Disorder-generated multifractals and Random Matrices: freezing phenomena and extremes [1]

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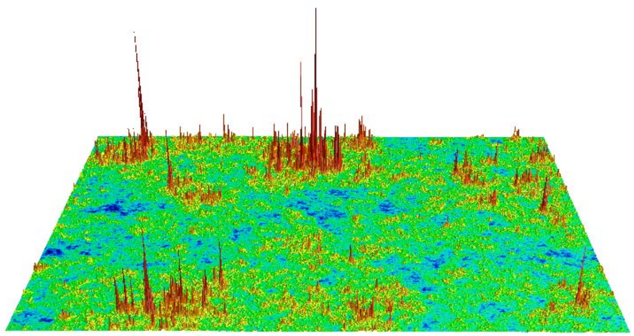
[1]Based on: **YVF**, **P Le Doussal** and **A Rosso** J Stat Phys: **149** (2012), 898-920
YVF, **G Hiary**, **J Keating** Phys. Rev. Lett. **108** , 170601 (2012) & **arXiv:1211.6063**
YVF, **B. A. Khoruzhenko** and **N. Simm** in preparation

Disorder-generated multifractals:

Intensity patterns in systems with disorder frequently display high variability over a wide range of space or time scales, associated with huge fluctuations which can be visually detected.



Intensity of a multifractal wavefunction at the point of Integer Quantum Hall Effect.

Courtesy of F. Evers, A. Mirlin and A. Mildenberger.

Multifractality characterizes such patterns in a lattice of $M = (L/a)^d \gg 1$ sites by attributing different scaling of intensities $h_i \sim M^{x_i}$ at different lattice sites $i = 1, \ldots M$, with exponents x_i forming a dense set such that

$$\rho_M(x) = \sum_{i=1}^{M} \delta\left(\frac{\ln h_i}{\ln M} - x\right) \approx c_M(x)\sqrt{\ln M}\, M^{f(x)}, \quad M \gg 1 \quad ,$$

which is frequently referred to as the **multifractal Ansatz**. Whereas the **singularity spectrum** $f(x)$ is typically self-averaging, there are essential sample-to-sample fluctuations of the **prefactor** $c_M(x)$ in different realizations of disorder, as well as fluctuations in the number and height of **extreme peaks** of the pattern. Those fluctuations will be the subject of our interest.

From disorder-generated multifractals to log-correlated fields:

Disorder-generated multifractal patterns of intensities $h(\mathbf{r})$ are typically **self-similar**

$$\mathbb{E}\left\{h^q(\mathbf{r_1})h^s(\mathbf{r_2})\right\} \propto \left(\frac{L}{a}\right)^{y_{q,s}}\left(\frac{|\mathbf{r}_1-\mathbf{r}_2|}{a}\right)^{-z_{q,s}}, \quad q,s\geq 0, \quad a \ll |\mathbf{r}_1-\mathbf{r}_2| \ll L$$

and **spatially homogeneous**

$$\mathbb{E}\left\{h^q(\mathbf{r_1})\right\} \sim \frac{1}{M}\sum_{\mathbf{r}} h^q(\mathbf{r}) \propto \left(\frac{L}{a}\right)^{d(\zeta_q-1)}$$

where ζ_q and $f(x)$ are related by the **Legendre transform**:

$$f'(y_*) = -q \text{ and } \zeta_q = f(y_*) + q\,y_*.$$

The consistency of the two conditions for $|\mathbf{r}_1-\mathbf{r}_2| \sim a$ and $|\mathbf{r}_1-\mathbf{r}_2| \sim L$ implies:

$$y_{q,s} = d(\zeta_{q+s}-1), \quad z_{q,s} = d(\zeta_{q+s}-\zeta_q-\zeta_s+1)$$

If we now introduce the field $V(\mathbf{r}) = \ln h(\mathbf{r}) - \mathbb{E}\left\{\ln h(\mathbf{r})\right\}$ and exploit the identities $\frac{d}{ds}h^s|_{s=0} = \ln h$ and $\zeta_0 = 1$ we arrive at the relation:

$$\mathbb{E}\left\{V(\mathbf{r_1})V(\mathbf{r_2})\right\} = -g^2\ln\frac{|\mathbf{r}_1-\mathbf{r}_2|}{L}, \quad g^2 = d\frac{\partial^2}{\partial s\partial q}\zeta_{q+s}|_{s=q=0}$$

Conclusion: logarithm of a multifractal intensity is a log-correlated random field.

