Topological and arithmetic intersection numbers attached to real quadratic cycles

Henri Darmon, McGill University

Jan Vonk, McGill University

Workshop, IAS, November 8

This is joint work with Jan Vonk

Preamble

Arithmetic quotients of symmetric spaces, and topological cycles on them, often behave "as if" they were algebraic.

For instance, a modular curve $\mathbf{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ is equipped with a canonical collection of:

• CM zero cycles, which *are* algebraic, and defined over ring class fields of imaginary quadratic fields.

• geodesic cycles attached to ideal classes of real quadratic quadratic fields, which are not algebraic.

Claim: These geodesic cycles (and their quaternionic analogues) encode the valuations of a richer collection of invariants, suitable for generating class fields of real quadratic fields.

Singular moduli

A *singular modulus* is a value of $j(z)$ at a quadratic imaginary argument (CM point) in the Poincaré upper half plane $\mathcal{H}$.

The *theory of complex multiplication* asserts that these values are *algebraic integers*.

Examples:

$j(i) = 1728; \qquad j\left(\frac{1+\sqrt{-3}}{2}\right) = 0; \qquad j\left(\frac{1+\sqrt{-7}}{2}\right) = -3375.$

$j\left(\frac{1+\sqrt{-23}}{2}\right) = w, \quad$ where

$$w^3 + 3491750w^2 - 5151296875w + 12771880859375 = 0.$$

It generates the Hilbert class field of $\mathbb{Q}(\sqrt{-23})$.

Differences of singular moduli and their factorisations

Gross, Zagier (1984). For all τ_1, τ_2 quadratic imaginary, the quantity

$$J_\infty(\tau_1, \tau_2) \quad := \quad j(\tau_1) - j(\tau_2) \in H_{12} := H_{\tau_1} H_{\tau_2}$$

is a smooth algebraic integer with an explicit factorisation.

All the primes q dividing $\mathrm{Norm} J_\infty(\tau_1, \tau_2)$ are $\leq D_1 D_2/4$.

The valuation $\mathrm{ord}_q \, \mathrm{Norm} J_\infty(\tau_1, \tau_2)$ is related to the *topological intersection* of certain CM 0-cycles on a zero-dimensional Shimura variety, attached to the definite quaternion algebra ramified at q and ∞.

Modular generating series

Let

$$J_\infty(D_1, D_2) := \prod J_\infty(\tau_1, \tau_2), \qquad \mathrm{disc}(\tau_1) = D_1,\ \mathrm{disc}(\tau_2) = D_2,$$

Kudla, Rapoport, Yang. The quantity $c(D_2) := \log J_\infty(D_1, D_2)$ (with D_1 fixed) is the $D_1 D_2$-th fourier coefficient of a *mock modular form* of weight $3/2$.

Real quadratic fields

If τ is a real quadratic irrationality, then $j(\tau)$ is not defined...!.

It is a part of "Kronecker's jugendtraum" or Hilbert's twelfth problem, to "make sense" of $j(\tau)$ in this setting.

Goals of this lecture: For τ_1 and τ_2 real quadratic,

• construct $J_p(\tau_1, \tau_2) \overset{?}{\in} H_{12}$ by p-adic analytic means;

• relate $\mathrm{ord}_q J_p(\tau_1, \tau_2)$ to the *topological intersection* of certain real quadratic geodesics on Shimura curves.

• interpret the generating series for $\log_p J_p(\tau_1, \tau_2)$ in terms of certain "p-adic mock modular forms".

Drinfeld's p-adic upper-half plane

The Drinfeld p-adic upper half plane $\mathcal{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$ offers a tempting framework for "real multiplication theory", since, unlike $\mathcal{H}$, it contains an abundance of real quadratic irrationalities.

Definition

A point on $\tau \in \mathcal{H}_p$ is called a *real multiplication (RM) point* if it belongs to $\mathcal{H}_p \cap K$ for some real quadratic field K.

Hope: A p-adic analogue of j leads to singular moduli for real quadratic $\tau \in \mathcal{H}_p$.

Question: What is this p-adic analogue?

Rigid meromorphic functions on $\mathcal{H}_p$

Classical setting: Meromorphic functions on $\mathbf{SL}_2(\mathbb{Z})\backslash\mathcal{H}$.

