$$S = (s_{f,g}) \quad ; \quad s_{fg} = \begin{cases} -\delta_{f\hat{g}} + \dfrac{2}{\deg(v)} & \text{if } g \text{ follows } f \text{ through } v \\ 0 & \text{otherwise} \end{cases}$$

HERE deg(v) is its degree.

S IS UNITARY.

SPECTRAL OR SECULAR POLYNOMIAL OF G:

$$P_G(z_1, z_2, \ldots, z_N) := \det\left(I - U(z_1, \ldots, z_N) S\right).$$

WHICH WE CONSIDER AS A LAURENT POLYNOMIAL IN $z_1, \ldots, z_N$.

IMMEDIATE PROPERTIES OF P_G:

(i) $P_G(z)$ IS DEGREE $2N$ AND IS OF DEGREE TWO IN EACH z_j.

(ii) LET $P^{\iota}(z_1, \ldots, z_N) = P(1/z_1, 1/z_2, \ldots, 1/z_N)$

THEN BOTH P_G AND P_G^{ι} ARE "$D = \{z : |z| < 1\}$ STABLE" THAT IS THEY DON'T VANISH FOR z WITH $z_j \in D$, FOR ALL j (FOLLOWS FROM THE UNITARITY OF S).

- THE CONNECTION TO COMPUTING SPEC(X) IS: (BARRA/GASPARD)

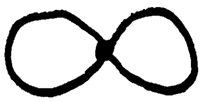
$$\text{spec}(X) = \{\text{ZEROS WITH MULTIPLICITY OF } k \longrightarrow P_G(e^{ikl_1}, e^{ikl_2}, \ldots, e^{ikl_N})\}.$$

CLEARLY THE ALGEBRAIC VARIETY

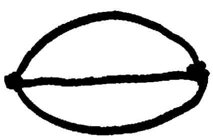
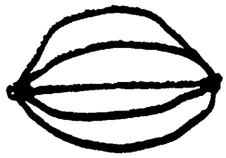
$$Z_G = \{z : P_G(z) = 0\} \subset (\mathbb{C}^*)^N$$

PLAYS A CENTRAL ROLE AND IN PARTICULAR THE QUESTION OF ~~ITS~~ THE FACTORIZATION OF P_G (OVER $\mathbb{C}$).

SPECIAL EXAMPLES:

$G =$  FIGURE EIGHT; $P_G(z_1,z_2) = (z_1-1)(z_2-1)(z_1z_2-1)$

Z_G IS A UNION OF THREE SUBTORI.

$G =$  W_3 ; OR MORE GENERALLY W_N: 

$$P_G(z_1,z_2,z_3) = \left(z_1z_2z_3 + \tfrac{1}{3}(z_1z_2+z_1z_3+z_2z_3) - \tfrac{1}{3}(z_1+z_2+z_3) - 1\right)\left(z_1z_2z_3 - \tfrac{1}{3}(z_1z_2+z_1z_3+z_2z_3) - \tfrac{1}{3}(z_1+z_2+z_3) + 1\right)$$

FACTORIZATION CORRESPONDS TO THE SYMMETRY: REFLECTION THRU THE MIDPOINT OF EACH EDGE.

THEOREM 1 (KURASOV-S):

ASSUME THAT G IS NOT W_N THEN

(i) $$P_G(z) = Q_G(z) . \prod_{e \text{ A LOOP}} (z_e - 1)$$

WHERE THE PRODUCT IS OVER ALL LOOP EDGES IN G, AND $Q_G(z)$ IS ABSOLUTELY IRREDUCIBLE.

(ii) Z_{Q_G} DOES NOT CONTAIN AN $N-1$ DIMENSIONAL SUBTORUS OR TRANSLATE THEREOF UNLESS G IS THE FIGURE EIGHT.

REMARK: PART (i) WAS CONJECTURED BY COLIN DE VERDIERE.

X IS A METRIC GRAPH ON G

THEOREM 2 (K-S) ADDITIVE STRUCTURE OF SPEC(X)

(i) $SPEC(X) = L_1(X) \sqcup L_2(X) \ldots \sqcup L_\nu(X) \sqcup N(X)$ (WITH MULT)

WHERE $L_j(X)$ IS A FULL INFINITE ARITHMETIC PROGRESSION AND THE NON-LINEAR PART $N(X)$ IF NOT EMPTY, SATISFIES

- $\#(N(X) \cap [-T,T]) = \alpha T + O(1)$ FOR T LARGE

WITH $\alpha = \frac{2}{\pi}(\ell_1 + \ell_2 \ldots + \ell_N) - \left(\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_\nu}\right)$, d_j COMMON DIFFERENCE

$\alpha > 0$.

- $DIM_{\mathbb{Q}}\, SPAN(N(X)) = \infty$

- THERE IS $C = C(G) < \infty$ SUCH THAT ANY ARITHMETIC PROGRESSION IN $\mathbb{R}$ MEETS $N(X)$ IN AT MOST C POINTS.

- $SPEC(X)$ IS UNIFORMLY DISCRETE IFF $Z_G \cap T(\ell_1, \ldots, \ell_N)$ IS A SMOOTH $\dim T - 1$ DIMENSIONAL MANIFOLD WHERE $T(\ell_1, \ldots, \ell_N)$ IS THE REAL CONNECTED TORUS WHICH IS THE TOPOLOGICAL CLOSURE OF $k \to (e^{i\ell_1 k}, \ldots, e^{i\ell_N k})_{k \in \mathbb{R}}$, IN $(S^1)^N$.

