Global existence and convergence of solutions to gradient systems and applications to Yang-Mills flow

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Outline

Łojasiewicz-Simon gradient inequality for analytic functionals

Łojasiewicz-Simon gradient inequality for analytic functionals

We shall discuss the

- Łojasiewicz gradient inequality on Euclidean space and some motivations for its development and applications;
- Łojasiewicz-Simon gradient inequalities for analytic functionals on Banach spaces.

Łojasiewicz gradient inequality on Euclidean space

Łojasiewicz gradient inequality on Euclidean space I

The gradient inequality was discovered by Stanisław Łojasiewicz [43] around 1960 in his research on semianalytic and subanalytic sets.

Our interest in his gradient inequality is due to its application to the study of non-linear evolution equations, a very powerful idea developed by Leon Simon [61] in 1983.

Simon's idea is based in turn on a paradigm due to Łojasiewicz [44] for proving global existence and convergence of solutions to ordinary differential equations in Euclidean space.

Łojasiewicz gradient inequality on Euclidean space II

Theorem 1.1 (Global existence and convergence of solutions to gradient systems in $\mathbb{R}^n$)

(Łojasiewicz [44, Theorem 1], [45, p. 1592]) Let $\mathscr{E}$ be an analytic, non-negative function on a neighborhood of the origin in $\mathbb{R}^n$ such that $\mathscr{E}(0) = 0$. Then there exists a neighborhood, $U = \{x \in \mathbb{R}^n : |x| < \sigma\}$, of the origin such that each trajectory, $u_{x_0}(t)$, with $u_{x_0}(0) = x_0 \in U$, of the system,

$$\dot{u}(t) = -\mathscr{E}'(u(t)), \tag{1}$$

is defined on $[0, \infty)$, has finite length, and converges uniformly to a point $u_{x_0}(\infty) \in \mathrm{Crit}(\mathscr{E}) := \{z \in U : \mathscr{E}'(z) = 0\}$ as $t \to \infty$. For a constant $\theta \in (0, 1)$ depending only on $\mathscr{E}$, one has

$$|u_{x_0}(t) - x_0| \leq \int_0^t |\dot{u}_{x_0}(s)|\, ds \leq \frac{\mathscr{E}(x_0)^{1-\theta}}{1-\theta},$$

$$|u_{x_0}(\infty) - u_{x_0}(t)| \leq \int_t^\infty |\dot{u}_{x_0}(s)|\, ds \leq \frac{(1+t)^{\theta-1}}{1-\theta}, \quad \text{for } 0 \leq t < \infty.$$

Łojasiewicz gradient inequality on Euclidean space III

To prove Theorem 1.1, Łojasiewicz applied the following version of his gradient inequality [43]:

Theorem 1.2 (Finite-dimensional Łojasiewicz gradient inequality)

(Łojasiewicz [43]) Let $U \subset \mathbb{R}^n$ be an open subset, $z \in U$, and let $\mathscr{E} : U \to \mathbb{R}$ be a real-valued function. If $\mathscr{E}$ is real analytic on a neighborhood of z and $\mathscr{E}'(z) = 0$, then there exist constants $\theta \in (0, 1)$ and $\sigma > 0$ such that

$$|\mathscr{E}'(x)| \geq |\mathscr{E}(x) - \mathscr{E}(z)|^\theta, \quad \forall x \in \mathbb{R}^n, \ |x - z| < \sigma. \tag{2}$$

Theorem 1.2 was stated by Łojasiewicz in [42] and proved by him as [43, Proposition 1, p. 92] and Bierstone and Milman as [3, Proposition 6.8]; see also Chill and Jendoubi [11, Proposition 5.1 (i)] and Łojasiewicz [45, p. 1592].

Łojasiewicz gradient inequality on Euclidean space IV

Simon was motivated to generalize Łojasiewicz's results from finite to infinite dimensions because many nonlinear evolution equations can be viewed as gradient systems for suitable energy functionals.

In geometric analysis, applications include

1. Harmonic map gradient flow for maps $(M, g) \to (N, h)$;
2. Mean curvature flow;
3. Ricci curvature flow;
4. Yamabe scalar curvature flow;

and in mathematical physics, applications include

1. Cahn-Hilliard model for dynamics of pattern formation;
2. Ginzburg-Landau models for superconductivity;

Łojasiewicz gradient inequality on Euclidean space V

3. Models in fluid dynamics.

The main questions of interest for any of these systems include

- Do smooth solutions exist globally for all time $t \in [0, \infty)$?
- Do solutions converge to a smooth critical point $t \to \infty$?
- At what rate do global solutions converge?
- What are stability properties of local minima?

Infinite-dimensional Łojasiewicz gradient inequalities can also be used to prove that the critical values of energy functionals are discrete, rather than forming continua, and thus prove *energy gap* or *energy discreteness* results for physical systems.

Łojasiewicz-Simon gradient inequalities for analytic functionals on Banach spaces

Łojasiewicz-Simon for functionals on Banach spaces I

We begin with the following generalization of Simon's infinite-dimensional version [61, Theorem 3] of the Łojasiewicz gradient inequality [43].

Łojasiewicz-Simon for functionals on Banach spaces II

Theorem 1.3 (Łojasiewicz-Simon gradient inequality for analytic functionals on Banach spaces)

(F. and Maridakis [24]) Let $\mathscr{X}$ be a Banach space that is continuously embedded in a Hilbert space $\mathscr{H}$. Let $\mathscr{U} \subset \mathscr{X}$ be an open subset, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ be an analytic function, and $x_\infty \in \mathscr{U}$ be a critical point of $\mathscr{E}$, that is, $\mathscr{E}'(x_\infty) = 0$. Assume that $\mathscr{E}''(x_\infty) : \mathscr{X} \to \mathscr{X}^*$ is a Fredholm operator with index zero. Then there are positive constants, Z, σ, and $\theta \in [1/2, 1)$, with the following significance. If $x \in \mathscr{U}$ obeys

$$\|x - x_\infty\|_{\mathscr{X}} < \sigma, \tag{3}$$

then

$$\|\mathscr{E}'(x)\|_{\mathscr{X}^*} \geq Z|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^\theta. \tag{4}$$

Theorem 1.3 was stated without proof by Huang as [34, Theorem 2.4.5].