To understand statistics of high values and extremes of general logarithmically correlated random fields we consider the simplest $1D$ case of such process: the Gaussian $1/f$ noise.

Ideal Gaussian periodic 1/f noise:

We will mainly study an idealized periodic model for Gaussian **1/f** noise defined as

$$V(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[v_n e^{int} + \overline{v}_n e^{-int} \right] , \quad t \in [0, 2\pi)$$

where $v_n, \overline{v}_n$ are **complex standard Gaussian i.i.d.** with $\mathbb{E}\{v_n \overline{v}_n\} = 1$. It implies the formal covariance structure:

$$\mathbb{E}\left\{V(t_1)V(t_2)\right\} = -2\ln|2\sin\frac{t_1 - t_2}{2}|, \quad t_1 \neq t_2$$

The corresponding process is a **random distribution** and in applications should be regularized. There are several alternative regularizations. E.g. one can replace the function $V(t)$, $t \in [0, 2\pi)$ with a sequence of $M \gg 1$ random mean-zero Gaussian variables $V_k \equiv V\left(t = \frac{2\pi}{M}k\right)$ with the covariance matrix $\mathbb{E}\left\{V_k V_m\right\}$ given by

$$\mathbb{E}\left\{V_k V_m\right\} = -2\ln|2\sin\frac{\pi}{M}(k - m)|, \quad C_{kk} = \mathbb{E}\left\{V_k^2\right\} > 2\ln M, \quad \forall k = 1, \ldots, M$$

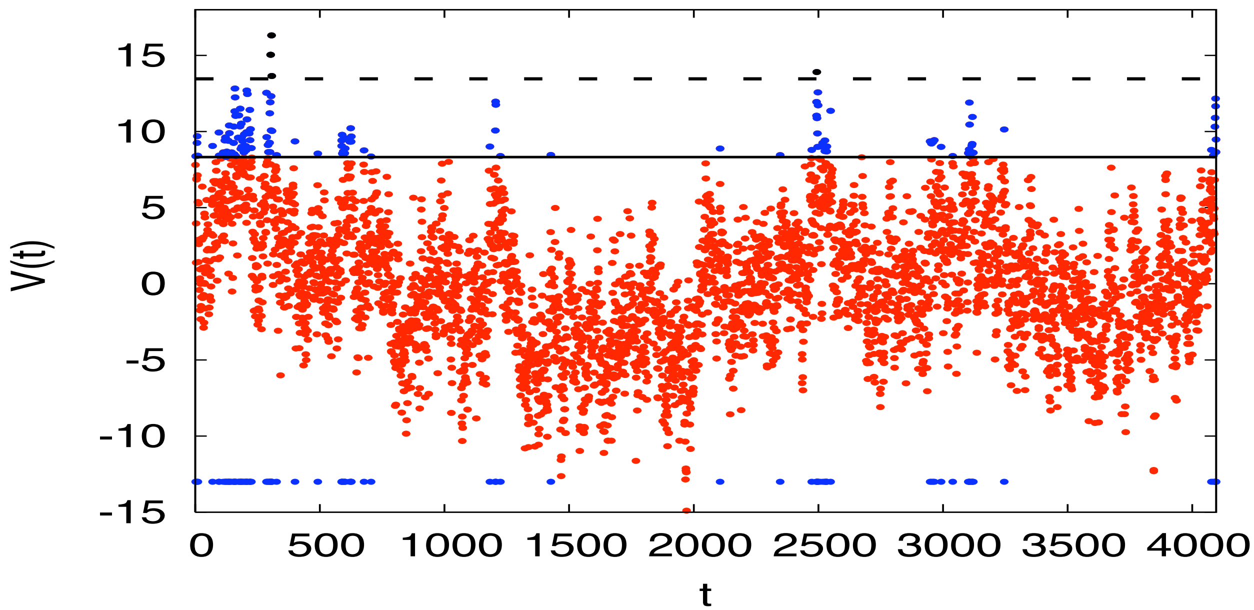
One also may give *bona fide* mathematical definition as e.g. 1D "projection" of the $2D$ **Gaussian Free Field**. We shall see that one may further consider aperiodic **stationary** logarithmically-correlated processes as well as similar processes with **stationary increments**:

$$\mathbb{E}\left\{V(t_1)V(t_2)\right\} \propto -\log|t_1 - t_2| \quad \text{or} \quad \mathbb{E}\left\{\left[V(t_1) - V(t_2)\right]^2\right\} \propto \log|t_1 - t_2|$$

The **multifractal intensity pattern** is then generated by setting $h_i = e^{V_i}$ for each $i = 1, \ldots, M$.