- IF $\ell_1, \ell_2, \ldots, \ell_N \in \mathbb{Q}$ (PROJECTIVELY) THEN $N(X) = \phi$.

IF $\ell_1, \ell_2, \ldots, \ell_N$ ARE LINEARLY INDEPENDENT OVER $\mathbb{Q}$, THE EXCEPT FOR THE FIGURE EIGHT ν IS EQUAL TO THE NUMBER OF LOOPS IN G, $\dim_{\mathbb{Q}}(SPEC\ X) = \infty$, AND IF G HAS NO LOOPS, $SPEC(X) = N(X)$.

SUMMATION FORMULA FOR X

FOR METRIC GRAPHS THE SUMMATION FORMULA TAKES AN EXACT FORM (ROTH, KOTTOS/SMILANSKY, KURASOV)

$$\widehat{\sum_{k \in \mathrm{spec}(X)} \delta_k} = \frac{2(\ell_1 + \dots \ell_n)}{\pi} \delta_0 + \frac{1}{\pi} \sum_{p \in P} \ell(\mathrm{prim}\, p) \left[S_v(p) \delta_{\ell(p)} + \overline{S_v(p)} \delta_{-\ell(p)} \right]$$

WHERE :

- P IS THE SET OF ORIENTED PERIODIC PATHS IN G UP TO CYCLIC EQUIVALENCE (BACKTRACKING ALLOWED)
- $\ell(p)$ IS THE LENGTH OF THE PATH
- $\mathrm{prim}(p)$ IS THE PRIMITIVE PART OF p (GOING AROUND ONCE)
- $S_v(p)$ IS THE PRODUCT OF THE SCATTERING COEFF ~~AT~~ THE VERTICES ENCOUNTERED ON TRAVERSING p.

$\widehat{\mu_X}$ IS SUPPORTED IN

$$\Lambda = \{ m_1 \ell_1 + m_2 \ell_2 + \dots + m_N \ell_N : m_j \geq 0 \text{ IN } \mathbb{Z} \}$$

WHICH IS A DISCRETE SET, BUT NOT LOCALLY UNIFORMLY BOUNDED.

CRYSTALLINE MEASURES (ALSO KNOWN AS FOURIER QUASI-CRYSTALS)

$$\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \quad ; \quad \hat{\mu} = \sum_{s \in S} b_s \delta_s$$

WITH μ TEMPERED AND Λ, S DISCRETE IN $\mathbb{R}$.

MAIN EXAMPLE: POISSON SUM OR A FINITE UNION OF SUCH ~~SO~~SUMS CALLED DIRAC COMBS.

ARE THERE OTHERS ?

• EXPLICIT FORMULA IN PRIME NUMBER THEORY (RIEMANN, GUINAND, WEIL, ...)

χ_1, χ_2 REAL EVEN DIRICHLET CHARACTERS OF CONDUCTORS q_1, q_2; DENOTE THE NONTRIVIAL ZEROS OF $L(s,\chi_1), L(s,\chi_2)$ BY

$$\tfrac{1}{2} + i\gamma_{\chi_j}$$

$$\mu := -\frac{1}{2}\sum_{\gamma_{\chi_1}} \delta_{\gamma_{\chi_1}} + \frac{1}{2}\sum_{\gamma_{\chi_2}} \delta_{\gamma_{\chi_2}}$$

$$\hat{\mu} = \frac{1}{2}\log(q_1/q_2)\delta_0 + \sum_{p,\, m \geq 1} \frac{(\chi_1(p^m) - \chi_2(p^m))\log p}{p^{m/2}} \delta_{m \log p}$$

NOTE: μ IS TEMPERED IFF RH HOLDS, $|\hat{\mu}|$ IS NOT TEMPERED.

• DYSON'S "BIRDS AND FROGS" SUGGESTS THE CLASSIFICATION OF ONE DIMENSIONAL CRYSTALLINE MEASURES AS A FROGS APPROACH TO RH.

THERE ARE THEOREMS GIVING CONDITIONS ON μ WHICH ENSURE THAT μ IS A DIRAC COMB (MEYER, CORDOBA, LEV-OLEVSKII, FAVOROV, ...)

Y. MEYER: IF $a_\lambda \in F$ WITH F A FINITE SET, AND $|\hat{\mu}|$ IS TRANSLATION BOUNDED ($\sup_{x \in \mathbb{R}} |\hat{\mu}|(x+[0,1]) < \infty$) THEN μ IS A DIRAC COMB.

OUR μ_X's WHEN $N(X) \neq \phi$ ARE FAR FROM DIRAC COMBS.

THEOREM 3 (K-S): ASSUME THAT $N(X) = \mathrm{SPEC}(X)$

(i) μ_X IS A CRYSTALLINE MEASURE
(ii) $\mu_X \geq 0$, $|\hat{\mu}_X|$ IS TEMPERED

} TRUE FOR ANY X

(iii) $\mathrm{DIM}_{\mathbb{Q}} \Lambda = \infty$; $\mathrm{DIM}_{\mathbb{Q}} S < \infty$