Łojasiewicz-Simon for functionals on Banach spaces III

Simon [61, Theorem 3] generalized the Łojasiewicz gradient inequality for analytic functions on Euclidean space to certain analytic functionals on the *Banach space of* $C^{2,\alpha}$ *sections of vector bundles over a closed, Riemannian manifold*.

Many researchers have adapted or extended Simon's [61, Theorem 3] in the intervening years, including Chill, Feireisl, Haraux, Huang, Jendoubi, Råde, Takáč, and others.

Our proof of Theorem 1.3 generalizes that of Feireisl and Takáč for their [26, Proposition 6.1] in the case of the Ginzburg-Landau energy functional.

The [9, Theorem 3.10 and Corollary 3.11] and [10, Corollary 3] due to Chill provide versions of the Łojasiewicz-Simon gradient

Łojasiewicz-Simon for functionals on Banach spaces IV

inequality for an analytic functional on a Banach space, but the hypotheses of Theorem 1.3 are simpler and more general.

Remark 1.4 (Gradient flows on metric spaces)

In recent years, there has been increasing interest in gradient flows on metric spaces and one might speculate as to whether one could further generalize Theorem 1.3 to such settings.

However, our proof of Theorem 1.3 uses the (Analytic) Implicit Function Theorem on Banach spaces and while that has replacements in more general settings (for example, the Nash-Moser implicit function theorem [28] on Fréchet spaces or that of Hofer, Wysocki, and Zehnder [32] on sc-Banach spaces, generalizations of the Łojasiewicz-Simon gradient inequality to such settings appear difficult.

Łojasiewicz-Simon gradient inequality for the Yang-Mills energy functional and discreteness of energies for Yang-Mills connections

Łojasiewicz-Simon for Yang-Mills connections I

We now summarize consequences of Theorem 1.3 for the Yang-Mills L^2 energy functional.

Łojasiewicz-Simon for Yang-Mills connections II

Definition 2.1 (Yang-Mills L^2 energy functional)

Let (X, g) be a closed, Riemannian, smooth manifold of dimension $d \geq 2$, and G be a compact Lie group, and P a smooth principal G-bundle over X. One defines the Yang-Mills L^2-energy functional by

$$\mathscr{E}_g(A) := \frac{1}{2}\int_X |F_A|^2 \, d\,\mathrm{vol}_g, \tag{5}$$

for all smooth connections, $A \in \mathscr{A}(P) = A_{\mathrm{REF}} + \Omega^1(X; \mathrm{ad}P)$, where $F_A \in \Omega^2(X; \mathrm{ad}P) \equiv C^\infty(X; \Lambda^2 \otimes \mathrm{ad}P)$ is the curvature of A and $\mathrm{ad}P := P \times_{\mathrm{ad}} \mathfrak{g}$ denotes the real vector bundle associated to P by the adjoint representation of G on its Lie algebra, $\mathrm{Ad} : G \ni u \to \mathrm{Ad}_u \in \mathrm{Aut}(\mathfrak{g})$, with fiber metric defined through the Killing form on $\mathfrak{g}$.

Łojasiewicz-Simon for Yang-Mills connections III

The gradient of the Yang-Mills L^2 energy functional $\mathscr{E}_g$ in (5) with respect to the L^2 metric on $\Omega^1(X; \mathrm{ad}P)$,

$$\left(\mathscr{E}_g'(A), a\right)_{L^2(X,g)} := \left.\frac{d}{dt}\mathscr{E}_g(A + ta)\right|_{t=0},$$

for all $a \in \Omega^1(X; \mathrm{ad}P)$, is given by

$$\left(\mathscr{E}_g'(A), a\right)_{L^2(X,g)} = (d_A^* F_A, a)_{L^2(X)}$$

where $d_A^* = d_A^{*,g} : \Omega^2(X; \mathrm{ad}P) \to \Omega^1(X; \mathrm{ad}P)$ is the L^2 adjoint of the exterior covariant derivative $d_A : \Omega^1(X; \mathrm{ad}P) \to \Omega^2(X; \mathrm{ad}P)$.

Łojasiewicz-Simon for Yang-Mills connections IV

One calls A a Yang-Mills connection (with respect to the Riemannian metric g on X) if it is a *critical point* for $\mathscr{E}_g$, that is,

$$\mathscr{E}_g'(A) = 0,$$

Thus A is a critical point of $\mathscr{E}_g$ if and only if it obeys the Yang-Mills equation,

$$d_A^{*,g} F_A = 0 \quad \text{on } X, \tag{6}$$

since $d_A^{*,g} F_A = \mathscr{E}_g'(A)$ when the gradient of $\mathscr{E} = \mathscr{E}_g$ is defined by the L^2 metric.

As our first application of Theorem 1.3, we have the following generalization of Råde's [54, Proposition 7.2], where X was assumed to have dimension $d = 2$ or 3.

Łojasiewicz-Simon for Yang-Mills connections V

Theorem 2.2 (Łojasiewicz-Simon inequality for the Yang-Mills functional)

(F., [21, Theorem 22.8]; (F. and Maridakis, [25, Theorem 2]) Let (X, g) be a closed, Riemannian, smooth manifold of dimension d, and G be a compact Lie group, and P be a smooth principal G-bundle over X. Let A_1 be a connection of class C^∞ on P, and A_∞ a Yang-Mills connection for g of class $W^{1,q}$, with $q \in [2, \infty)$ obeying $q > d/2$. If $p \in [2, \infty)$ obeys $d/2 \leq p \leq q$ and, in addition $p \geq 4d/(d+4)$ for $d = 2, 3$, and $p' \in (1, \infty)$ is the dual exponent defined by $1/p + 1/p' = 1$,

Łojasiewicz-Simon for Yang-Mills connections VI

Theorem 2.2 (Łojasiewicz-Simon inequality for the Yang-Mills functional)

then the gradient map,

$$\mathscr{E}_g' : W^{1,p}_{A_1}(X;\Lambda^1 \otimes \operatorname{ad}P) \to W^{-1,p'}_{A_1}(X;\Lambda^1 \otimes \operatorname{ad}P),$$

is real analytic and there are positive constants $Z \in [1,\infty)$, $\sigma \in (0,1]$, and $\theta \in [1/2,1)$, depending on A_1, A_∞,, g, G, p, and q with the following significance. If A is a $W^{1,q}$ connection on P obeying

$$\|A - A_\infty\|_{W^{1,p}_{A_1}(X)} < \sigma, \tag{7}$$

then the Yang-Mills energy functional (5) *obeys*

$$\|\mathscr{E}_g'(A)\|_{W^{-1,p'}_{A_1}(X)} \geq Z|\mathscr{E}_g(A) - \mathscr{E}_g(A_\infty)|^\theta. \tag{8}$$

