III: Key developments leading up to mid-century

Of course, a great deal contributed to the development of the calculus of variations. We have at the very least the following, all of which are needed in the modern formulations.

A. Banach and Hilbert spaces of functions
B Definition of Fredholm and compact operators
C Understanding of the role of boundary values and how the Dirichlet and Neuman boundary value problems occur in the calculus of variations
D Manifolds
E. Weak convergence and lower semicontinuity

IV. The Plateau Problem

A major milestone in carrying out Hilbert's ideas was the solution of the Plateau problem in 1930. Plateau's problem is due to Lagrange in 1760. He asked the shape of a surface whose boundary was a wire in three space. It is named after Plateau, who experimented with soap films. It was solved by Douglas and Rado , who jointly received Field's medal for this work. Here is a brief description:

Let $s : S^1 \to \mathbb{R}^3$ be a fixed embedded curve, and u be a map from the disk to $\mathbb{R}^3$. We minimize the energy

$$E(u) = \iint_{x^2+y^2 \le 1} |\operatorname{grad} u|^2 \, dx\, dy$$

subject to the constraints on u

(a) the energy is finite

(b) $u(1,\theta) = s(\sigma(\theta))$

(c) $\sigma : S^1 \to S^1$, $\sigma'(\theta) \ge 0$

(d) $\sigma\left(\frac{n 2\pi}{3}\right) = \frac{n 2\pi}{3}$

The minimum is a conformally immersed surface of minimal area.

We note that

(a) defines a Hilbert space

(b) boundary value prescribed

(c) Understanding of the conformal mapping theor

The process of minimization involves weak compactness and lower semi-continuity

V. Morse theory

In his book (1934).”The Calculus of Variations in the Large” Marston Morse (of IAS) formulated and proved a theorem on the relationship between the number of geodesics connecting two points on a complete manifold M, and the topology of the loop space of the manifold M. There are a number of interesting points about this development.

A. The book is unreadable. It is painfully evident that homology theory was in its infancy; Eilenberg and Steenrod was decades in the future; hence the topology as being developed as well as the analysis.

B. Morse theory was developed for the calculus of variations. The application to finite dimensional manifolds, and the decomposition of manifolds into cell complexes using a generic function was a spin off.

C. The original proof of Bott periodicity (1957) describing the topology of Lie groups, despite the claim of topologists that this is not the “right” proof, was via Morse theory.

D. Morse theory on geodesics uses minimal energy, not minimal length. This parameterizes geodesics by arc length The proof is via finite dimensional approximation.

i) Solve the problem locally of finding a shortest curve between two nearby points. This is done in a coordinate chart on the manifold N.

ii) Take an arbitrary curve s(x) with a bound E on energy. Then distance squared from s(x) to s(y) is bounded by |x-y|E.

iii). For an arbitrary curve s(v) with a bound on energy E, divide [0,1] into small intervals of length e^2/E. Then the endpoints are of distance smaller than e. In each interval find the shortest geodesic between the two endpoint as in i).

iv). Retract the curve onto the broken geodesic defined above. We now have a function on NxNx….xN, a finite dimensional manifold. We then use the critical points and calculus to construct a cell complex.

VI. Puzzlement at Mid Century

The subjects of topology, analysis on manifolds and partial differential equations developed nicely. The calculus of variations was a bit stuck. It had been a subject in the standard curriculum for decades (Bob Williams remembers taking it as a graduate student) but it now disappeared from the syllabus. Differential geometry was a different subject than it is now with very little analysis in it.

A Morse theory, which was thriving as a tool for studying manifolds and the geodesics in them, was out of reach for multivariable problems. I can remember being instructed seriously by an eminent mathematician that it was by its very nature limited to single integral variational problems.

B. Valiant attempts were made to find finite dimensional approximations to multivariable integrals. Of course, this is now standardly done in numerical analysis, but at the time did not at the time give any insight into the topological questions. Also, computation via computer remained comparatively primitive for decades.

C. As a graduate student, I read a paper of Guisti and Miranda (1968) constructing minima to variational problems in high dimension with singular set, which also gave an estimate on the Hausdorff dimension of the singular set. Singularities for minimal volume surfaces in high dimension were also known, bur there was no real understanding of the phenomenon.

VII. The Promise of Global Analysis

During my graduate student years global analysis blossomed. Calculus in infinite dimensions came to the fore. Implicit function theorems, existence of solutions to ordinary differential equations, manifolds, especially manifolds of maps between manifolds can be constructed with metrics and tangent bundles and all and last but not least, integrals in the calculus of variations became functions on an infinite dimensional manifold. Suddenly all things were possible

A.The Atiyah-Singer index theorem(1963) linked the index of an elliptic operator to the topology of the underlying manifold and the symbol of the operator.
B. Eels and Sampson(1963) proved the existence and uniqueness of harmonic maps into a manifold of negative curvature.
C. Smale's infinite dimensional version of Sard's theorem gave a whole new meaning to the word "generic".