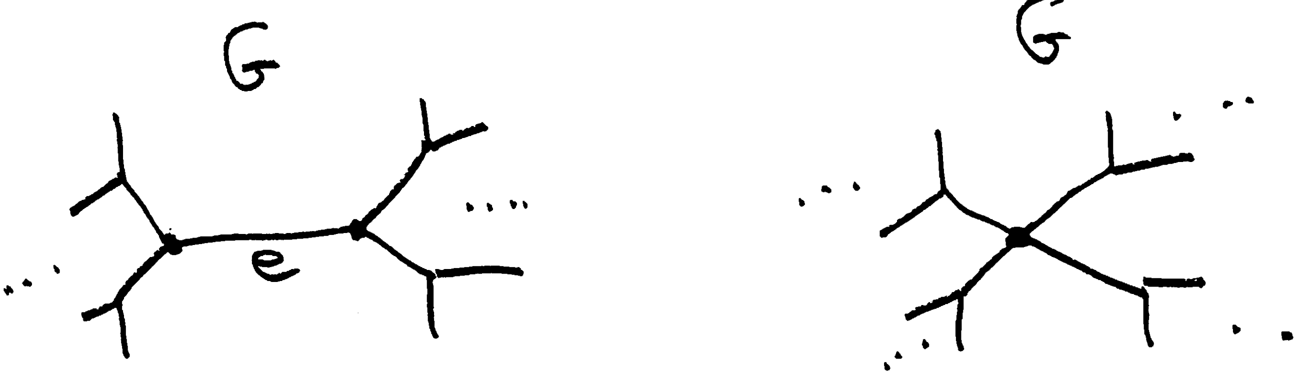
(iv) $\Lambda = \mathrm{SPEC}(X)$ MEETS EVERY ARITHMETIC PROGRESSION IN AT MOST $C(G)$ POINTS.

OTHER CONSTRUCTIONS OF CRYSTALLINE MEASURES WHICH ARE NOT DIRAC COMBS HAVE BEEN GIVEN (GUINAND, MEYER, KOLOUNTZAKIS, LEV-OLEVSKII), HOWEVER THE μ_X's IN THEOREM 3 ARE THE FIRST SUCH WHICH ARE POSITIVE AS WELL AS SATISFYING OTHER TECHNICAL CONDITIONS.

ONE CAN PRODUCE SIMILAR SUCH EXOTIC CRYSTALLINE MEASURES USING ANY SEVERAL VARIABLE $\underline{P}(z_1,\dots,z_N)$ FOR WHICH $\underline{P}$ AND $\underline{P}^{L}$ ARE D-STABLE. FOR EXAMPLE FROM SUCH POLYNOMIALS ARISING IN THE LEE-YANG THEOREM AND THE THEORY OF HYPERBOLIC POLYNOMIALS, WHERE THE PROOF OF STABILITY IS NOT A CONSEQUENCE OF A DETERMINANTAL FORMULA AND UNITARITY.

OUTLINE OF PROOFS:

THE PROOF OF THEOREM 1 IS BASED ON EDGE CONTRACTION



G IS CONTRACTED TO G' BY REMOVING e AND IDENTIFYING THE END POINTS. WE ALLOW THE INTRODUCTION DEGREE TWO VERTICES, LOOPS...

• THE KEY LEMMA ASSERTS THAT IN SUCH A CONTRACTION P_G AND $P_{G'}$ ARE RELATED BY SPECIALIZING THE VARIABLE z_e TO 1.

IN THIS WAY ONE CAN FOLLOW THE FACTORIZATION PROPERTIES OF P_G UNDER REPEATED CONTRACTION. THE "WATER MELLON" GRAPHS WN APPEAR AS END POINTS THAT NEED SPECIAL ATTENTION, AND OTHERWISE ONE NAVIGATES, THE CONTRACTIONS TO A FINITE (SMALL) NUMBER OF CONFIGURATIONS THAT ARE EXAMINED DIRECTLY.

THEOREM 2 IS BASED ON SOME ADVANCED RESULTS IN DIOPHANTINE ANALYSIS ON TORI.

LANG'S $\mathbb{G}_m$ CONJECTURES:

THERE ARE TWO FLAVORS; VERTICAL AND HORIZONTAL, WE NEED BOTH.

$$\mathbb{G}_m = \text{MULTIPLICATIVE GROUP } \mathbb{C}^*$$

$T = (\mathbb{C}^*)^N$ IS AN N-TORUS, IT IS AN ALGEBRAIC GROUP UNDER COORDINATE PRODUCT.

$V \subset (\mathbb{C}^*)^N$ AN ALGEBRAIC SUBVARIETY

GIVEN BY THE ZERO SET OF LAURENT POLYNOMIALS.

$$\mathrm{tor}(T) = \{(z_1, .., z_N) : z_j \text{ IS A ROOT OF UNITY FOR ALL } j=1,..,N\}$$

$\mathrm{tor}(T)$ CONSISTS OF ALL POINTS IN T OF FINITE ORDER.

VERTICAL LANG CONJ:

GIVEN $V \subset T$ AS ABOVE, THERE ARE FINITELY MANY SUBTORI OR TRANSLATES THEREOF BY TORSION POINTS, $T_1, T_2, \ldots, T_\nu$ CONTAINED IN V SUCH THAT

$$\mathrm{tor}(T) \cap V = \mathrm{tor}(T) \cap (T_1 \cup T_2 \ldots \cup T_\nu).$$

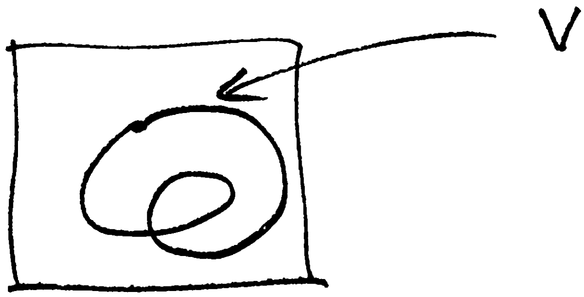
SO WHAT APPEARS TO BE A NON-LINEAR COMPLICATED PROBLEM IS IN FACT VERY STRUCTURED IN THAT TORSION POINTS CAN ONLY LIE ON A FINITE NUMBER OF COSETS OF SUBGROUPS. NOTE THE T_j's MAY BE ZERO DIMENSIONAL IN WHICH CASE THEY ARE TORSION POINTS.

