A Tale of Two Bases

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2020-09-23

IAS Postdoc Talk

About

G a complex, algebraic, simply-laced, semisimple, adjoint group

Representation theory

- perfect bases
- algebra map

Algebraic geometry

- varieties
- multidegrees

Combinatorics

- crystal bases
- flag functions

Thesis work, a bird's eye view

Question are perfect bases unique?

Expected discrepancy at classical counterexample

Method The algebra map $\overline{D}$

Findings ideals of MV cycles in type A via a recipe for generating matrix varieties from tableaux

Zooming in

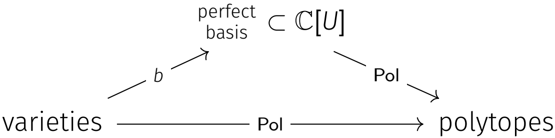
Units highest weight representations of G viewed in $\mathbb{C}[U]$

$$\bigcup_{\lambda \in P_+} L(\lambda) \to \bigoplus_{\nu \in Q_+} \mathbb{C}[U]_{-\nu}$$

Why because the MV basis and the dual semicanonical basis have **compatible polytope models**

Why?

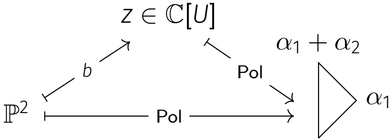
Both bases are $B(\infty)$ crystal bases, hence admit maps to polytopes. Remarkably, their polytopes can also be obtained from the geometric spaces defining them



([BKT14])

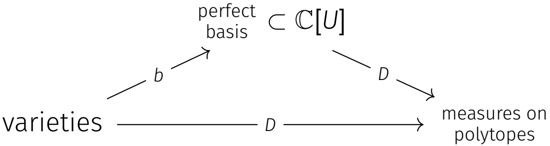
such that $\mathsf{Pol}(Z) = \mathsf{Pol}(Y)$ whenever $\mathsf{Pol}(b(Z)) = \mathsf{Pol}(b(Y))$.

For example



Question

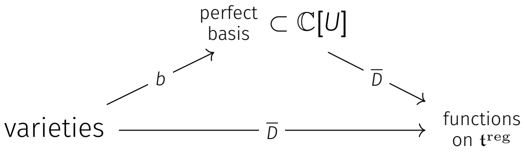
Do equal polytopes imply equal perfect basis vectors?



Baumann, Kamnitzer and Knutson in [BKK19] ask a weaker question by associating to elements f of $\mathbb{C}[U]$ measures $D(f)$ on $\mathfrak{t}^*_{\mathbb{R}}$ which can again be defined intrinsic the Z's and Y's.

Question

$D(f)$ is replaced by the constant coefficient of its Fourier transform, a quantity which is denoted $\overline{D}(f)$.



$\overline{D} : \mathbb{C}[U] \to \mathbb{C}[\mathfrak{t}^{\mathrm{reg}}]$ is an algebra map

$$\overline{D}(f) = \sum_{\mathbf{i} \in \mathrm{Seq}(\nu)} \langle e_{\mathbf{i}}, f \rangle \overline{D}_{\mathbf{i}} \qquad \overline{D}_{\mathbf{i}} = \prod_{k=1}^{p} \frac{1}{\alpha_{i_1} + \cdots + \alpha_{i_k}} \qquad \overline{D}(f)(x) = f(n_x)$$

where $n_x \in U$ is such that $\mathrm{Ad}_{n_x}(x) = x + e$ with e a sum of root vectors.

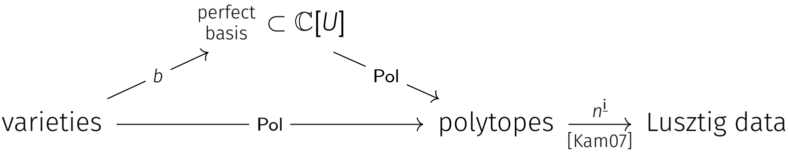
Do we have $\overline{D}(b(Z)) = \overline{D}(b(Y))$ whenever $\mathrm{Pol}(Z) = \mathrm{Pol}(Y)$?

Means to compute on the MV basis

The Mirković–Vybornov isomorphism [MVy07] relates slices in the nilpotent cone of $\mathfrak{gl}_N$ and slices in affine Grassmannians Gr_{GL_m} and restricts to an isomorphism of MV cycles and generalized orbital varieties, which can be labeled by semistandard Young tableaux according to a Spaltenstein recipe.