The p-adic setting: A *rigid meromorphic function* is a ratio of rigid analytic functions.

It is natural to consider rigid meromorphic functions with good transformation properties under $\mathbf{SL}_2(\mathbb{Z})$.

In fact it turns out to be appropriate to work with an even larger group of symmetries: the p-modular group

$$\Gamma := \mathbf{SL}_2(\mathbb{Z}[1/p]).$$

Functions on $\Gamma\backslash\mathcal{H}_p$

The action of Γ, or even of $\mathbf{SL}_2(\mathbb{Z})$, on $\mathcal{H}_p$ is not discrete in the p-adic topology. The subgroup of translations $z \mapsto z + n$, with $n \in \mathbb{Z}$, already has non-discrete orbits!

Let $\mathcal{M}$:= the space of rigid meromorphic functions on $\mathcal{H}_p$, endowed with the translation action of Γ:

$$f|\gamma = f\left(\frac{az+b}{cz+d}\right).$$

There are no non-constant $\mathbf{SL}_2(\mathbb{Z})$ or Γ-invariant elements in $\mathcal{M}$:

$$H^0(\Gamma, \mathcal{M}) = \mathbb{C}_p.$$

Rigid meromorphic cocycles

Let $\mathcal{M}^\times :=$ the multiplicative group of non-zero elements of $\mathcal{M}$.

Since $H^0(\Gamma, \mathcal{M}^\times) = \mathbb{C}_p^\times$, consider its higher cohomology instead!

Definition

A *rigid meromorphic cocycle* is a class in $H^1(\Gamma, \mathcal{M}^\times)$.
It is said to be *parabolic* if its restrictions to the parabolic subgroups of Γ are trivial.

Elementary but key observation: rigid meromorphic cocycles can be meaningfully *evaluated* at RM points.

Evaluating a modular cocycle at an RM point

An element $\tau \in \mathcal{H}_p$ is an RM point if and only if

$$\mathrm{Stab}_\Gamma(\tau) = \langle \pm\gamma_\tau \rangle$$

is an infinite group of rank one.

Definition

If $J \in H^1(\Gamma, \mathcal{M}^\times)$ is a rigid meromorphic cocycle, and $\tau \in \mathcal{H}_p$ is an RM point, then the *value* of J at τ is

$$J[\tau] := J(\gamma_\tau)(\tau) \in \mathbb{C}_p \cup \{\infty\}.$$

The quantity $J[\tau]$ is a well-defined numerical invariant, independent of the cocycle representing the class of J, and

$$J[\gamma\tau] = J[\tau], \qquad \text{for all } \gamma \in \Gamma.$$

Rigid meromorphic cocycles and RM points

Let S be the standard matrix of order 2 in $\mathbf{SL}_2(\mathbb{Z})$.

Theorem (Jan Vonk, D)

If J is a rigid meromorphic cocycle, then $j := J(S) \in \mathcal{M}^\times$ has its poles concentrated in finitely many Γ-orbits of RM points.

$H_\tau :=$ ring class field attached to the prime-to-p-part of $\mathrm{disc}(\tau)$.

Definition

The *field of definition of J*, denoted H_J, is the compositum of H_τ as τ ranges over the poles of $j(z)$.

The main conjecture of real multiplication

The main assertion of complex multiplication:

Theorem (Kronecker, ...)

Let J be a meromorphic modular function on $\mathbf{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ with fourier coefficients in a field H_J.
For all imaginary quadratic $\tau \in \mathcal{H}$, the value $J(\tau)$ belongs to the compositum of H_J and H_τ.

Conjecture (Jan Vonk, D)

Let J be a rigid meromorphic cocycle on $\mathbf{SL}_2(\mathbb{Z}[1/p])\backslash\mathcal{H}_p$, and let H_J denote its field of definition.
For all real quadratic $\tau \in \mathcal{H}_p$, the value $J[\tau]$ belongs to the compositum of H_J and H_τ.

Example of rigid meromorphic cocycles

For real quadratic $\tau \in \mathcal{H}_p$, the orbit $\Gamma\tau$ is dense in $\mathcal{H}_p$.

The set $\Sigma_\tau := \{w \in \Gamma\tau \text{ such that } ww' < 0\}$ is discrete.