The proof of Theorem 2.2 has two main ingredients:

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Łojasiewicz-Simon for Yang-Mills connections VII

Prove a slice theorem that says if

$$\|A - A_\infty\|_{W^{1,p}_{A_1}(X)},$$

then there is a $W^{2,q}$ gauge transformation $u \in \mathrm{Aut}(P)$ such that

$$d^*_{A_\infty}(u(A) - A_\infty) = 0$$

and

$$\|u(A) - A_\infty\|_{W^{1,p}_{A_1}(X)} \leq C\|A - A_\infty\|_{W^{1,p}_{A_1}(X)}.$$

This standard when $p > d/2$, but difficult when $p = d/2$ (the case we want when $d = 4$) because $W^{2,\frac{d}{2}}(X) \not\subset C(X)$ and is not a Banach algebra.

Łojasiewicz-Simon for Yang-Mills connections VIII

Apply the abstract Łojasiewicz-Simon gradient inequality (Theorem 1.3) to the Yang-Mills energy functional, noting that the restriction of the Hessian $\mathscr{E}''(A_\infty)$ to the slice $\operatorname{Ker} d_{A_\infty}^* \cap W_{A_1}^{1,p}(X; \Lambda^1 \otimes \operatorname{ad} P)$ takes the form

$$\mathscr{E}''(A_\infty) = d_{A_\infty}^* d_{A_\infty} + d_{A_\infty} d_{A_\infty}^* + \text{lower-order terms},$$

and can be shown to be Fredholm with index zero, as required.

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Remark 2.3 (Quasi-conformally invariant norms)

Theorem 2.2 is especially interesting when $d = 4$ and one has energy bubbling. When $p = 2$, the norm on $W_A^{1,p}(X; \Lambda^1 \otimes \operatorname{ad} P)$,

$$\|a\|_{W_{A,g}^{1,p}(X)} = \left(\int_X \left(|\nabla_A^g a|_g^p + |a|_g^p \right) \, d\operatorname{vol}_g \right)^{1/p},$$

is quasi-invariant with respect to conformal diffeomorphisms of $(S^4, g_{\mathrm{round}})$ in the following sense: There exists $C \in [1, \infty)$ such that

$$C^{-1} \|a\|_{W_A^{1,2}(S^4)} \leq \|h^* a\|_{W_{h^*A}^{1,2}(S^4)} \leq C \|a\|_{W_A^{1,2}(S^4)},$$

for all $h \in \operatorname{Conf}(S^4, g_{\mathrm{round}})$, $A \in \mathscr{A}(S^4; P)$, and $a \in W_A^{1,2}(S^4; \Lambda^1 \otimes \operatorname{ad} P)$.

Łojasiewicz-Simon for Yang-Mills connections X

Remark 2.3 (Quasi-conformally invariant norms)

On the other hand, $W_A^{1,2}(X; \Lambda^1 \otimes \mathrm{ad}P)$ is a convenient Sobolev norm to use when possible for connections with good control of energy.

Remark 2.4 (Łojasiewicz-Simon gradient inequalities for coupled Yang-Mills L^2-energy functionals)

Versions of Theorem 2.2 hold for coupled Yang-Mills L^2-energy functionals for pairs, (A, Φ), consisting of a connection A on P and a section Φ of a vector bundle (F. and Maridakis [24]).

For examples, see Bradlow [6, 7], Bradlow and García-Prada [8], Hitchin [31], F. and Leness [23], Hong [33], Li and Zhang [38], Jost, Peng, and Wang [35], Parker in [51, Section 2], and Simpson [62].

Discreteness of the energy spectrum for Yang-Mills connections over four-dimensional manifolds

Discreteness of energies for Yang-Mills connections I

Theorem 2.5 (Discreteness of the energy spectrum for Yang-Mills connections over four-dimensional manifolds)

(See F. [20, Theorem 1]) Let G be a compact Lie group and P be a smooth principal G-bundle over a closed, four-dimensional, oriented, smooth manifold, X, endowed with a smooth Riemannian metric, g. Then the subset of critical values of the L^2-energy functional, $\mathscr{E}_g : \mathscr{A}(P) \to [0, \infty)$, is closed and discrete, depending at most on g, G, and the homotopy equivalence class, $[P]$. In particular, if $\{c_i\}_{i\in\mathbb{N}} \subset [0, \infty)$ denotes the strictly increasing sequence of critical values of $\mathscr{E}_g$ and A is a g-Yang-Mills connection on P with

$$c_i \leq \mathscr{E}_g(A) < c_{i+1}, \tag{9}$$

for some $i \geq 0$, then $\mathscr{E}_g(A) = c_i$.

Discreteness of energies for Yang-Mills connections II

Theorem 2.5 generalizes *gap theorems* for Yang-Mills connections with energies suitably close to the *ground state* due to Bourguignon, Lawson, and Simons [4, 5], Min-Oo [46], Nakajima [48], Parker [51], and others.

Previous gap theorems required some positivity hypothesis on the Riemann curvature tensor for g.

Discreteness of energies for Yang-Mills connections III

Remark 2.6 (Discreteness of critical values of coupled Yang-Mills L^2-energy functionals)

A version of Theorem 2.5 should hold more generally for solutions to coupled Yang-Mills equations for pairs, (A, Φ), consisting of a connection A on P and a section Φ of an associated vector bundle, provided one has a Uhlenbeck (more specifically, bubble-tree) compactness result for the space of solutions modulo gauge transformations.

Theorem 2.5 was proved by Råde [54] when X has dimension 2 or 3.

