Random walk on groups

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Set-up

- G a topological (finite) group, with Borel sigma algebra $\mathscr{B}$
- $\mathscr{P}(G)$ the set of Borel probability measures on G
- For $\mu, \nu \in \mathscr{P}(G)$, $f \in C_c(G)$,

$$\langle f, \mu * \nu \rangle = \int_G \int_G f(xy) d\mu(x) d\nu(y)$$

- Consider, for $\mu \in \mathscr{P}(G)$, the large n behavior of μ^{*n} as a weak-* limit in one of several function spaces, e.g. $L^\infty(G)$, Lipschitz functions, Sobolev spaces, etc., and also the growth of $\mathrm{supp}(\mu^{*n})$
- We seek quantitative statements, e.g. a rate of convergence

Example: riffle shuffling

Let $N > 1$ and consider the following random walk on the symmetric group $\mathfrak{S}_N$ (Gilbert-Shannon-Reeds)

- μ is the distribution on $\mathfrak{S}_N$ given by
  - Choose $1 \leq n \leq N$ according to the binomial distribution $\mathrm{Prob}(n) = \frac{\binom{N}{n}}{2^N}$
  - Conditioned on the value of n, the measure is uniform over all permutations which preserve the relative order of the first n and last $N - n$ cards
- Convergence to uniform is observed after $\frac{3}{2}\log_2 N + O(1)$ steps in the total variation (L^1) metric [1], [2]

Motivation

- Rich source of Markov chains which can be rigorously analyzed
- Useful in computational group theory software
- Source of expander graphs [3] with applications in number theory [4]
- Gives a simplified model for the dynamics of e.g. flows in linear groups
- Non-commutative analogues of the classical theorems of probability theory in Euclidean space
  1. Central and local limit theorems
  2. Simple random walk

Sample theorem

Let $U_m(\mathbb{Z})$, $m \geq 3$ be the upper triangular group of $m \times m$ matrices

$$U_m(\mathbb{Z}) = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} & \cdots & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} & \cdots & \mathbb{Z} \\ \vdots & & \ddots & \ddots & \vdots \\ & & & 1 & \mathbb{Z} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}_{m \times m}$$

- $Z_{1,m}$ denotes the upper right corner (central) coordinate.
- M_j is the matrix with 1 at jth position in the first super-diagonal, $M_j = I + e_j \otimes e_{j+1}$.
- Measure $\mu_m \in \mathscr{M}(U_m(\mathbb{Z}))$ is uniform on the set $\{I_m, M_1^{\pm}, ..., M_{m-1}^{\pm}\}$

Sample theorem

Theorem (Diaconis, H. 2015)

Let $m \geq 3$ and let $c > 0$. As prime $p \to \infty$, for $n \sim cp^{\frac{2}{m-1}}$

$$\sum_{x \bmod p} \left| \mu_m^{*n}(Z_{1,m} \equiv x \bmod p) - \frac{1}{p} \right| \ll e^{-c}.$$

Our method can handle more general measures and (in progress) gives a local limit theorem extending [5] to discrete nilpotent groups, e.g. for all $a, b, c \in \mathbb{Z}$,

$$\mu_3^{*n}\left(\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}\right) = \nu\left(\begin{pmatrix} 1 & \frac{a}{\sqrt{n}} & \frac{c}{n} \\ 0 & 1 & \frac{b}{\sqrt{n}} \\ 0 & 0 & 1 \end{pmatrix}\right) + O(n^{-2-\delta}).$$

where ν is the density function of a Gaussian measure on $U_3(\mathbb{R})$.

Ideas in the proof

- Let $\underline{w} = w_1 w_2 \cdots w_{Mk}$ be a typical word on generators. We use a product group action on segments of generators to demonstrate smoothness in the measure at various scales

$$\overbrace{w_1 \cdots w_k}^{G\circlearrowleft} \overbrace{w_{k+1} \cdots w_{2k}}^{G\circlearrowleft} \cdots \overbrace{w_{(M-1)k+1} \cdots w_{Mk}}^{G\circlearrowleft}$$

  via commutator calculus.
- For $n \geq 5$ the separate factors in the product group act non-locally (dependently). The non-local behavior is eliminated by applying the Gowers-Cauchy-Schwarz inequality to the characteristic function of the central coordinate.
- Stronger rates are achieved via concentration of measure techniques (Azuma's inequality).

Survey of recent work

- Varjú [5] gives a quantitative local limit theorem for random walk on the group $\mathrm{Isom}(\mathbb{R}^n)$ of isometries of $\mathbb{R}^n$, with applications to self-similar measures [3]
- Helfgott-Seress-Zuk [1] show that for random $g, h \in \mathfrak{S}_n$, the mixing time of random walk generated by $\{g, h, g^{-1}, h^{-1}\}$ is $O(n^3(\log n)^{O(1)})$
- Pillai-Smith [4] show a total-variation mixing time of $O(n \log n)$ for the Kac random walk (modelling a Boltzman gas) on the n-sphere

Techniques include: spectral gap estimates, coupling, comparison with classical Markov chains

Open problems

- Classical limit and local limit theorems on Lie groups make restrictive conditions on the generating measure, e.g. that it be absolutely continuous w.r.t. Haar measure, or have support generating a dense subgroup. Can these be removed?
- The distance to uniformity with lower order error term is known for some examples of conjugation invariant measures. It's desirable to extend these results to less symmetric measures. For instance, it would be nice to demonstrate a sharp phase transition in L^1 for the Kac walk; presently a mean field version is known [2].
- Green and Tao [6] give quantitative equidistribution results for polynomial sequences in a nilmanifold. It's desirable to have similar results for unipotent flows on other homogeneous spaces.

Thanks for listening!

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