Theorem (Vonk, D)

Let $p = 2, 3, 5, 7, 11, 17, 19, 23, 29, 31, 41, 47, 59,$ or 71. (I.e., p divides the cardinality of the Monster sporadic group!) For each real quadratic $\tau \in \mathcal{H}_p$, there is a unique rigid meromorphic cocycle J_τ^+ for which $j_\tau^+ := J_\tau^+(S)$ is given by

$$j_\tau^+(z) \sim \prod_{\substack{w\in\Sigma_\tau \\ |w|_p\leq 1}} \left(\frac{z-w}{z-pw}\right)^{\mathrm{sgn}(w)} \times \prod_{\substack{w\in\Sigma_\tau \\ |w|_p>1}} \left(\frac{z/w-1}{z/pw-1}\right)^{\mathrm{sgn}(w)}.$$

Computational aspects

Rigid meromorphic cocycles are amenable to explicit numerical calculations on the computer, for the following reasons:

• The rigid meromorphic cocycle J is completely determined by a single rigid analytic function $j := J(S) \in \mathcal{O}_{\mathcal{H}_p}$.

• The value $J[\tau]$ can be expressed as a product of values of the form $j(w)$ where w belongs to the "standard affinoid" $\mathcal{A} \subset \mathcal{H}_p$, namely, the complement of the $p+1$ mod p residue discs centered at the points in $\mathbb{P}_1(\mathbb{F}_p)$.

• The image of $j(z)$ in the Tate algebra $\mathcal{O}_{\mathcal{A}}$ can be computed with an accuracy of p^{-M} in time that is polynomial in M.

An example

Let $\varphi = \frac{-1+\sqrt{5}}{2}$ be the golden ratio.

The p-adic J_φ^+ for $p = 2, 3, 7, 13, 17, 23$, or 47, is the "simplest instance" of a rigid meromorphic cocycle.

The RM point $\tau = \sqrt{223}$ of discriminant 223 has class number 6, and $J_\varphi^+[\sqrt{223}]$ appears to satisfy:

$$p = 7. \quad 282525425x^6 + 27867770x^5 + 414793887x^4 - 128906260x^3 + 414793887x^2 + 27867770x + 282525425,$$

$$p = 13. \quad 464800x^6 + 1275520x^5 + 1614802x^4 + 1596283x^3 + 1614802x^2 + 1275520x + 464800,$$

$$p = 47. \quad 4x^6 + 4x^5 + x^4 - 2x^3 + x^2 + 4x + 4.$$

An aside on rational modular cocycles

Rigid meromorphic cocycles are analogous to *rational modular cocycles*: elements $\Phi \in H^1(\mathbf{SL}_2(\mathbb{Z}), \mathcal{R}^\times)$, where $\mathcal{R}^\times$ is the multiplicative group of rational functions on $\mathbb{P}_1$.

• These objects were studied and classified by Marvin Knopp, Avner Ash, Youngju Choie and Don Zagier.

• Bill Duke, Ozlem Imamoglu, Arpad Toth: the RM values of rational modular cocycles are related to the topological linking numbers of real quadratic geodesics on $\mathbf{SL}_2(\mathbb{Z})\backslash\mathbf{SL}_2(\mathbb{R})$.

Classification of rigid meromorphic cocycles

Guided by the Knopp-Choie-Zagier classification, we have:

Theorem (Jan Vonk, D)

For any RM point $\tau \in \mathcal{H}_p$, there is a unique $J_\tau \in H^1(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$ whose poles are concentrated on $\Gamma\tau$.
Every rigid meromorphic cocycle is a product of powers of finitely many of these J_τ, modulo scalars.

The definition of J_τ is very similar to that of J_τ^+.

Remark: $H^1_{\mathrm{par}}(\Gamma, \mathbb{C}_p^\times)$ is trivial, so a rigid meromorphic cocycle is determined by its image in $H^1(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$.

p-adic intersection numbers

The work of Duke, Imamoglu and Toth on linking number of modular geodesics immediately suggests the following definition:

Definition: The quantity $J_p(\tau_1, \tau_2) := J_{\tau_1}[\tau_2] \stackrel{?}{\in} H_{12}$ is called the *p-adic intersection number* of τ_1 and τ_2.