Energy bubbling for harmonic maps and Yang-Mills connections

Energy bubbling for maps and connections I

Four is the critical dimension for the Yang-Mills equation since the Yang-Mills L^2 energy functional (5),

$$\mathscr{E}_g(A) = \frac{1}{2}\int_X |F_A|^2 \, d\,\mathrm{vol}_g,$$

is invariant with respect to conformal changes in the Riemannian metric g on X when $\dim X = 4$, leading to the phenomenon of energy bubbling or concentration, first analyzed by Uhlenbeck [68] and later Taubes [65, 64].

Energy bubbling for maps and connections II

Similarly, two is the critical dimension for harmonic map equation for maps of Riemannian manifolds, $f : (M, g) \to (N, h)$, since the harmonic map energy functional (10),

$$\mathscr{E}_{g,h}(f) = \frac{1}{2} \int_M |df|_{g,h}^2 \, d\,\mathrm{vol}_g,$$

is invariant with respect to conformal changes in the Riemannian metric g on M when $\dim M = 2$.

Energy bubbling was first discovered by Sacks and Uhlenbeck in the context of harmonic maps from Riemann surfaces into Riemannian manifolds [55, 56].

Energy bubbling for maps and connections III

It is this highly *non-compact* phenomenon of energy bubbling occurring in critical dimensions that makes it difficult to answer questions such as

- Global existence or convergence of gradient flows,
- Discreteness of energies,

for these energy functionals.

However, in the context of proving discreteness of the energy spectra for Yang-Mills connections or harmonic maps, this non-compactness can be partly 'tamed' by the use of bubble-tree compactifications for their moduli spaces, due to Parker and Wolfson [52, 53] (for harmonic maps and pseudoholomorphic curves) and Taubes [65] and F. [22] (for Yang-Mills connections).

Energy bubbling for maps and connections IV

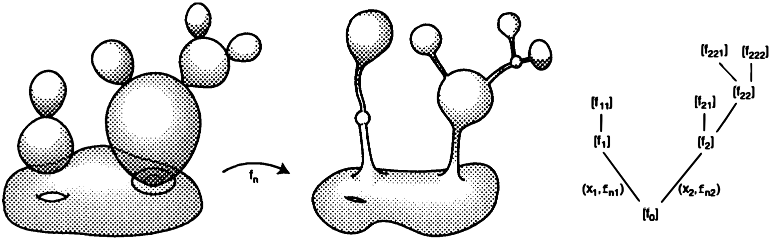

Figure: Formation of bubble trees via iterated conformal blow-ups

Energy bubbling for maps and connections V

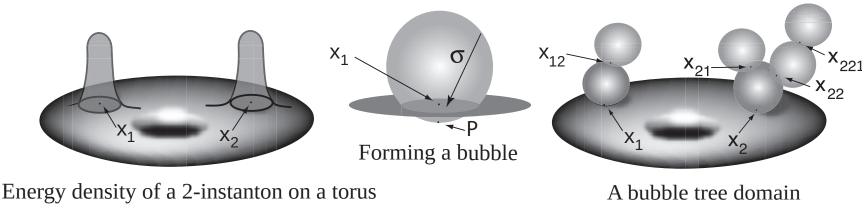

Figure: Summary of the bubble-tree convergence process

Łojasiewicz-Simon gradient inequalities for the harmonic map energy functional and discreteness of energies for harmonic maps

Łojasiewicz-Simon inequalities for harmonic maps I

Next, we describe a consequence of Theorem 1.3 for the harmonic map L^2-energy functional.

Definition 3.1 (Harmonic map energy functional)

Let (M, g) and (N, h) be a pair of closed, Riemannian, smooth manifolds. One defines the *harmonic map L^2-energy functional* by

$$\mathscr{E}_{g,h}(f) := \frac{1}{2}\int_M |df|_{g,h}^2 \, d\,\mathrm{vol}_g, \tag{10}$$

for smooth maps, $f : M \to N$, where $df : TM \to TN$ is the differential map.

To define the gradient of the energy functional $\mathscr{E}_{g,h}$ in (10) with respect to the L^2 metric on $C^\infty(M; N)$, we first choose an

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Łojasiewicz-Simon inequalities for harmonic maps II

isometric embedding, $(N, h) \hookrightarrow \mathbb{R}^n$ for a sufficiently large n (see Nash [49]), and recall that

$$\left(\mathscr{E}'_{g,h}(f), u\right)_{L^2(M)} := \frac{d}{dt}\mathscr{E}_{g,h}(f + tu)\Big|_{t=0} = \left(\Delta_g f - A_h(df, df), u\right)_{L^2(M)},$$

for all $u \in C^\infty(M; N)$, that is,

$$\mathscr{E}'_{g,h}(f) = \Delta_g f - A_h(df, df).$$

Here, A_h denotes the second fundamental form of the isometric embedding, $(N, h) \hookrightarrow \mathbb{R}^n$ and

$$\Delta_g := -\operatorname{div}_g \operatorname{grad}_g f = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^\beta}\left(\sqrt{\det g}\,\frac{\partial f}{\partial x^\alpha}\right)$$

Łojasiewicz-Simon inequalities for harmonic maps III

denotes the Laplace-Beltrami operator for (M, g).

One says that a smooth map $f : M \to N$ is *harmonic* if it is a critical point of the L^2 energy functional (10), that is

$$\mathscr{E}'_{g,h}(f) = \Delta_g f - A_h(df, df) = 0.$$

A choice of isometric embedding $(N, h) \hookrightarrow \mathbb{R}^n$ may be used to define Sobolev norms of $f \in C^\infty(M; N)$ via the inclusion $C^\infty(M; N) \subset C^\infty(M; \mathbb{R}^n)$.

If (N, h) is real analytic, then the isometric embedding $(N, h) \hookrightarrow \mathbb{R}^n$ may also be chosen to be analytic by the analytic isometric embedding theorem (see Nash [50] or Greene and Jacobowitz [27]).

Łojasiewicz-Simon inequalities for harmonic maps IV

As an application of Theorem 1.3, we obtain the

Theorem 3.2 (Analyticity and Łojasiewicz-Simon gradient inequality for the harmonic map energy functional)

(F. and Maridakis [24]) Let $d \geq 2$, $k \geq 1$, and $p \in [1, \infty)$ obey

$$kp > d \quad \text{or} \quad k = d \text{ and } p = 1.$$

Let (M, g) and (N, h) be closed, Riemannian, smooth manifolds, with M of dimension d. If (N, h) is real analytic (respectively, C^∞) and $f \in W^{k,p}(M; N)$, then the gradient map,

$$\mathscr{E}'_{g,h} : W^{k,p}(M; f^*TN) \to W^{-k,p'}(M; f^*TN),$$

is a real analytic (respectively, C^∞) map of Banach spaces.