Matching MV cycles and generalized orbital varieties

Add to the [BKT14] diagram a more appropriate “combinatorial fingerprint”



because it is readily available for tableaux.

Theorem. [D19] Let $\tau \in \mathcal{T}(\lambda)_\mu$ and let Z be the MV cycle with $n(Z) = n(\tau)$. Then (up to a certain translate) Z is equal to the closure of the image of the generalized orbital variety labeled by τ.

Comparing

Proposition. [BKK19] Let Z be an MV cycle of Lusztig datum n and let X_τ be the corresponding generalized orbital variety. Then

$$\overline{D}(b(Z)) = \varepsilon^T_{L_0}(Z) = \varepsilon^T_{L_\mu}(t^\mu Z) = \frac{\mathrm{mdeg}_{\mathbb{T}_\mu \cap \mathfrak{n}}(X_\tau)}{\prod_{\Delta_+} \beta}$$

In other words, $\overline{D}(b(Z))$ can be computed in terms of the multidegree of the corresponding generalized orbital variety. To find the multidegree of X_τ we need to know the generators of its ideal, which we find using a Spaltenstein recipe.

Counterexample

Let $G^\vee = SL_6$ and

$$\tau = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 5 & 5 \\ \hline 4 & 4 \\ \cline{1-2} 6 & 6 \\ \cline{1-2} \end{array}$$

and suppose $Y \in \operatorname{Irr} \Lambda$ and $Z \in \mathrm{Gr}$ have Lusztig data equal to $n(\tau)$.

Theorem (6.5.1)

There exists $Y' \in \operatorname{Irr} \Lambda(\nu)$ such that

$$\overline{D}(b(Z)) = \overline{D}(b(Y)) - 2\overline{D}(b(Y')) .$$

In particular, $\overline{D}(b(Z)) \neq \overline{D}(b(Y))$, and therefore $b(Y) \neq b(Z)$ even though $\mathrm{Pol}(Y) = \mathrm{Pol}(Z)$.

Categorical $\overline{D}$?

The Mirković–Vybornov slice admits a quantization called a truncated shifted Yangian. Kamnitzer, Tingley, Webster, Weekes and Yacobi in [KTWWY19] show that its category $\mathcal{O}$ is equivalent to a category of KLR modules, whose simples also admit MV polytopes. Can we use $\overline{D}$ to understand supports of simples as unions of MV cycles, and compare MV basis and canonical basis?

Some further problems

Derived $\overline{D}$?

[BKK19] offers a proof of Muthaih's conjecture that $L(\lambda)_0 \to \mathbb{C}[\mathfrak{t}^{\mathrm{reg}}]$ is W-equivariant. Can we generalize this result to a quasi-equivariant map $L(\lambda)_0 \to \mathbb{C}[\mathfrak{t}^{\mathrm{reg}} \times \mathbb{C}]$ by generalizing $\overline{D}$ to a map which manifests as a $T \times \mathbb{C}^{\times}$ multidegree?

Thank you for listening

Labeling generalized orbital varieties

To tableau $\tau \in \mathcal{T}(\lambda)_\mu$ we can associate a matrix $A \in \mathbb{T}_\mu \cap \mathfrak{n}$ such that for all $1 \leq i \leq m$ the upper submatrix made of the first $i \times i$ blocks has Jordan type $\lambda^{(i)} =$ shape of $\tau\big|_{1,2,\dots,i}$

$$A_{(i)} \in \mathbb{O}_{\lambda^{(i)}} \rightsquigarrow \mathring{X}_\tau$$

By example,

$$\tau = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \supset \tau\big|_{1,2} = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \supset \tau\big|_{1} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}$$

defines a matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a & b & c \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad a, d = 0 \text{ and } b, c \neq 0$$

Lusztig data of tableaux

By example, if

$$\tau = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline \end{array}$$

then $n(\tau) = (1, 0, 2)$ is got by considering the GT pattern of τ

$$\begin{array}{ccccc} & & \lambda_1^{(1)} & & \\ & \lambda_1^{(2)} & & \lambda_2^{(2)} & \\ \lambda_1^{(3)} & & \lambda_2^{(3)} & & \lambda_3^{(3)} \end{array} \quad = \quad \begin{array}{ccccccc} & & & 2 & & & \\ & 3 & & 1 & & 1 & \\ 3 & & 0 & 3 & 2 & & 0 \end{array}$$