Conjecture (Jan Vonk, D)

The quantity $J_p(\tau_1, \tau_2)$ behaves in many key respects like the classical $J_\infty(\tau_1, \tau_2) = j(\tau_1) - j(\tau_2)$ of Gross-Zagier.

A few values of $J_p(\sqrt{2}, \tau)$ with $\tau \in \mathbb{Z}[\sqrt{2}]$

τ	$p = 3$	$p = 5$	$p = 13$
$2\sqrt{2}$	$\frac{7+24\sqrt{-1}}{2\cdot5^2}$	$\frac{-7+4\sqrt{-2}}{3^2}$	1
$4\sqrt{2}$	$\frac{-7+24\sqrt{-1}}{2\cdot5^2}$	$\frac{-7+4\sqrt{-2}}{3^2}$	1
$7\sqrt{2}$	$\frac{-97247+24675\sqrt{-7}}{2^3\cdot11^4}$	$\frac{-2719+5763\sqrt{-7}}{2^7\cdot11^2}$	$\frac{31+3\sqrt{-7}}{2^5}$
$8\sqrt{2}$	$\frac{2047+3696\sqrt{-1}}{5^2\cdot13^2}$	$\frac{511+680\sqrt{-2}}{3^2\cdot11^2}$	$\frac{7+4\sqrt{-2}}{3^2}$
$11\sqrt{2}$	$\frac{-17005256513+1565252064\sqrt{-22}}{13^2\cdot19^4\cdot29^2}$	$\frac{28463+504\sqrt{-22}}{13^4}$	$\frac{-8071+2363\sqrt{-11}}{2\cdot3^2\cdot5^4}$
$16\sqrt{2}$	$\frac{985306661831273376-335876326171960619 3\sqrt{-1}}{5^6\cdot13^2\cdot29^4\cdot37^4}$	$\frac{651578431+788458960\sqrt{-2}}{3^6\cdot11^6}$	$\frac{-7+4\sqrt{-2}}{3^2}$

Gross-Zagier factorisations

$J_\infty(\tau_1, \tau_2) := j(\tau_1) - j(\tau_2) \in H_{12} = H_1 H_2$.

Fix embeddings of H_{12} into $\mathbb{C}$ and into $\bar{\mathbb{Q}}_q$, for each q.

We can then talk about $\mathrm{ord}_q\, J_\infty(\tau_1, \tau_2)$.

Gross and Zagier gave an algebraic formula for this quantity, involving the **definite** quaternion algebra $B_{q\infty}$ satisfying:

$\bullet B_{q\infty} \otimes \mathbb{R} \simeq H$, where $H =$ Hamilton quaternions;

$\bullet B_{q\infty} \otimes \mathbb{Q}_q \simeq H_q$, the unique division algebra of rank 4 over $\mathbb{Q}_q$;

$\bullet B_{q\infty} \otimes \mathbb{Q}_\ell \simeq M_2(\mathbb{Q}_\ell)$, for all $\ell \neq \infty, q$.

Quaternionic embeddings

A CM point $\tau \in \mathcal{H}$ of discriminant $D < 0$ corresponds to an embedding of the order $\mathcal{O}$ into $M_2(\mathbb{Z})$, the maximal order in the split quaternion algebra $M_2(\mathbb{Q})$.

Definition: An optimal embedding of $\mathcal{O}$ into $B_{q\infty}$ is a pair (φ, R) where R is a maximal order in $B_{q\infty}$ and $\varphi : K \longrightarrow B_{q\infty}$ satisfies $\varphi(K) \cap R = \varphi(\mathcal{O})$.

The group $B_{q\infty}^\times$ acts on $\mathrm{Emb}(\mathcal{O}, B_{q\infty})$ by conjugation:

$$b * (\varphi, R) = (b\varphi b^{-1}, bRb^{-1}).$$

$$\Sigma(\mathcal{O}, B_{q\infty}) := B_{q\infty}^\times \backslash \mathrm{Emb}(\mathcal{O}, B_{q\infty}).$$

Key Fact: Both $\mathbf{SL}_2(\mathbb{Z})\backslash\mathcal{H}^D$ and $\Sigma(\mathcal{O}, B_{q\infty})$ are endowed with simply transitive G_D-actions.