Łojasiewicz-Simon inequalities for harmonic maps V

Theorem 3.2 (Analyticity and Łojasiewicz-Simon gradient inequality for the harmonic map energy functional)

If $f_\infty \in W^{k,p}(M;N)$ is a harmonic map, then there are positive constants $Z \in [1,\infty)$, and $\sigma \in (0,1]$, and $\theta \in [1/2,1)$, depending on f_∞, g, h, k, p, with the following significance. If $f \in W^{k,p}(M;N)$ obeys

$$\|f - f_\infty\|_{W^{k,p}(M)} < \sigma, \tag{11}$$

then the harmonic map energy functional (10) obeys,

$$\|\mathscr{E}'(f)\|_{W^{-k,p'}(M)} \geq Z|\mathscr{E}(f) - \mathscr{E}(f_\infty)|^\theta. \tag{12}$$

Theorem 3.2 generalizes earlier results due to Kwon [37, Theorem 4.2], Liu and Yang [41, Lemma 3.3], Simon [61, Theorem 3], and Topping [67, Lemma 1].

Łojasiewicz-Simon inequalities for harmonic maps VI

Remark 3.3 (Conformally invariant norms)

Again, Theorem 3.2 is especially interesting when $d = 2$ and one has energy bubbling. When $p = 1$, the norm on $W^{2,1}(M;\mathbb{R}^n)$,

$$\|f\|_{\bar{W}^{2,1}(M,g)} := \|\Delta_g f\|_{L^1(M,g)} + \|df\|_{L^2(M,g)} + \|f\|_{L^\infty(M)},$$

for $f \in C^\infty(M;N)$ with $N \subset \mathbb{R}^n$, is invariant with respect to conformal changes of the Riemannian metric g on M:

$$\|f\|_{\bar{W}^{2,1}(M,e^{2w}g)} = \|f\|_{\bar{W}^{2,1}(M,g)}.$$

The Sobolev Embedding Theorem implies that the preceding norm is equivalent to the standard norm on $W^{2,1}(M;\mathbb{R}^n)$,

$$\|f\|_{W^{2,1}(M,g)} := \|\nabla^g df\|_{L^1(M,g)} + \|df\|_{L^1(M,g)} + \|f\|_{L^1(M,g)}.$$

Discreteness of the energy spectrum for harmonic maps from a Riemann surface into a closed Riemannian manifold

Discrete energy spectrum for harmonic maps I

We have the following analogue of our Theorem 2.5, a discrete energy result for Yang-Mills connections over closed, four-dimensional, Riemannian, smooth manifolds.

Discrete energy spectrum for harmonic maps II

Theorem 3.4 (Discreteness for critical values of the L^2 energies of harmonic maps from a Riemann surface to a closed, real analytic, Riemannian manifold)

(F. [19, Theorem 1]) Let (M, g) be a closed Riemann surface and (N, h) be a closed, real analytic, Riemannian manifold. Then the subspace in $\mathbb{R}$ of critical values of the L^2-energy functional $\mathscr{E}_{g,h} : W^{1,2}(M; N) \to [0, \infty)$ is closed and discrete. In particular, if $\{c_i\}_{i\in\mathbb{N}} \subset [0, \infty)$ denotes the strictly increasing sequence of critical values of $\mathscr{E}_{g,h}$ and $f : M \to N$ is a harmonic map with

$$c_i \leq \mathscr{E}_{g,h}(f) < c_{i+1},$$

for some $i \geq 0$, then $\mathscr{E}_{g,h}(f) = c_i$.

Discrete energy spectrum for harmonic maps III

Theorem 3.4 may be viewed, in part, as a generalization of the following energy gap result due to Sacks and Uhlenbeck (1981), who do not require that the target manifold be real analytic.

Theorem 3.5 (Energy gap near the constant map)

[55, Theorem 3.3] Let (M, g) be a closed Riemann surface and (N, h) be a closed, Riemannian, smooth manifold. There exists a constant $\varepsilon > 0$ such that if $f \in C^\infty(M; N)$ is a harmonic map and $\mathscr{E}_{g,h}(f) < \varepsilon$, then f is a constant map and $\mathscr{E}_{g,h}(f) = 0$.

The proof of Theorem 3.4 follows by adapting (and simplifying) our proof of discreteness of energies for Yang-Mills connections (Theorem 2.5).

Discrete energy spectrum for harmonic maps IV

Theorem 3.4 gives a positive answer to a long-standing conjecture on the discreteness of the L^2 energies of harmonic maps from the sphere into a closed, real analytic, Riemannian manifold posed variously by Adachi and Sunada [1], Eells and Sampson [18], Hartman [30], and Lin [40, Conjecture 5.7], Simon, and Valli [71, Corollary 8].

Li and Wang (2015) provide the following counterexample when the hypothesis on *analyticity* of the metric h on N is relaxed.

Discrete energy spectrum for harmonic maps V

Example 3.6 (Non-discreteness of energy spectrum for harmonic maps from S^2 into a smooth Riemannian manifold with boundary)

(See Li and Wang [39, Section 4]) There exists a smooth Riemannian metric h on $N = S^2 \times (-1, 1)$ such that energies of harmonic maps from $(S^2, g_{\mathrm{round}})$ to (N, h) have an accumulation point at energy level 4π, where, g_{round} denotes the standard round metric of radius one.

Schmidt and Sutton [58] prove similar results for discreteness for lengths of closed geodesics in a closed, real analytic, Riemannian manifold and give counterexamples when the hypothesis on *analyticity* is relaxed.

Global existence and convergence of smooth solutions to Yang-Mills gradient flow on compact four-dimensional manifolds

Yang-Mills gradient flow over four-manifolds I

The following conjecture essentially goes back to Atiyah and Bott [2], Sedlacek [59], Taubes [64, 65, 66], and Uhlenbeck [68, 69]; it appears explicitly in an article by Schlatter, Struwe, and Tahvildar-Zadeh [57, p. 118] and elsewhere.