Ideal Gaussian periodic 1/f noise:

An example of the $1/f$ signal sequence generated for $M = 4096$ for the discretized version of periodic **1/f** noise is given in the figure.



The upper line marks the typical value of the **extreme value threshold** $V_m = 2\ln M - \frac{3}{2}\ln\ln M$.
The lower line is the level $\frac{1}{\sqrt{2}}V_m$ and blue dots mark points supporting $V_i > \frac{1}{\sqrt{2}}V_m$.

Questions we would like to answer: How many points are typically above a given level of the noise? How strongly does this number fluctuate for $M \gg 1$ from one realization to the other? How to understand the typical position V_m and statistics of the **extreme values** (maxima or minima), etc. And, after all, what parts of the answers are **universal** and what is the universality class?

Characteristic polynomial of random CUE matrix and periodic 1/f noise:

Let U_N be a $N \times N$ **unitary matrix**, chosen at random from the unitary group $\mathcal{U}(N)$. Introduce its **characteristic polynomial** $p_N(\theta) = \det\left(1 - U_N\, e^{-i\theta}\right)$ and further consider $V_N(\theta) = -2\log|p_N(\theta)|$. Following **Hughes, Keating & O'Connell** 2001 one can employ the following representation

$$V_N^{(U)}(\theta) = -2\log|p_N(\theta)| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[e^{-in\theta} v_n^{(N)} + \text{comp. conj.}\right]$$

where $\quad v_n^{(N)} = \frac{1}{\sqrt{n}}\text{Tr}\left(U_N^{-n}\right)$.

According to **Diaconis & Shahshahani** 1994 the coefficients $v_{n_1}^{(N)}, \ldots, v_{n_k}^{(N)}$ for any fixed finite set $n_1, \ldots, n_k$ tend in the limit $N \to \infty$ to i.i.d. complex gaussian variables with zero mean and variance $\mathbb{E}\{|\zeta_n|^2\} = 1$. This suggests that for finite N **Log-Mod** of the characteristic polynomial of CUE matrices is just a **certain regularization** of the stationary random **Gaussian Fourier series** of the form

$$V(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[v_n e^{int} + \overline{v}_n e^{-int}\right], \quad t \in [0, 2\pi)$$

where $v_n, \overline{v}_n$ are **complex standard Gaussian i.i.d.** with $\mathbb{E}\{v_n \overline{v}_n\} = 1$. It implies

$$\mathbb{E}\left\{V(t_1)V(t_2)\right\} = 2\sum_{n=1}^{\infty} \frac{1}{n} \cos n(t_1 - t_2) \equiv -2\ln\left|2\sin\frac{t_1 - t_2}{2}\right|, \quad t_1 \neq t_2.$$

Characteristic polynomial of random GUE matrix and aperiodic 1/f noise:

Let $\mathbf{H}$ be a $N \times N$ **Hermitian** GUE random matrix: $\mathcal{P}(\mathbf{H}) \propto \exp\left\{-\frac{N}{2}Tr\mathbf{H}^2\right\}$ such that the mean eigenvalue density $\rho_N(u) = \mathbb{E}\left\{\frac{1}{N}Tr\delta(u-\mathbf{H})\right\}$ tends to the Wigner semicircular law in $x \in (-2,2)$. Introduce its **characteristic polynomial** $\pi_N(x) = \det\left(x - \mathbf{H}\right)$. Let $T_n(u) = \cos\left(n \arccos u\right)$ be **Tschebyshev** polynomials orthogonal in $u \in (-1,1)$ with the weight $1/\sqrt{1-u^2}$. It turns out that the following representation is valid:

$$f_N(x) = \mathbb{E}\left\{\log|\pi_N(x)|\right\} - \log|\pi_N(x)| = \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} \left[w_n^{(N)} + R_n^{(N)}\right] T_n\left(\frac{x}{2}\right)$$

where $\quad w_n^{(N)} = \sqrt{\frac{2}{n}}\left[Tr\left\{T_n\left(\frac{\mathbf{H}}{2}\right)\right\} - N\int_{-\infty}^{\infty} T_n\left(\frac{u}{2}\right)\rho_N(u)\,du\right].$