Arithmetic intersection multiplicities

- Given $(\varphi_1, R_1) \in \mathrm{Emb}(\mathcal{O}_1, B_{q\infty})$ and $(\varphi_2, R_2) \in \mathrm{Emb}(\mathcal{O}_2, B_{q\infty})$,

$$\text{let} \quad [\varphi_1, \varphi_2]_q = 0 \text{ if } R_1 \neq R_2,$$

and, if $R_1 = R_2 =: R$,

$$[\varphi_1, \varphi_2]_q := \mathrm{Max}_t \text{ such that } \varphi_1(\mathcal{O}_1) = \varphi_2(\mathcal{O}_2) \text{ in } R/q^{t-1}R.$$

- Given $(\varphi_1, R_1) \in \Sigma(\mathcal{O}_1, B_{q\infty})$ and $(\varphi_2, R_2) \in \Sigma(\mathcal{O}_2, B_{q\infty})$, set

$$(\varphi_1, \varphi_2)_q := \sum_{b \in B_{q\infty}^{\times}} [b\varphi_1 b^{-1}, \varphi_2]_q.$$

The Gross-Zagier factorisation

Theorem (Gross-Zagier)

Let $q \nmid D_1 D_2$ be a prime. If D_1 or D_2 is a square modulo q, then $\mathrm{ord}_q J_\infty(\tau_1, \tau_2) = 0$. *Otherwise, there exists bijections*

$$\mathbf{SL}_2(\mathbb{Z})\backslash\mathcal{H}^{D_1} \leftrightarrow \Sigma(\mathcal{O}_{D_1}, B_{q\infty}), \qquad \mathbf{SL}_2(\mathbb{Z})\backslash\mathcal{H}^{D_2} \leftrightarrow \Sigma(\mathcal{O}_{D_2}, B_{q\infty}),$$

compatible with the G_{D_1} and G_{D_2}-actions, for which

$$\mathrm{ord}_q J_\infty(\tau_1, \tau_2) = (\varphi_1, \varphi_2)_q,$$

for all $\tau_1 \in \mathcal{H}^{D_1}$ and $\tau_2 \in \mathcal{H}^{D_2}$, associated to φ_1 and φ_2 respectively.

Factorisations of real quadratic singular moduli

We now consider the factorisation of $J_p(\tau_1, \tau_2) \overset{?}{\in} H_{12} = H_1 H_2$.

Fix embeddings of H_{12} into $\mathbb{C}$ and into $\bar{\mathbb{Q}}_q$, for each q.

We can then talk about $\mathrm{ord}_q J_p(\tau_1, \tau_2)$.

Our conjectural formula for this quantity, involves... the **indefinite** quaternion algebra B_{qp} ramified at q and p:

•$B_{qp} \otimes \mathbb{Q}_q \simeq H_q$, $B_{qp} \otimes \mathbb{Q}_p \simeq H_p$, the unique division algebra of rank 4 over $\mathbb{Q}_q$ and $\mathbb{Q}_p$;

•$B_{qp} \otimes \mathbb{Q}_\ell \simeq M_2(\mathbb{Q}_\ell)$, for all $\ell \neq p, q$.

•$B_{qp} \otimes \mathbb{R} \simeq M_2(\mathbb{R})$;

Shimura curves

Because B_{qp} in indefinite, it has a *unique* maximal order R, up to conjugation.

The group $\Gamma_{pq} = R_1^\times \subset \mathbf{SL}_2(\mathbb{R})$ acts discretely and co-compactly on $\mathcal{H}$;

The Riemann surface $\Gamma_{pq}\backslash\mathcal{H}$ is called the *Shimura curve* attached to the pair (p, q).

Given embeddings $\varphi_1 \in \mathrm{Emb}(\mathcal{O}_1, R)$ and $\varphi_2 \in \mathrm{Emb}(\mathcal{O}_2, R)$, let γ_1 and γ_2 be the hyperbolic geodesics on $\mathcal{H}$ joining the fixed points for $\varphi_1(\mathcal{O}_1^\times)$ and $\varphi_2(\mathcal{O}_2^\times)$ respectively.

The geodesics γ_1 and γ_2 map to closed geodesics $\bar{\gamma}_1$ and $\bar{\gamma}_2$ on the Shimura curve $\Gamma_{pq}\backslash\mathcal{H}$.