Yang-Mills gradient flow over four-manifolds II

Conjecture 4.1 (Global existence and convergence of Yang-Mills gradient flow over closed four-dimensional manifolds)

Let G be a compact Lie group and P a principal G-bundle over a closed, connected, four-dimensional, smooth manifold, X, with Riemannian metric, g. If A_0 is a smooth connection on P, then there is a smooth solution, $A(t)$ for $t \in [0, \infty)$, to the gradient flow,

$$\frac{\partial A}{\partial t} = -d_{A(t)}^{*_g} F_{A(t)}, \tag{13a}$$

$$A(0) = A_0, \tag{13b}$$

for the Yang-Mills energy functional (5) with respect to the L^2 Riemannian metric on the affine space of connections on P. Moreover, as $t \to \infty$, the flow, $A(t)$, converges to a smooth Yang-Mills connection, A_∞, on P.

Evidence for and against long-time existence and convergence of Yang-Mills gradient flow

Evidence for long-time existence and convergence I

Struwe [63, Theorem 2.3] and Kozono, Maeda, and Naito [36] established existence of solutions to Yang-Mills gradient flow (13), up to a finite time $T_1 > 0$ characterized by energy bubbling singularities.

Donaldson's proof of Conjecture 4.1 in the case of a Hermitian, rank-two vector bundle, E, over a Kähler surface, X, makes essential use of the Kähler structure of X ($\implies$ *global existence*) and stability of E ($\implies$ *convergence*) [16, Theorem 1], [17, Theorem 6.1.5].

Donaldson's existence results (for Hermitian Yang-Mills connections) were extended by Uhlenbeck and Yau using purely elliptic PDE techniques [70].

Evidence for long-time existence and convergence II

It follows from results of Daskalopoulos and Wentworth in [13, 14] that one can construct examples of *unstable* holomorphic vector bundles, E, and initial connections, A_0, such that the Yang-Mills gradient flow necessarily develops bubble singularities at $T = \infty$.

When X has dimension $d = 2$ or 3, Råde [54, Theorems 1, 1′, and 2] has shown that Conjecture 4.1 is *true*, as has G. Daskalopoulos [12] when X has dimension two.

Naito [47, Theorem 1.3] has shown that Conjecture 4.1 is *false* when X has dimension $d \geq 5$, even when $X = S^d$: If $P \to S^d$ is a non-trivial principal G-bundle, then there is a constant $\varepsilon_0 > 0$ such that for any initial connection, A_0, with $\|F_{A_0}\|_{L^2(S^d)} < \varepsilon_0$, the flow $A(t)$ blows up in finite time.

Global existence and convergence of Yang-Mills gradient flow near a local minimum

Global existence and convergence of Yang-Mills flow I

Theorem 4.2 (Global existence and convergence of Yang-Mills gradient flow near a local minimum)

(See F. [21, Theorem 1]) Let G be a compact Lie group and P a principal G-bundle over a closed, connected, oriented, smooth manifold, X, of dimension $d \geq 2$ and with Riemannian metric, g. Let A_1 and $A_{\min}$ be C^∞ connections on P, with $A_{\min}$ being a local minimum, and let $p = 2$ for $2 \leq d \leq 4$ and $p > d/2$ for $d \geq 5$. Then there are constants $c \in [1, \infty)$, and $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$, depending on $(A_1, A_{\min}, g, p)$, with the following significance.

Global existence and convergence of Yang-Mills flow II

Theorem 4.2 (Global existence and convergence of Yang-Mills gradient flow near a local minimum)

1. *Global existence: There is a constant* $\varepsilon \in (0, \sigma/4)$*, depending on* $(A_1, A_{\min}, g, p)$*, with the following significance. If* A_0 *is a* C^∞ *connection on* P *such that*

$$\|A_0 - A_{\min}\|_{W^{1,p}_{A_1}(X)} < \varepsilon,$$

then there exists a solution, $A(t) = A_0 + a(t)$ *for* $t \in [0, \infty)$*, with*

$$a \in C^\infty([0,\infty) \times X; \Lambda^1 \otimes \operatorname{ad} P),$$

to the Yang-Mills gradient flow (13) *with initial data,* $A(0) = A_0$*, and*

$$\|A(t) - A_{\min}\|_{W^{1,p}_{A_1}(X)} < \sigma/2, \quad \forall\, t \in [0, \infty).$$

Global existence and convergence of Yang-Mills flow III

Theorem 4.2 (Global existence and convergence of Yang-Mills gradient flow near a local minimum)

2. *Dependence on initial data: The solution, $A(t)$ for $t \in [0, \infty)$, varies continuously with respect to A_0 in the $C_{\mathrm{loc}}([0, \infty); W^{1,p}_{A_1}(X; \Lambda^1 \otimes \operatorname{ad} P))$ topology and, more generally, smoothly for all non-negative integers, k, l, in the $C^l_{\mathrm{loc}}([0, \infty); H^k_{A_1}(X; \Lambda^1 \otimes \operatorname{ad} P))$ topology.*
3. *Convergence: As $t \to \infty$, the flow, $A(t)$, converges strongly with respect to the norm on $W^{1,p}_{A_1}(X; \Lambda^1 \otimes \operatorname{ad} P)$ to a Yang-Mills connection, A_∞, of class C^∞ on P, and the gradient-flow line has finite length in the sense that*

$$\int_0^\infty \left\| \frac{\partial A}{\partial t} \right\|_{W^{1,p}_{A_1}(X)} dt < \infty.$$

If $A_{\min}$ is a cluster point of the orbit, $O(A) = \{A(t) : t \geq 0\}$, then $A_\infty = A_{\min}$.

Global existence and convergence of Yang-Mills flow IV

Theorem 4.2 (Global existence and convergence of Yang-Mills gradient flow near a local minimum)

4. *Convergence rate: For all $t \geq 1$,*

$$\|A(t) - A_\infty\|_{W^{1,p}_{A_1}(X)} \leq \begin{cases} \dfrac{1}{c(1-\theta)}\left(c^2(2\theta-1)(t-1) + (\mathscr{E}(A_0) - \mathscr{E}(A_\infty))^{1-2\theta}\right)^{-(1-\theta)/(2\theta-1)}, \\ \dfrac{2}{c}\sqrt{\mathscr{E}(A_0) - \mathscr{E}(A_\infty)}\exp(-c^2(t-1)/2), \end{cases}$$

for $1/2 < \theta < 1$ and $\theta = 1/2$, respectively.