D. Richard Palais and Steve Smale(1963) proved the existence of a Morse theory on a complete separable Banach manifold X, given certain conditions, including the now famous condition c), (coming naturally after conditions a) and b)).

c) [for smooth $J : X \to R$] If $\{p_j\}$ is a sequence of points on X for which $|J(p_j| < K$ and $\lim |dJ(p_j)| \to 0$, then a subsequence of the p_j converges to a critical point of J.

E. The infinite dimensional approach gives a direct proof of Morse's theory without the approximation. Condition c) can be verified for a number of multiple integral problems.

F. The mountain pass lemma was proved by quite a few people (there were some arguments here). This is a now familiar minimax problem.

There was quite naturally a backlash. Global analysis had its limitations.

G. Cumbersome abstract machinery with very few applications to interesting problems. Not up to advertisement.
H. The function spaces X used in direct minimization methods cannot be used to define manifolds
I. Integrals of interest do not satisfy the Palais-Smale condition.
J. Can do calculus, but cannot integrate on infinite dimensional manifolds. Hence useless in quantum field theory.

In the end, global analysis won out. It absorbed the technical tools of "hard" estimates and a deep understanding of geometry. The beginning of the new era was S T Yau's solution of the Calabi conjecture (1977) and he has been at the forefront ever since. The new "hard" version of global analysis became geometric analysis, to distinguish it from the earlier "softer" version, and is still thriving.

VIII. Renormalizable, unrenormalizable and scale invariant problems

The understanding which comes from the Palais-Smale condition agrees with the classifications given to the problem by physicists, and was developed at about the same time, in the 1970's, although the physicists have a much more ambitious program: construct a measure versus understanding of the classical solutions. The crucial number is the dimension n of the ambient manifold M. When n = 1, all the familiar problems are renormalizeable (quantum mechanics).

Renormalizable, or. below the critical dimension, means that the dimension n and the analysis line up to construct manifolds and a functional which satisfy the Palais-Smale condition. Hence we expect smooth solutions, and as many as the topology calls for. In physics, it means the scheme to construct a quantum field theory converges up to the usual "infinities".

Unrenormalizable or above the critical dimension, means that it may be possible to construct solutions (usually absolute minima), they will have singularities, and the Hausdorff dimension of the singular set is at most n - (critical dimension).

Scale invariant, or at the critical dimension is easy to check for, as it is the problem where scale is not seen. Partial topological results apply, but the Morse theory is spoiled by “bubbling” or small neighborhoods in the ambient manifold M behaving like R^n due to scale invariance. Solutions are smooth, without singularities.

We move on to the three examples.

IX. The three classic geometric Problems

There are many other problems, but at least historically the ones first treated were as follows.

A. Harmonic maps: u:M—>N a map between Riemannian manifolds. The integral is

$$J(u) = \int_M 1/2\, |du|^2 *1 , \quad \dim(M) = n.$$

We look for critical points of J subject to the constraint that u = f is fixed on the boundary of M (if it is non-empty). It is easy to see that n = 2 is the scale invariant or critical dimension. Then n = 1 < 2 is well behaved, and the Palais-Smale condition is easily verified if N is compact or N is complete and M = [0,1]. This is the classic example of Morse theory proved by Morse by using finite dimensional approximation as described earlier.

For n > 2, absolute minima exist at least for the case when M has a boundary but have singularities of dimension n - 3.

The interesting case is the scale invariant one or $n = 2$. It is related to minimal surfaces, by adding an additional variation over the conformal structure of the surface M. The original proof of Douglas and Radio prevents "bubbling" by fixing the three points on the boundary of the disk.

B. The second example is a gift from our colleagues in physics. In a fiber bundle with compact structure group we integrate the norm squared of the curvature

$$F_A = [D_A, D_A]$$

of a connection D_A.

$$J(D_A) = \int |F_A|^2 *1.$$

For this example, the critical dimension is $n = 4$. The integral is particularly well-behaved for $n = 2 < 4$. For $n > 4$, it is an open problem of whether or not it is possible to construct absolute minima, and as for harmonic maps, the interesting case is n = 4, which we will visit revisit in the next section.

C. The Yamabe problem has a slightly different variational formulation. Mathematically it can be phrased as a constraint problem. Here u is simply a function on M. We minimize

$$J(u) = \int |du|^2 *1$$

subject to the constraint that $\int |u|^p *1 = 1$.

An easy check by rescaling both integrals shows that the scale invariant case is p = 2n/(n-2). For p < 2n/(n-2), this forms a nice Hilbert manifold, and by dividing out by u = - u, we get a problem on RP^(infinity). This problem will have an infinite number of critical points, since the integral satisfies the Palais Smale condition.

There is an interesting difference, as below and above the critical dimension are reversed in p because it appears in the constraint.

For $p > 2n/(n-2)$, the constraint will not be preserved in the limiting process and we get very little insight from this formulation.
The case $p = 2n/(n-2)$, if we add a lower order terms in curvature to the integral, is the Yamabe problem. Again, we get a solution by observing that the minimum will be taken on modulo "bubbling", which yields a solution on $R^n = S^n - \{p\}$, and the minimum will be taken on if the minimum of $J(u)$ is less than that of the sphere.