THERE ARE A NUMBER OF PROOFS OF THIS VERTICAL CASE AND THE PROOF CAN BE MADE EFFECTIVE IN THAT THE T_j's ARE DETERMINED.

ONE PROOF PROCEEDS AS FOLLOWS:

N=2:

$V \cap (S^1 \times S^1) \subset V \cap T$



IF $\zeta = (\zeta_1, \zeta_2) \in tor(T) \cap V$, $\zeta_1^{m_1} = 1$, $\zeta_2^{m_2} = 1$

AND $\sigma \in GAL(K(\zeta_1, \zeta_2)/K)$ WHERE K IS THE FIELD OF DEFINITION OF V; THEN

$$\sigma((\zeta_1, \zeta_2)) \in tor(T) \cap V.$$

NOW THESE GALOIS ORBITS GROW FAST AS THE ORDER OF ζ INCREASES

$$\deg[\mathbb{Q}(\zeta_m) : \mathbb{Q}] = \phi(m) \gg m^{1-\varepsilon}.$$

HENCE IF ONE CAN ESTABLISH A SUITABLE NON TRIVIAL UPPER BOUND FOR THE NUMBER OF TORSION POINTS OF SUCH ORDER ON V (ASSUMING V DOES NOT CONTAIN SUBTORI) THEN ONE IS LED TO THERE BEING NO SUCH POINTS OF LARGE ORDER.

SUCH UPPER BOUNDS CAN BE GIVEN IN THIS TORUS CASE BY ELEMENTARY METHODS.

THIS UPPER BOUND VS GALOIS ORBIT METHOD HAS PROVEN TO BE ROBUST FOR OTHER VERTICAL PROBLEMS:

- BOMBIERI-PILA; GIVE UPPER BOUNDS SHARP UP TO EXPONENT FOR TRANSCENDETAL. CURVES IN THE PLANE; FOR RATIONAL POINTS

- PILA-WILKIE GIVE SHARP UPPER BOUNDS FOR RATIONAL POINTS ~~[illegible]~~ ON THE TRANSCENTAL PARTS OF DEFINABLE SETS IN O-MINIMAL STRUCTURES IN $\mathbb{R}^n$.

- PILA-ZANNIER PROVE THE VERTICAL ABELIAN VARIETY VERSION OF LANG'S CONJ, ALSO KNOWN AS THE MANIN-MUMFORD CONJ.

- THE VERTICAL ANALOGUE IN SHIMURA VARIETIES OF TORSION POINTS ARE "CM-POINTS" AND THESE LIE ON FINITELY MANY SHIMURA SUBVARIETIES "ANDRE-OORT" CONJ.

- PROVED FOR PRODUCTS OF MODULAR CURVES BY PILA
- PROVED FOR $\mathcal{A}_g$ BY PILA AND TSIMERMAN.

HORIZONTAL LANG G_m CONJ FOR $T=(\mathbb{C}^*)^N$

IF $V \subset T$ IS AS ABOVE AND Γ IS A FINITELY GENERATED SUBGROUP OF T, THERE FINITELY MANY TRANSLATES OF SUBTORI $T_1, T_2, \ldots, T_\nu$ IN V, SUCH THAT

$$\Gamma \cap V = \Gamma \cap (T_1 \cup T_2 \cdots \cup T_\nu).$$

THIS LIES DEEPER AND IT WAS PROVEN BY M. LAURENT. THE KEY INPUT IS THE SCHMIDT SUBSPACE THEOREM WHICH IS A STRIKING HIGHER DIMENSIONAL VERSION OF THE THUE-SIEGEL-ROTH THEOREM.

SIMPLEST VERSION (SCHMIDT)

LET $L_1(x), L_2(x), \ldots, L_n(x)$ BE n linearly INDEPENDENT LINEAR FORMS IN $(x_1, \ldots x_n) = x$ WITH REAL ALGEBRAIC COEFFICIENTS; THEN FOR $\varepsilon > 0$ THE SET OF SOLUTIONS WITH $x \in \mathbb{Z}^n$ OF

$$|L_1(x) L_2(x) \cdots L_n(x)| < \|x\|^{-\varepsilon}$$

LIE IN FINITELY MANY PROPER $\mathbb{Q}$-LINEAR SUBSPACES OF $\mathbb{Q}^n$.

NOTE: THE PROOF YIELDS AN EFFECTIVE BOUND FOR THE NUMBER OF SUBSPACES BUT NOT FOR THEIR DETERMINATION

VERTICAL AND HORIZONTAL:

TO COMBINE THE TWO LET $\overline{\Gamma}$ BE THE DIVISION GROUP OF Γ

$$\overline{\Gamma} = \{ z \in T : z^{\ell} \in \Gamma \text{ FOR SOME } \ell \geq 1 \}$$

(SO $\overline{1} = \mathrm{tor}(T)$).