5. *Stability: As an equilibrium of the Yang-Mills gradient flow (13), the point A_∞ is Lyapunov stable; if A_∞ is isolated or a cluster point of the orbit $O(A)$, then A_∞ is uniformly asymptotically stable.*

Global existence and convergence of Yang-Mills flow V

Theorem 4.2 (Global existence and convergence of Yang-Mills gradient flow near a local minimum)

6. *Uniqueness: Any two solutions are equivalent modulo a path of gauge transformations,*

$$u \in C^\infty([0,\infty) \times X; \mathrm{Ad}P), \quad u(0) = \mathrm{id}_P.$$

Global existence and convergence of Yang-Mills flow VI

Remark 4.3

1. Theorem 4.2 was proved for X of dimension 2 or 3 by Råde [54] for initial data A_0 of arbitrary energy.

2. When X has dimension 4, the hypothesis that A_0 be $W^{1,2}$-norm close to a local minimum (for example, an anti-self-dual Yang-Mills connection) can be relaxed to the energy condition

$$\|F_A^{+,g}\|_{L^2(X,g)} < \varepsilon,$$

in the presence of various combinations of additional hypotheses on G, the topology of P, or the Riemannian metric, g, that guarantee *existence of anti-self-dual Yang-Mills connections* on P.

Global existence and convergence of Yang-Mills flow VII

Remark 4.3

3. Global existence in the case of X of dimension 4 and initial data A_0 of arbitrary energy (not close to a local minimum $A_{\min}$) remains open.

Main ideas in the proof of global existence and convergence

1. Short-time well-posedness for Yang-Mills heat and gradient-flow equations;
2. Łojasiewicz-Simon gradient inequality;
3. Growth estimates, existence, convergence, and stability for solutions to abstract gradient systems;
4. *A priori* estimates for $\int_\delta^T \|\dot{A}(t)\|_{W_{A_1}^{1,2}(X)}\,dt$ in terms of $\int_0^T \|\dot{A}(t)\|_{L^2(X)}\,dt$.

We shall discuss a few of these items in the following slides.

Short-time well-posedness for the Yang-Mills heat and gradient-flow equations

Short-time well-posedness for the Yang-Mills equations I

There are essentially two methods:

- Analytic semi-group theory for positive sectorial operators on Banach spaces, following ideas of Kozono, Maeda, and Naito [36] and the theory of non-linear evolution equations in Banach spaces, Sell and You [60];
- Contraction-mapping, following Struwe [63], based on existence of strong solutions,

$$a \in L^2(0, T; H^2_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P)) \cap H^1(0, T; L^2(X; \Lambda^1 \otimes \mathrm{ad}P)) \cap L^\infty(0, T; H^1_{A_1}(X; \Lambda^1 \otimes \mathrm{ad}P)),$$

to the linear heat equation, $\partial_t a + \Delta_{A_1} a = f$ in $\Omega^1(X; \mathrm{ad}P)$, where A_1 is a fixed, C^∞ reference connection on P.

Short-time well-posedness for the Yang-Mills equations II

The methods give different (non-overlapping) initial possibilities for the regularity of the initial data, A_0, and the regularity of the solution, $A(t) = A_1 + a(t)$ on a short time interval, $[0, \tau)$.

All of these methods rely on Donaldson's version [16], [17, Equation (6.3.3)] of the DeTurck Trick for Ricci flow [15] to convert the Yang-Mills *gradient flow* equation in $\Omega^1(X; \Lambda^1 \otimes \operatorname{ad} P)$,

$$\frac{\partial a}{\partial t} + d^*_{A(t)} F_{A(t)} = 0, \quad \text{for } t > 0, \tag{14}$$

with initial data $a(0) = a_0 \in \Omega^1(X; \operatorname{ad} P)$, to the Yang-Mills *heat* equation, a quasi-linear parabolic equation,

$$\frac{\partial a}{\partial t} + d^*_{A(t)} F_{A(t)} + d_{A(t)} d^*_{A(t)} a(t) = 0, \quad \text{for } t > 0, \tag{15}$$

Short-time well-posedness for the Yang-Mills equations III

For $a(t)$ solving the (*parabolic*) Yang-Mills heat equation (15), one defines a family of gauge transformations, $u(t) \in \operatorname{Aut} P$, by solving

$$u^{-1}(t) \circ \frac{\partial u(t)}{\partial t} = -d_{A(t)}^* a(t), \quad \forall\, t \in (0, \infty), \quad u(0) = \mathrm{id}_P. \quad (16)$$

and finding that $\widetilde{A}(t) = u(t)^* A(t)$ solves the Yang-Mills gradient flow equation (14).

Short-time well-posedness for the Yang-Mills equations IV

Remark 4.4 (Nash-Moser implicit function theorem)

As is well-known, Hamilton applied the Nash-Moser implicit function theorem to prove short-time well-posedness for the Ricci flow equation [29], before his approach was superseded by the DeTurck Trick.

It is likely that the Nash-Moser implicit function theorem could also be used to prove short-time well-posedness for the Yang-Mills gradient flow equation.

Growth estimates, global existence, convergence, and stability for abstract gradient systems

Global existence and convergence for gradient systems I

The collection of results we shall now describe comprise a 'toolkit' that may be directly and easily applied to analyze a wide range of gradient systems in geometric analysis, including

- harmonic map gradient flow,
- knot energy flow,
- mean curvature flow,
- Ricci flow,
- Yamabe flow, and
- (coupled and pure) Yang-Mills flow,

Global existence and convergence for gradient systems II

as well as numerous other gradient systems in applied mathematics and mathematical physics.