According to **Johansson** 1998 in the limit $N \to \infty$ the coefficients $w_{n_1}^{(N)}, \ldots, w_{n_k}^{(N)}$ tend to a set of i.i.d. real gaussian variables with zero mean and unit variance. We also can show that $R_n^{(N)} \to 0$ in probability. This suggests that for finite N **Log-Mod** of the characteristic polynomial of GUE matrices is just a **certain regularization** of a random **Gaussian series** of the form

$$f(x) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{n}}\, w_n\, T_n\left(\frac{x}{2}\right)\,, \quad x \in (-2,2)$$

where w_n are **real standard** Gaussian i.i.d variables. It implies

$$\mathbb{E}\left\{f(x_1)f(x_2)\right\} = \sum_{n=1}^{\infty} \frac{2}{n} T_n\left(\frac{x_1}{2}\right) T_n\left(\frac{x_2}{2}\right) \equiv -\ln|x_1 - x_2|, \quad x_1 \neq x_2.$$

Mesoscopic regime of GUE and fractional Brownian motion with $H=0$:

Define parameter d_N such that $1 \ll d_N \ll N$ for $N \gg 1$, and consider for $\eta > 0$

$$W_{N,x}^{(\eta)}(t) = -\log \left| \det \left(\mathcal{H} - x - \frac{t+i\eta}{d_N} \right) \right| + \log \left| \det \left(\mathcal{H} - x - \frac{i\eta}{d_N} \right) \right|,$$

One can show that

$$W_{N,x}^{(\eta)}(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \, e^{-\eta\omega} \left[v_{N,x}(\omega) \, \frac{e^{i\omega t}-1}{2} + \overline{v}_{N,x}(\omega) \, \frac{e^{-i\omega t}-1}{2} \right]$$

where $v_{N,x}(\omega) = \frac{1}{\sqrt{\omega}} e^{ixd_N \omega} \, \mathrm{Tr} \left[e^{-i\omega d_N \mathbf{H}} \right]$.

It can be further verified that for $N \to \infty$ and $x \in (-2,2)$ the Fourier coefficients $v_{N,x}(\omega)$ tend to the standard complex **Gaussian "white noise"**:

$$\mathbb{E}\{v_{N,x}(\omega)\} \to 0, \quad \mathbb{E}\{v_{N,x}(\omega_1) \, \overline{v_{N,x}(\omega_2)}\} \to \delta\left(\omega_1 - \omega_2\right)$$

which implies that $W_{N,x}^{(\eta)}(t) \to B_0^{(\eta)}(t)$ such that

$$\mathbb{E}\left\{ \left[B_0^{(\eta)}(t_1) - B_0^{(\eta)}(t_2) \right]^2 \right\} = \frac{1}{\pi} \ln \frac{(t_1-t_2)^2 + 4\eta^2}{4\eta^2}.$$

This is a **properly regularized** limit $H \to 0$ of the **Fractional Brownian Motion** $B_H(t)$ such that $\mathbb{E}\left\{B_H(t_1) B_H(t_2)\right\} = \frac{\sigma_H^2}{2} \left(|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H} \right)$.

1/f Noises, Disordered Energy Landscapes, and Burgers Turbulence:

In the area of **Statistical Mechanics** of **Disordered Systems** logarithmically correlated fields and **1/f** noises have been recently identified as **potential energy landscapes** underlying an intriguing phenomenon of the **freezing transition** which takes place at some finite temperature $T = T_c$ (**Carpentier** $\&$ **Le Doussal** 2001; **YVF** $\&$ **Bouchaud** 2008; **YVF , Le Doussal** $\&$ **Rosso** 2009). In a related development, it was shown that a freezing transition shows up also in the problem of **decaying Burgers turbulence**, i.e. analysis of the Burgers equation $\partial_t v + (v\nabla)v = \nu\nabla^2 v, \quad \nu > 0$ with random initial conditions given by the gradient of the **1/f** noise (**YVF, Le Doussal** $\&$ **Rosso** 2010 $\&$ unpublished).