Topological intersections

$[\gamma_1, \gamma_2]_\infty :=$ signed intersection of γ_1 and γ_2.

Fact. The topological intersection multiplicity of $\bar{\gamma}_1$ and $\bar{\gamma}_2$ on the Shimura curve $\Gamma_{pq}\backslash\mathcal{H}$ is

$$(\bar{\gamma}_1, \bar{\gamma}_2)_\infty := \sum_{b \in \mathcal{O}_2^\times \backslash \Gamma_{pq} / \mathcal{O}_1^\times} [b\gamma_1 b^{-1}, \gamma_2]_\infty.$$

Definition. The q-weighted intersection number of φ_1 and φ_2 is

$$(\varphi_1, \varphi_2)_{q\infty} := \sum_{b \in \mathcal{O}_2^\times \backslash \Gamma_{pq} / \mathcal{O}_1^\times} [b\varphi_1 b^{-1}, \varphi_2]_q \cdot [b\gamma_1 b^{-1}, \gamma_2]_\infty.$$

A Gross-Zagier-style factorisation

Conjecture (Jan Vonk, D)

Let $q \nmid D_1 D_2$ *be a prime. If* D_1 *or* D_2 *is a square modulo* q*, then* $\mathrm{ord}_q\, J_p(\tau_1, \tau_2) = 0$*. Otherwise, there exists bijections*

$$\Gamma \backslash \mathcal{H}_p^{D_1} \leftrightarrow \Sigma(\mathcal{O}_{D_1}, R), \qquad \Gamma \backslash \mathcal{H}_p^{D_2} \leftrightarrow \Sigma(\mathcal{O}_{D_2}, R),$$

which are compatible with the G_{D_1} *and* G_{D_2}*-actions, and for which*

$$\mathrm{ord}_q\, J_p(\tau_1, \tau_2) = (\varphi_1, \varphi_2)_{q\infty},$$

for all $\tau_1 \in \mathcal{H}_p^{D_1}$ *and* $\tau_2 \in \mathcal{H}_p^{D_2}$*, associated to* φ_1 *and* φ_2 *respectively.*

An example

James Rickards has developed and implemented efficient algorithms for computing the q-weighted topological intersection numbers of real quadratic geodesics on Shimura curves.

An example: $D_1 = 13$, $D_2 = 285 = 3 \cdot 5 \cdot 19$, $p = 2$

Vonk, D: $J_2(\tau_1, \tau_2)$ satisfies (to 800 digits of 2-adic precision)

$$7736097284175893694750297399823 9x^4 + 1401810704388908317213141350998 03x^3$$

$$+209895619549791255199413489899292x^2 + 140181070438890831721314135099803x$$

$$+773609728417589369475029739982 39,$$

James Rickards: $e_{q2} := \frac{1}{2} \sum_{\tau_1, \tau_2} |(\varphi_{\tau_1}, \varphi_{\tau_2})_{q\infty}|$ on $\Gamma_{2q} \backslash \mathcal{H}$.

q	7	19	31	73	109	151	163	397	457	463
e_{q2}	7	2	2	1	2	2	1	1	1	1

But: $773609728417589369475029739982 39 =$

$$7^7 \cdot 19^2 \cdot 31^2 \cdot 73 \cdot 109^2 \cdot 151^2 \cdot 163 \cdot 397 \cdot 457 \cdot 463.$$

Norms of singular moduli

Let $q \equiv 3 \pmod 4$ be a prime, and for all *negative* D,

$$J_\infty(-q, D) := \prod_{\substack{\mathrm{disc}(\tau_1)=-q, \\ \mathrm{disc}(\tau_2)=D}} J_\infty(\tau_1, \tau_2) \in \mathbb{Z}.$$

Gross-Zagier, Kudla-Rapoport-Yang: The quantity $c(D) := \log J_\infty(-q, -D)$ for $D > 0$ is the D-th fourier coefficient of a non-holomorphic modular form of weight $3/2$.

This assertion is a very special case of the "Kudla program", predicting that quantities like $c(D)$, which describe the arithmetic intersections of natural cycles on Shimura varieties, can be packaged into a modular generating series.