THE ULTIMATE VERSION WHICH IS ALSO UNIFORM OVER THE DEFINING FIELDS AND QUANTITATIVE IN THE ~~rank~~ r OF Γ IS DUE TO EVERSTE/SCHLICKEWEI/SCHMIDT:

THEOREM: $V \subset (\mathbb{C}^*)^N$, Γ A FINITELY GENERATED SUBGROUP OF RANK r; THERE ARE $T_1, T_2, \ldots T_\nu$ ~~an~~ TRANSLATES OF SUBTORI CONTAINED IN V SUCH THAT

$$\overline{\Gamma} \cap V = \overline{\Gamma} \cap (T_1 \cup T_2 \cdots \cup T_\nu)$$

AND $$\nu \leq (c(V))^{r}.$$

REMARK: THE CONSTANT $c(V)$ CAN BE GIVEN EXPLICITLY, HOWEVER THE ACTUAL SAY ZERO DIMENSIONAL T_j's CANNOT IN GENERAL BE DETERMINED BY THIS PROOF.

THE PROOF INVOLVES SPECIALIZATION ARGUMENTS REDUCING TO $\Gamma \subset T(\bar{\mathbb{Q}})$ AND ABSOLUTE VERSIONS OF THE SCHMIDT SUBSPACE THEOREM, AS WELL AS A STUDY OF POINTS OF SMALL HEIGHT AND LARGE HEIGHT . . .

AFTER ANALYZING OUR SUBVARIETIES Z_G AND APPLYING THE DIOPHANTINE ANALYSIS WE ARRIVE AT:

- GIVEN G THERE IS $\varepsilon(G) > 0$ SUCH THAT FOR ANY t DISTINCT POINTS IN $N(X)$, $x_1, x_2, \ldots, x_t$

$$\dim_{\mathbb{Q}} \operatorname{span}(x_1, \ldots, x_t) \geq \varepsilon(G) \log t .$$

WHICH LEADS TO THEOREM 2.

We conclude that if a hypertoric factor is a degree two in any of the variables it depends on, then it is a degree two polynomial in all other variables it depends on and is given by

$$T(z_1, \ldots, z_n) = z_1^2 z_2^2 \ldots z_m^2 - 1, \tag{49}$$

which is obviously factorizable as in the case of one variable

$$(z_1 z_2 \ldots z_m - 1)(z_1 z_2 \ldots z_m + 1).$$

We conclude that any (irreducible) hypertoric factor is a first degree polynomial in all variables it depends on. □

Let us study how first order hypertoric factors may look like.

Theorem 5. *Let G be a finite connected graph without degree two vertices, not a segment and not figure-eight graph, then any irreducible hypertoric factor in the secular polynomial is of the form*

$$T(\vec{z}) = z_j - 1, \tag{50}$$

and occurs if and only if the edge e_j forms a loop in G. If G is a segment or a figure-eight graph, then the secular polynomials are

$$P(z_1) = (z_1 - 1)(z_1 + 1) \quad \textit{and} \quad P(z_1, z_2) = (z_1 - 1)(z_2 - 1)(z_1 z_2 - 1) \tag{51}$$

respectively and contain additional hypertoric factors $z_1 + 1$ and $z_1 z_2 - 1$.

Proof. Consider arbitrary connected graph G which is not a watermelon and the corresponding secular polynomial. Let G have d loops formed by $e_1, \ldots, e_d$, then the secular polynomial contains hypertoric factors $(z_j - 1)$, $j = 1, 2, \ldots, d$ in accordance with Theorem 4. The irreducible factor $Q_G(\vec{z})$ appearing in the factorisation (45) is a first degree polynomial in $z_1, \ldots, z_d$ and second degree in all other variables. But Lemma 5 states that any hypertoric factor is a first degree polynomial in all variables it depends on. Hence d coincides with the number of edges in G, *i.e.* all edges in G form loops. In other words, G is a flower graph $\mathbf{F}_d$. The factor $Q_{\mathbf{F}_d}$ is hypertoric if and only if $d = 1, 2$ when G is a loop or a figure-eight graph.
In the case G is a watermelon the two factors are not hypertoric unless $d = 2$ corresponding to the loop graph on two edges, which contains degree two vertices and therefore is excluded. □

8. Arithmetic properties of the spectrum

9. Spectral measures and crystalline measures.

The spectra of metric graphs yield exotic measures related to the theory of quasi-crystals. We review briefly some of the relevant theory following the recent paper [35] before examining the properties of the spectral measures of metric graphs.

Definition 4. ([35]) A tempered distribution μ is a crystalline measure if μ and $\hat{\mu}$ are of the form

$$\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda, \quad \hat{\mu} = \sum_{s \in S} b_s \delta_s, \tag{52}$$

with Λ and S discrete subsets of $\mathbb{R}$.