Though most of the above considerations are not yet rigorous, as the result we by now have a qualitative, and sometimes, quite precise quantitative understanding of statistics of **high** and **extreme** values of such random processes: the statistics of the **number of points** in a pattern above a **given level**, and the distribution of the **highest intensity** V_m in the pattern. In particular, for the periodic Gaussian $1/f$ noise we have a prediction $V_m = 2\ln M - \frac{3}{2}\ln\ln M - x$, where x is a random variable with the density $p(x) = 2e^x K_0(2e^{x/2})$. This is manifestly different from the ubiquitous double-exponential **Gumbel distribution** $p_{Gumb}(x) = -\frac{d}{dx}\exp\{-e^x\}$ universally valid for maxima of **short-range correlated** random variables.

Statistics of the counting function $\mathcal{N}_M(x)$ and threshold of extreme values:

By relating moments of the counting function $\mathcal{N}_M(x) = \int_x^\infty \rho_M(y)\,dy$ for log-correlated $1/f$ **noise** to **Selberg integrals** we conjecture that the probability density for the (scaled) counting function $n = \mathcal{N}_M(x)/\mathcal{N}_t(x)$ is given by:

$$\mathcal{P}_x(n) = \frac{4}{x^2}\, e^{-n^{-\frac{4}{x^2}}}\, n^{-\left(1+\frac{4}{x^2}\right)}, \quad n \ll n_c(x), \quad 0 < x < 2\,.$$

with $n_c \to \infty$ for $M \to \infty$ and the **characteristic scale** $\mathcal{N}_t(x)$ given by

$$\mathcal{N}_t(x) = \frac{M^{f(x)}}{x\sqrt{\pi \ln M}}\frac{1}{\Gamma(1-x^2/4)} = \mathbb{E}\left\{\mathcal{N}_M(x)\right\}\frac{1}{\Gamma(1-x^2/4)}, \quad f(x) = 1 - x^2/4$$

Note: For $x \to 2$ the **typical** value of the counting function $e^{\mathbb{E}\{\ln \mathcal{N}_M(x)\}} \sim \mathcal{N}_t(x)$ and hence is **parametrically smaller** than the mean value $\mathbb{E}\left\{\mathcal{N}_M(x)\right\}$. In particular, the position x_m of the typical threshold of **extreme values** determined from the condition $\mathcal{N}_t(x) \sim 1$ is given by

$$x_m = 2 - c\,\frac{\ln\ln M}{\ln M} + O(1/\ln M) \text{ with } c = 3/2.$$

In contrast, for **short-ranged** random sequences **mean**=**typical**. Had we instead decided to use the condition $\mathbb{E}\left\{\mathcal{N}_M(x)\right\} \sim 1$ that would give the above but with $c = 3/2 \to c = 1/2$. The value $c = \frac{3}{2}$ is conjectured to be a universal feature of systems with **logarithmic** correlations (cf. **Bramson & Zeitouni**).

Threshold of extreme values for self-similar multifractal fields:

Apart from 1/f noise and its incarnations (like modulus of characteristic polynomials of random matrices) the new universality class is believed to include: the $2D$ Gaussian free field, branching random walks $\&$ polymers on disordered trees, modulus of zeta-function along the critical line, fluctuations of shapes of random Young tableaux sampled with Plancherel measure, some models in turbulence and financial mathematics and, with due modifications the disorder-generated multifractals.

Namely, consider a multifractal random **probability measure** $p_i \sim M^{-\alpha_i}$, $i = 1, \ldots, M$ such that $\sum_{i=1}^{M} p_i = 1$ characterized by a general non-parabolic **singularity spectrum** $f(\alpha)$ with the left endpoint at $\alpha = \alpha_- > 0$. Then very similar consideration based on insights from **Mirlin** $\&$ **Evers** 2000 suggests that the **extreme value threshold** should be given by $p_m = M^{-\alpha_m}$, where α_m

$$\alpha_m \approx \alpha_- + \frac{3}{2}\frac{1}{f'(\alpha_-)}\frac{\ln\ln M}{\ln M} \quad \Rightarrow -\ln p_m \approx \alpha_- \ln M + \frac{3}{2}\frac{1}{f'(\alpha_-)}\ln\ln M$$

For branching random walks this is indeed rigorously proved: L. Addario-Berry $\&$ B. Reed 2009; E. Aidekon 2012