The incoherent Eisenstein series of Kudla-Rapoport-Yang

Let $\chi_q : (\mathbb{Z}/q\mathbb{Z})^\times \longrightarrow \pm 1$ be the odd quadratic Dirichlet character.

Non-homomorphic Eisenstein series:

$$E_-(\tau, s) = y^{s/2} \sum_{(c,d)} (c\tau + d)^{-1} |c\tau + d|^{-s} \Phi_q^-(c, d).$$

Functional equation: $E_-(\tau, -s) \sim -E_-(\tau, s)$.

Hence $E_-(\tau, 0) = 0$.

Definition. The *incoherent Eisenstein series* of Kudla-Rapoport-Yang is the derivative

$$\Phi_{KRY} := \frac{d}{ds}(E_-(\tau, s))_{s=0}.$$

It is a non-holomorphic modular form of weight one.

The theorem of Kudla-Rapoport Yang

Theorem (Kudla, Rapoport, Yang)

The quantity $c(D) := \log J_\infty(-q, -D)$ *is essentially the* D*th fourier coefficient of* $\Phi_{KRY}(4\tau) \times \theta(\tau)$*, where* $\theta(q)$ *is the standard unary theta series of weight* $1/2$.

This theorem has been extended to the setting where weight one theta-series are replaced by a weight one cusp form g, by Bill Duke+Yingkun Li, Stephan Ehlen, Maryna Viazovska, and Pierre Charollois+Yingkun Li.

The role of the incoherent Eisenstein series of weight one of KRY is played by a *mock modular form of weight one having* g *as its shadow*.

Twisted norms of real quadratic singular moduli

Now let $\psi : G_q \longrightarrow L^\times$ be any class character, $q = 1 + 4m$.

The set $\Gamma \backslash \mathcal{H}_p^{\mathrm{disc}=q}$ is endowed with a simple transitive G_q-action, and can thus be identified with G_q.

For all *positive* D, let

$$J_p(\psi, D) := \prod_{\substack{\mathrm{disc}(\tau_1)=q \\ \mathrm{disc}(\tau_2)=D}} J_p(\tau_1, \tau_2)^{\psi^{-1}(\tau_1)} \overset{?}{\in} (H_q^\times \otimes L)^\psi.$$

Conjecture (Jan Vonk, D)

The quantity $c_\psi(D) := \log_p J_p(\psi, D)$ is the Dth fourier coefficient of a "p-adic mock modular form" of weight $3/2$.

A p-adic Kudla-Rapoport-Yang theorem

Theorem (Alan Lauder, Victor Rotger, D)

There exists a "p-adic mock modular form" Φ_ψ of weight one whose fourier coefficients are the p-adic logarithms of elements of $(H_q^\times \otimes L)^\psi$. It exhibits many of the same properties as Φ_{KRY} and of the forms arising in Duke-Li, Ehlen, Viazovska, Charollois-Li...

The modular form Φ_ψ is simply the derivative, with respect to the weight, of a Hida family of modular forms specialising to θ_{ψ_o} in weight one, where $\psi_o/\psi_o' = \psi$.

The proof of the theorem is very different, and *substantially simpler* from the approaches of Kudla-Rapoport-Yang, Duke-Li, Ehlen, Viazovska, Charollois-Li used in the Archimedean setting. It rests crucially on the deformation theory of modular forms and of p-adic Galois representations.

A more tracatable conjecture?

Conjecture (Jan Vonk, D)

The quantity $c_\psi(D) := \log_p J_p(\psi, D)$ *is essentially the* D*th fourier coefficient of* $\Phi_\psi(q^4) \times \theta(q)$*, where* $\theta(q)$ *is the standard unary theta series of weight* $1/2$*.*

This conjecture suggests a possible road map for proving the algebraicity of "real quadratic singular moduli"...

Conclusion

The RM values of rigid meromorphic multiplicative cocycles lead to *conjectural analogues* of singular moduli, with applications to

- explicit class field theory for real quadratic fields;
- Gross-Zagier style factorisation formulae;
- suggesting test cases for an eventual "p-adic Kudla program".

The experiments reveal a promising connection between the p-adic Kudla program and Hilbert's twelfth problem for real quadratic fields.

We are still *very far* from understanding this "real multiplication theory" as well as its classical counterpart!

Thank you for your attention!