The basic examples of such measures come from the Poisson summation formula which asserts that $\mu = \sum_{m\in\mathbb{Z}} \delta_m = \hat{\mu}$. Finite linear combinations of these are called Dirac combs and for these Λ and S are finite unions of arithmetic progressions.
Guinand [18] pointed to other crystalline measures and in particular ones coming from the explicit formula in the theory of prime numbers. If χ_1, χ_2 are primitive even real Diricihlet characters of conductors q_1 and q_2 and the non-trivial zeros of Diricihlet functions $L(s,\chi_1)$ and $L(s,\chi_2)$ are denoted by $\frac{1}{2}+i\gamma^{(\chi_1)}$ and $\frac{1}{2}+i\gamma^{(\chi_2)}$, then assuming the Riemann hypothesis (that is that the γ's are real) we have that for

$$\begin{aligned} \mu &= -\frac{1}{2}\sum_{\gamma^{(\chi_1)}} \delta_{\gamma^{(\chi_1)}} + \frac{1}{2}\sum_{\gamma^{(\chi_2)}} \delta_{\gamma^{(\chi_2)}}, \\ \hat{\mu} &= \frac{1}{2}\log\Big(\frac{q_1}{q_2}\Big)\delta_0 + \sum_{p,m} \frac{\big(\chi_1(p^m)-\chi_2(p^m)\big)\log p}{p^{m/2}}\delta_{m\log p}, \end{aligned} \tag{53}$$

the last sum being over $m \geq 1$ and p prime. Clearly μ is a crystalline measure. Similar crystalline measures can be constructed from the Selberg trace formula (and without any unproven hypotheses).
While μ is tempered in (53) and hence so is $\hat{\mu}$, note that $|\hat{\mu}|$ is not tempered, since there is an exponential in x number of $\log p$ in an interval $[x, x+1]$. The same applies to the crystalline measures coming from the Selberg trace formula. For our (one-dimensional) metric graphs the support of S is contained in the set of the lengths of the periodic orbits and μ is tempered, and even though there is an exponential number of closed orbits of a given large length inside $[x, x+1]$, $|\hat{\mu}|$ is tempered.[2] This points to a fundamental difference to the crystalline measures coming from the explicit formula.
One of the central questions is to understand the crystalline measures which are not Dirac combs. There is a number of results which show that under some additional conditions on μ and $\hat{\mu}$, that μ must be a Dirac comb. A couple of these that we will make us of are:

(1) [34] If μ is a crystalline measure and a_λ for $\lambda \in \Lambda$ takes values in a finite set and $|\hat{\mu}|$ is translation bounded (that is $\sup_{x\in\mathbb{R}} |\hat{\mu}|(x+[0,1])$ is finite), then μ is a Dirac comb (a key ingredient in the proof is the idempotent theorem in [8]).
(2) [32] [Theorem 2.1] If μ is a positive Fourier quasi-crystal and S is uniformly discrete (that is $|s-s'| \geq \epsilon > 0$ for some $\epsilon > 0$ and all $s \neq s'$ in S) then μ is a Dirac comb.

Various constructions of Fourier quasi-crystals from Dirac combs have been given recently [22, 31, 35, 36]. These are gotten from Voronoi summation formulae in odd dimensions, projections of higher dimensional lattices and delicate limits of Dirac combs. A basic question that has been open for some time is whether a positive crystalline measure or a Fourier quasi-crystal must be a Dirac comb. The next theorem gives the properties of the spectral measure μ_Γ of a metric graph Γ. It provides an answer to the last question as well as a number of others. Before stating the theorem we recall a few more definitions. A *distribution μ is almost periodic* if $\mu * \phi$ is a Bohr almost periodic function for every C^∞ function ϕ of compact support. A *measure μ is almost periodic* if $\mu * \phi$ is Bohr almost periodic

[2]Crystalline measures with $|\mu|$ and $|\hat{\mu}|$ tempered are often called 'Fourier quasi-crystals".

function for every continuous ϕ of compact support. Finally a discrete subset of $\mathbb{R}$ is a *Delone set* if it is uniformly discrete and relatively dense in $\mathbb{R}$, that is every interval of length R for some large enough R meets the set.
Our measures $\mu_\Gamma = \sum_{k\in\text{spec }(\Gamma)} \delta_k$ enjoy the following set of properties under specialization of Γ :

Theorem 6.

(1) μ_Γ *is crystalline.*
(2) $\mu_\Gamma \geq 0$*,* a_λ *takes on only finitely many positive integer values.* $|\mu_\Gamma|$ $(= \mu_\Gamma)$ *is translation bounded and distribution almost periodic.*
(3) $|\hat{\mu}_\Gamma|$ *is tempered.*
(4) $\dim_{\mathbb{Q}} S < \infty$.
(5) If $N(\Gamma) \neq \emptyset$ *then* $\dim_{\mathbb{Q}} \Lambda = \infty$ *and* $|\hat{\mu}_\Gamma|$ *is not translation bounded,* μ_Γ *is not an almost periodic measure, and* S *is not a Delone set.*
(6) If $N(\Gamma) = \text{spec }(\Gamma)$ *then there is* $C = C(G) < \infty$*, such that* Λ *contains no more than* C *elements in any arithmetic progression in* $\mathbb{R}$.
(7) If T *is a connected subtorus of* S^N *and* $\mathbf{Z}_T$ *is the restriction of* $\mathbf{Z}_G$ *to* T*, is smooth, then for any* $(\ell_1, \dots, \ell_N) \in T$ *the support* Λ *of* μ_Γ *is a Delone set. Moreover if* G *is a tree then* $a_\lambda = 1$ *for* $\lambda \in \Lambda$*, that is* μ_Γ *is "idempotent".*

The theorems in the earlier sections describe when Γ satisfies the conditions in the theorem including ones satisfying all conditions. Positive crystalline measures which are not Dirac combs are provided by any μ_Γ satisfying (5), answering the last questions in [35]. Part 3 of Question 11.2 in [32] asks for such a positive measure for which every arithmetic progression meets Λ in a finite set, this is provided by μ_Γ's with Γ satisfying (6). For Γ satisfying (7) the support of μ_Γ is Delone set but the support of $\hat{\mu}$ is not which answers the other question on page 3158 of [35] and Part 2 of Question 11.2 in [32]. Finally the idempotent measures in (7) give Bohr almost periodic Delone sets which are not ideal crystals answering Problem 4.4 in [28].