Work in progress: testing such a prediction for multifractal eigenvectors of a certain $N \times N$ random matrix ensemble introduced by E. Bogomolny $\&$ O. Giraud, *Phys. Rev. Lett.* **106 044101 (2011). Preliminary numerics is supportive of the theory.**

Distribution of the absolute maximum: partition function approach:

Given the sequence $\{V_i,\, i = 1, \ldots, M\}$ we are interested in finding the **distribution** of $V_{(m)} = \max(V_1, \ldots, V_M)$ that is

$$P(v) = \text{Prob}(V_{(m)} < v) = \text{Prob}(V_i < v,\, \forall i) = \mathbb{E}\left\{\prod_{i=1}^{M} \theta(v - V_i)\right\}$$

Next we use: $$\lim_{q\to\infty} \exp\left[-e^{-q(v-V_i)}\right] = \left\{\begin{array}{ll} 1 & v > V_i \\ 0 & v < V_i \end{array}\right. \equiv \theta(v - V_i)$$

which immediately shows that:

$$P(v) = \text{Prob}(V_{(m)} < v) = \lim_{q\to\infty} \mathbb{E}\left\{\exp\left[-e^{-qv} Z_q\right]\right\}, \quad \text{where} \quad Z_q = \sum_{i=1} e^{qV_i}$$

In the limit $\ln M \gg 1$ moments of Z_q for $|q| < 1$ can be again related to Selberg integrals, which allows to extract the function $G_q(v) = \mathbb{E}\left\{\exp\left[-e^{-qv} Z_q\right]\right\}$ for $q < 1$:

$G_q(v) = g_q\left(v - c_q \ln M\right)$ **where** $c_q = \left(q + \frac{1}{q}\right)$ **and** $g_q(v) = \int_0^\infty dt \exp\left\{-t - e^{-qv} t^{-q^2}\right\}$

One may further notice that not only $c_q = c_{q^{-1}}$ but the whole function satisfies a quite remarkable duality relation

$$g_q(v) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\left[e^{-nqv}\Gamma(1 - nq^2) + e^{-n\frac{v}{q}}\Gamma\left(1 - \frac{n}{q^2}\right)\right] = g_{\frac{1}{q}}(x)$$

THIS HOWEVER STILL DOES NOT ALLOW TO CONTINUE TO $q > 1$!

Freezing conjecture and the distribution of extremes:

Using certain analogy with the **Derrida-Spohn** model of polymers on disordered trees we conjecture the following **freezing scenario**: for the $log-$circular model the same sort of **freezing transition** takes place at $q = 1$. Namely, the function

$$g_{q<1}(v) = \int_0^\infty dt \exp\left\{-t - e^{-qv} t^{-q^2}\right\}$$

freezes to the **q−independent** profile $g_{q=1}(v) = 2e^{-v/2} K_1(2e^{-v/2})$ in the whole "glassy" phase $q > 1$.

Consequences:

(**i**) The latter profile then is precisely the distribution $P(v)$ of the (shifted) absolute maximum: $V_m = 2\ln M - \frac{3}{2}\ln\ln M + v$. This distribution is manifestly **non-Gumbel**, and shows the tail behaviour: $P(v \to -\infty) \approx 1 - |v|e^v$

(**ii**) The probability density of the partition function Z_q in the whole regime $q > 1$ must display a power-law forward tail of the form:

$$\mathcal{P}_{q>1}(Z) \propto Z^{-\left(1+\frac{1}{q}\right)} \ln Z$$

We conjecture that such a tail, including the meaningful **log-factor**, is to be **universal** for the whole class of logarithmically correlated processes.

Numerics for the maxima of CUE characteristic polynomials:

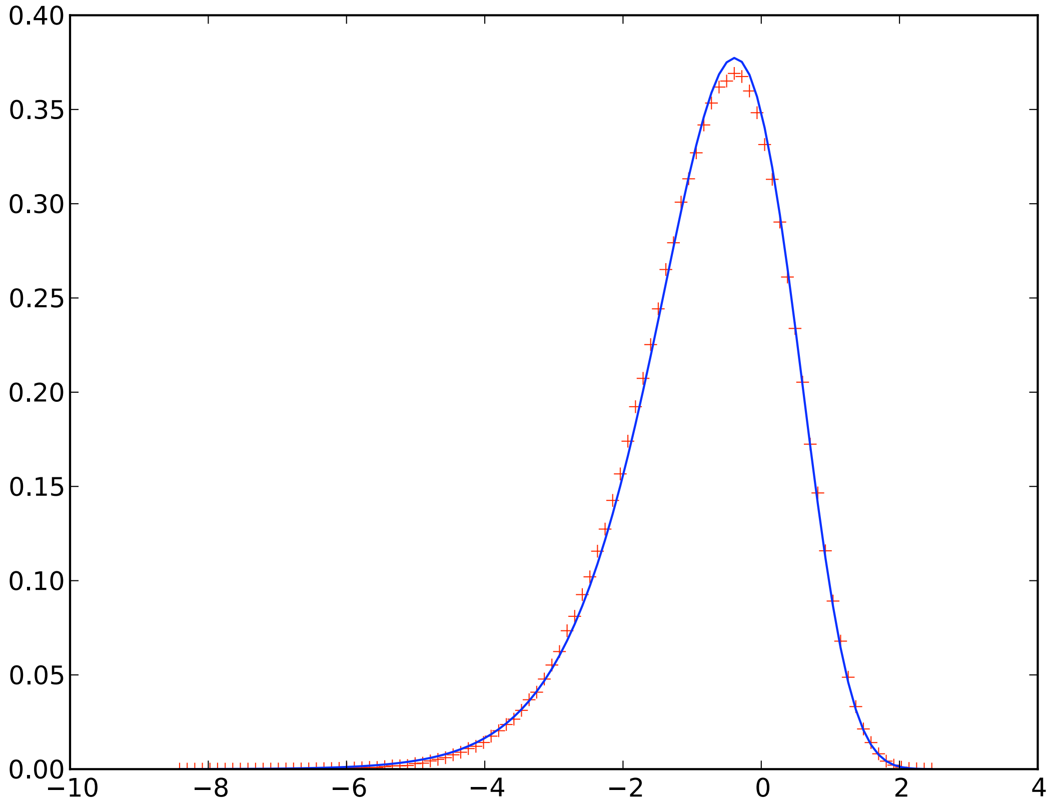


Figure 1: **Density of maxima for CUE polynomials (** $N = 50, 10^6$**samples) compared to periodic** $1/f$ **noise prediction** $p(v) = 2e^v K_0(2e^{v/2})$**.**

From $1/f$ noise to Riemann $\zeta(1/2 + it)$:

As is well-known, zeroes of the Riemann zeta-function $\zeta(s)$ along the "critical line" $s = 1/2 + it$, $t \in \mathbb{R}$ behave statistically as a sequence of eigenvalues of random Hermitian GUE matrices (**Montgomery** 1983). Following the ideas of **Keating** $\&$ **Snaith** 2000 one can expect that **log-mod** of the Riemann zeta-function $\zeta(1/2 + it)$ **locally** resembles **log-mod** of CUE characteristic polynomial, and hence a (non-periodic) version of the **1/f noise**, see also **P. Bourgade** 2010. One can exploit this fact to predict statistics of **moments** and **high values** of the Riemann zeta along the critical line using the previously exposed ideas (**YVF, Keating** 2012).

Our approach to statistics of $\zeta(1/2 + it)$:

We expect a **single** unitary matrix of size $N_T = \log(T/2\pi) \gg 1$ to model the Riemann zeta $\zeta(1/2 + it)$, statistically, over a range of $T \leq t \leq T + 2\pi$. We thus suggest splitting the **critical line** into ranges of **length** 2π, and making the statistics of $\zeta(1/2 + it)$ over the many ranges.

There are roughly N_T zeros in each range of length 2π. At heights $T \sim 10^{22} - 10^{28}$ we used samples that spans $\approx 10^7$ zeros.

Our predictions for $\zeta(1/2+it)$ and CUE characteristic polynomials:

We expect

$$\log \zeta_{max}(T) \approx \log N_T - \frac{3}{4}\log\log N_T - \frac{1}{2}\,x, \quad N_T = \log\left(T/2\pi\right)$$

where x is distributed with a probability density behaving in the tail as $\rho(x \to -\infty) \approx |x|\, e^{x}$.