10. Perspectives

References

BaLe17 [1] Ram Band and Guillaume Lévy, *Quantum graphs which optimize the spectral gap*, Ann. Henri Poincaré **18** (2017), no. 10, 3269–3323, DOI 10.1007/s00023-017-0601-2. MR3697195

BaGa [2] F. Barra and P. Gaspard, *On the level spacing distribution in quantum graphs*, J. Statist. Phys. **101** (2000), no. 1-2, 283–319, DOI 10.1023/A:1026495012522. Dedicated to Grégoire Nicolis on the occasion of his sixtieth birthday (Brussels, 1999). MR1807548

BoMaZa99 [3] E. Bombieri, D. Masser, and U. Zannier, *Intersecting a curve with algebraic subgroups of multiplicative groups*, Internat. Math. Res. Notices **20** (1999), 1119–1140, DOI 10.1155/S1073792899000628. MR1728021

BoBr09 [4] Julius Borcea and Petter Brändén, *The Lee-Yang and Pólya-Schur programs. I. Linear operators preserving stability*, Invent. Math. **177** (2009), no. 3, 541–569, DOI 10.1007/s00222-009-0189-3. MR2534100

BoBr09-2 [5] ———, *The Lee-Yang and Pólya-Schur programs. II. Theory of stable polynomials and applications*, Comm. Pure Appl. Math. **62** (2009), no. 12, 1595–1631, DOI 10.1002/cpa.20295. MR2569072

BeKu [6] Gregory Berkolaiko and Peter Kuchment, *Introduction to quantum graphs*, Mathematical Surveys and Monographs, vol. 186, American Mathematical Society, Providence, RI, 2013. MR3013208

Br14 [7] Petter Brändén, *The Lee-Yang and Pólya-Schur programs. III. Zero-preservers on Bargmann-Fock spaces*, Amer. J. Math. **136** (2014), no. 1, 241–253, DOI 10.1353/ajm.2014.0003. MR3163359

Co60 [8] Paul J. Cohen, *On a conjecture of Littlewood and idempotent measures*, Amer. J. Math. **82** (1960), 191–212, DOI 10.2307/2372731. MR133397

Co88 [9] Antonio Córdoba, *La formule sommatoire de Poisson*, C. R. Acad. Sci. Paris Sér. I Math. **306** (1988), no. 8, 373–376 (French, with English summary). MR934622

Co89 [10] A. Córdoba, *Dirac combs*, Lett. Math. Phys. **17** (1989), no. 3, 191–196, DOI 10.1007/BF00401584. MR995797

CdV [11] Yves Colin de Verdière, *Semi-classical measures on quantum graphs and the Gaußmap of the determinant manifold*, Ann. Henri Poincaré **16** (2015), no. 2, 347–364, DOI 10.1007/s00023-014-0326-4. MR3302601

DuGu75 [12] J. J. Duistermaat and V. W. Guillemin, *The spectrum of positive elliptic operators and periodic bicharacteristics*, Invent. Math. **29** (1975), no. 1, 39–79, DOI 10.1007/BF01405172. MR405514

Ev99 [13] Jan-Hendrik Evertse, *Points on subvarieties of tori*, A panorama of number theory or the view from Baker's garden (Zürich, 1999), Cambridge Univ. Press, Cambridge, 2002, pp. 214–230, DOI 10.1017/CBO9780511542961.015. MR1975454

EvSchSch02 [14] J.-H. Evertse, H. P. Schlickewei, and W. M. Schmidt, *Linear equations in variables which lie in a multiplicative group*, Ann. of Math. (2) **155** (2002), no. 3, 807–836, DOI 10.2307/3062133. MR1923966

Fa19 [15] S. Yu. Favorov, *Large Fourier quasicrystals and Wiener's theorem*, J. Fourier Anal. Appl. **25** (2019), no. 2, 377–392, DOI 10.1007/s00041-017-9576-0. MR3917950

Fr05-IJM [16] Leonid Friedlander, *Genericity of simple eigenvalues for a metric graph*, Israel J. Math. **146** (2005), 149–156, DOI 10.1007/BF02773531. MR2151598

Fr05-AIF [17] ———, *Extremal properties of eigenvalues for a metric graph*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 199–211 (English, with English and French summaries). MR2141695

Gu59 [18] A. P. Guinand, *Concordance and the harmonic analysis of sequences*, Acta Math. **101** (1959), 235–271, DOI 10.1007/BF02559556. MR107784

GuSm [19] Boris Gutkin and Uzy Smilansky, *Can one hear the shape of a graph?*, J. Phys. A **34** (2001), no. 31, 6061–6068, DOI 10.1088/0305-4470/34/31/301. MR1862642

Ga11 [20] L. Garding, *Introduction to hyperbolicity*, Hyperbolicity, C.I.M.E. Summer Sch., vol. 72, Springer, Heidelberg, 2011, pp. 73–123, DOI 10.1007/978-3-642-11105-1_3. MR3185906

KaMa [21] J.-P. Kahane and S. Mandelbrojt, *Sur l'équation fonctionnelle de Riemann et la formule sommatoire de Poisson*, Ann. Sci. École Norm. Sup. (3) **75** (1958), 57–80 (French). MR0106208

Ko16 [22] Mihail N. Kolountzakis, *Fourier pairs of discrete support with little structure*, J. Fourier Anal. Appl. **22** (2016), no. 1, 1–5, DOI 10.1007/s00041-015-9416-z. MR3448912

KoSm99 [23] Tsampikos Kottos and Uzy Smilansky, *Periodic orbit theory and spectral statistics for quantum graphs*, Ann. Physics **274** (1999), no. 1, 76–124, DOI 10.1006/aphy.1999.5904. MR1694731

KuBook [24] Pavel Kurasov, *Quantum graphs: specral theory and inverse problems*, Birkhäuser, 2020.

KuJFA [25] ———, *Schrödinger operators on graphs and geometry. I. Essentially bounded potentials*, J. Funct. Anal. **254** (2008), no. 4, 934–953, DOI 10.1016/j.jfa.2007.11.007. MR2381199

KuArk [26] ———, *Graph Laplacians and topology*, Ark. Mat. **46** (2008), no. 1, 95–111, DOI 10.1007/s11512-007-0059-4. MR2379686

KuNo [27] Pavel Kurasov and Marlena Nowaczyk, *Inverse spectral problem for quantum graphs*, J. Phys. A **38** (2005), no. 22, 4901–4915, DOI 10.1088/0305-4470/38/22/014. MR2148632

La00 [28] Jeffrey C. Lagarias, *Mathematical quasicrystals and the problem of diffraction*, Directions in mathematical quasicrystals, CRM Monogr. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2000, pp. 61–93. MR1798989

La84 [29] Michel Laurent, *Équations diophantiennes exponentielles*, Invent. Math. **78** (1984), no. 2, 299–327, DOI 10.1007/BF01388597 (French). MR767195

LeOl15 [30] Nir Lev and Alexander Olevskii, *Quasicrystals and Poisson's summation formula*, Invent. Math. **200** (2015), no. 2, 585–606, DOI 10.1007/s00222-014-0542-z. MR3338010

LeOl16 [31] ———, *Quasicrystals with discrete support and spectrum*, Rev. Mat. Iberoam. **32** (2016), no. 4, 1341–1352, DOI 10.4171/RMI/920. MR3593527

LeOl17 [32] ———, *Fourier quasicrystals and discreteness of the diffraction spectrum*, Adv. Math. **315** (2017), 1–26, DOI 10.1016/j.aim.2017.05.015. MR3667579

Li74 [33] Pierre Liardet, *Sur une conjecture de Serge Lang*, Journées Arithmétiques de Bordeaux (Conf., Univ. Bordeaux, Bordeaux, 1974), Soc. Math. France, Paris, 1975, pp. 187–210. Astérisque, Nos. 24-25 (French). MR0376688

Me70 [34] Yves Meyer, *Nombres de Pisot, nombres de Salem et analyse harmonique*, Lecture Notes in Mathematics, Vol. 117, Springer-Verlag, Berlin-New York, 1970 (French). Cours Peccot donné au Collège de France en avril-mai 1969. MR0568288

Me16 [35] Yves F. Meyer, *Measures with locally finite support and spectrum*, Proc. Natl. Acad. Sci. USA **113** (2016), no. 12, 3152–3158, DOI 10.1073/pnas.1600685113. MR3482845

Me17 [36] Yves Meyer, *Guinand's measures are almost periodic distributions*, Bull. Hellenic Math. Soc. **61** (2017), 11–20. MR3657425

Me17-2 [37] ———, *Measures with locally finite support and spectrum*, Rev. Mat. Iberoam. **33** (2017), no. 3, 1025–1036, DOI 10.4171/RMI/962. MR3713038

Ro84 [38] Jean-Pierre Roth, *Le spectre du laplacien sur un graphe*, Théorie du potentiel (Orsay, 1983), Lecture Notes in Math., vol. 1096, Springer, Berlin, 1984, pp. 521–539, DOI 10.1007/BFb0100128 (French). MR890375

Ru10 [39] David Ruelle, *Characterization of Lee-Yang polynomials*, Ann. of Math. (2) **171** (2010), no. 1, 589–603, DOI 10.4007/annals.2010.171.589. MR2630051

Sa87 [40] Peter Sarnak, *Determinants of Laplacians*, Comm. Math. Phys. **110** (1987), no. 1, 113–120. MR885573

SaAd(Ru)94 [41] Peter Sarnak and Scot Adams, *Betti numbers of congruence groups*, Israel J. Math. **88** (1994), no. 1-3, 31–72, DOI 10.1007/BF02937506. With an appendix by Ze'ev Rudnick. MR1303490

Sch99 [42] Wolfgang M. Schmidt, *The zero multiplicity of linear recurrence sequences*, Acta Math. **182** (1999), no. 2, 243–282, DOI 10.1007/BF02392575. MR1710183

StTe96 [43] H. M. Stark and A. A. Terras, *Zeta functions of finite graphs and coverings*, Adv. Math. **121** (1996), no. 1, 124–165, DOI 10.1006/aima.1996.0050. MR1399606

Su85 [44] Toshikazu Sunada, *L-functions in geometry and some applications*, Curvature and topology of Riemannian manifolds (Katata, 1985), Lecture Notes in Math., vol. 1201, Springer, Berlin, 1986, pp. 266–284, DOI 10.1007/BFb0075662. MR859591

Wa11 [45] David G. Wagner, *Multivariate stable polynomials: theory and applications*, Bull. Amer. Math. Soc. (N.S.) **48** (2011), no. 1, 53–84, DOI 10.1090/S0273-0979-2010-01321-5. MR2738906