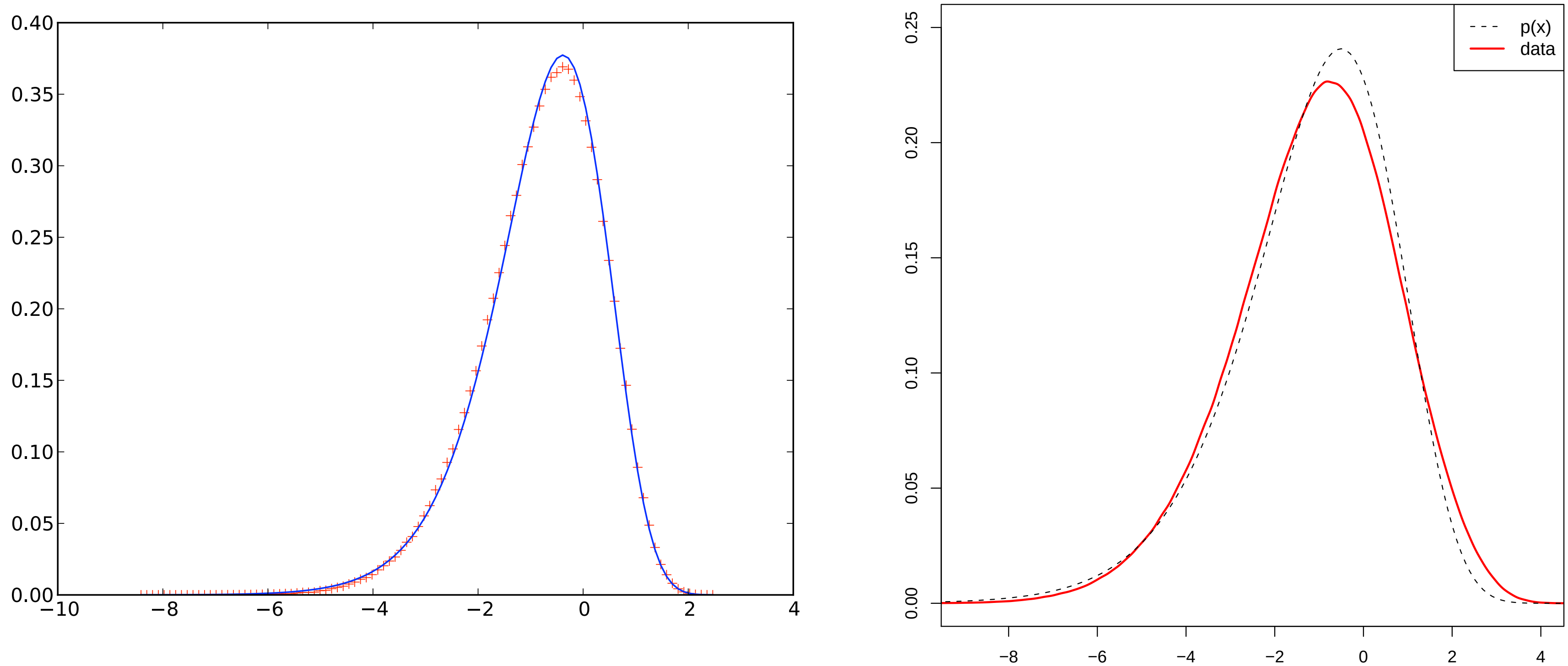


Figure 2: **Statistics of maxima for CUE polynomials (left: $N = 50, 10^6$samples) and $|\zeta(1/2+it)|$ (right: $N_T = 65,\ 10^5$ samples) compared to periodic $1/f$ noise prediction $p(x) = 2e^{x}K_0(2e^{x/2})$.**

Summary:

I. **Disorder-generated multifractal patterns are intimately connected to log-correlated random fields.**

II. **log-mod of characteristic polynomials of random matrices on the *global* scale are examples of log-correlated Gaussian 1/f noises. The same objects on *mesoscopic* scale are examples of Fractional Brownian Motion with $H = 0$.**

III. **Exploiting the methods of statistical mechanics of disordered systems we attempted to understand the statistics of minima/maxima of the CUE characteristic polynomial over various intervals, as well as the related moments and high values.**

IV. **The above picture can be translated into making non-trivial conjectures about statistics of moments and high values of (i) the Riemann zeta along the critical line (ii) a more general class of disorder-generated multifractal fields.**