Hypothesis 4.5 (*A priori* interior estimate for a trajectory)

Let $\mathscr{X}$ be a Banach space that is continuously embedded in a Hilbert space $\mathscr{H}$. If $\delta \in (0, \infty)$ is a constant, then there is a constant $C_1 = C_1(\delta) \in [1, \infty)$ with the following significance. If $S, T \in \mathbb{R}$ are constants obeying $S + \delta \leq T$ and $u \in C^\infty([S, T); \mathscr{X})$, we say that $\dot{u} \in C^\infty([S, T); \mathscr{X})$ obeys an *a priori interior estimate on* $(0, T]$ if

$$\int_{S+\delta}^{T} \|\dot{u}(t)\|_{\mathscr{X}}\, dt \leq C_1 \int_{S}^{T} \|\dot{u}(t)\|_{\mathscr{H}}\, dt. \tag{17}$$

Hypothesis 4.5 is an abstract version of the conclusion of Råde's [54, Lemma 7.3] for Yang-Mills gradient flow over a closed manifold of dimension two or three.

Global existence and convergence for gradient systems III

In applications, $u \in C^\infty([S,T);\mathscr{X})$ in Hypothesis 4.5 will often be a solution to a quasi-linear parabolic partial differential system, from which an *a priori* estimate (17) may be easily deduced.

More generally, Hypothesis 4.5 can be verified for a nonlinear evolution equation on a Banach space $\mathcal{V}$ of the form

$$\frac{du}{dt} + \mathcal{A}u = \mathcal{F}(t, u(t)), \quad t \geq 0, \quad u(0) = u_0, \tag{18}$$

where $\mathcal{A}$ is a positive, sectorial, unbounded operator on a Banach space, $\mathcal{W}$, with domain $\mathcal{V}^2 \subset \mathcal{W}$ and the nonlinearity, $\mathcal{F}$, has suitable properties.

We have the following analogue of Huang [34, Theorems 3.3.3 and 3.3.6] and abstract analogue of Simon [61, Corollary 2].

Global existence and convergence for gradient systems IV

Theorem 4.6 (Convergence of a subsequence implies convergence for a smooth solution to a gradient system)

Let $\mathscr{U}$ be an open subset of a real Banach space, $\mathscr{X}$, that is continuously embedded and dense in a Hilbert space, $\mathscr{H}$. Let $\mathscr{E} : \mathscr{U} \subset \mathscr{X} \to \mathbb{R}$ be an analytic function with gradient map $\mathscr{E}' : \mathscr{U} \subset \mathscr{X} \to \mathscr{H}$. Assume that $\varphi \in \mathscr{U}$ is a critical point of $\mathscr{E}$, that is $\mathscr{E}'(\varphi) = 0$. If $u \in C^\infty([0,\infty); \mathscr{X})$ solves

$$\dot{u}(t) = -\mathscr{E}'(u(t)), \quad t \in (0,\infty), \tag{19}$$

and the orbit $O(u) = \{u(t) : t \geq 0\} \subset \mathscr{X}$ is precompact and φ is a cluster point of $O(u)$, then $u(t)$ converges to φ as $t \to \infty$ in the sense that

$$\lim_{t\to\infty} \|u(t) - \varphi\|_{\mathscr{X}} = 0 \quad \text{and} \quad \int_0^\infty \|\dot{u}\|_{\mathscr{H}}\, dt < \infty.$$

Global existence and convergence for gradient systems V

Theorem 4.6 (Convergence of a subsequence implies convergence for a smooth solution to a gradient system)

Furthermore, if u satisfies Hypothesis 4.5 on $(0, \infty)$, *then*

$$\int_1^\infty \|\dot{u}\|_{\mathscr{X}}\, dt < \infty.$$

We next have the following abstract analogue of Råde's [54, Proposition 7.4], in turn a variant the *Simon Alternative*, namely [61, Theorem 2].

Global existence and convergence for gradient systems VI

Theorem 4.7 (Simon Alternative for convergence of a solution to gradient system)

Let $\mathscr{U}$ be an open subset of a real Banach space, $\mathscr{X}$, that is continuously embedded and dense in a Hilbert space, $\mathscr{H}$. Let $\mathscr{E} : \mathscr{U} \subset \mathscr{X} \to \mathbb{R}$ be an analytic function with gradient map $\mathscr{E}' : \mathscr{U} \subset \mathscr{X} \to \mathscr{H}$. Assume that

1. *$\varphi \in \mathscr{U}$ is a critical point of $\mathscr{E}$, that is $\mathscr{E}'(\varphi) = 0$; and*
2. *Given positive constants b, η, and τ, there is a constant $\delta = \delta(\eta, \tau, b) \in (0, \tau]$ such that if v is a smooth solution to (19) on $[t_0, t_0 + \tau)$ with $t_0 \in \mathbb{R}$ and $\|v(t_0)\|_{\mathscr{X}} \leq b$, then*

$$\sup_{t\in[t_0,t_0+\delta]} \|v(t) - v(t_0)\|_{\mathscr{X}} < \eta. \qquad (20)$$

Global existence and convergence for gradient systems VII

Theorem 4.7 (Simon Alternative for convergence of a solution to gradient system)

If (c, σ, θ) *are the Łojasiewicz-Simon constants for* $(\mathscr{E}, \varphi)$*, there is a constant*

$$\varepsilon = \varepsilon(c, C_1, \delta, \theta, \rho, \sigma, \tau, \varphi) \in (0, \sigma/4)$$

with the following significance. If $u : [0, \infty) \to \mathscr{U}$ *is a smooth solution to* (19) *that satisfies Hypothesis 4.5 on* $(0, \infty)$ *and there is a constant* $T \geq 0$ *such that*

$$\|u(T) - \varphi\|_{\mathscr{X}} < \varepsilon, \tag{21}$$

then either

1. $\mathscr{E}(u(t)) < \mathscr{E}(\varphi)$ *for some* $t > T$*, or*

Global existence and convergence for gradient systems VIII

Theorem 4.7 (Simon Alternative for convergence of a solution to gradient system)

2. *$u(t)$ converges in $\mathscr{X}$ to a limit $u_\infty \in \mathscr{X}$ as $t \to \infty$ in the sense that*

$$\lim_{t\to\infty} \|u(t) - u_\infty\|_{\mathscr{X}} = 0 \quad \text{and} \quad \int_1^\infty \|\dot{u}\|_{\mathscr{X}}\, dt < \infty.$$

If φ is a cluster point of the orbit $O(u) = \{u(t) : t \geq 0\}$, then $u_\infty = \varphi$.

In applications, the short-time estimate (20) for $v \in C^\infty([t_0, t_0 + \tau); \mathscr{X})$ will usually follow from the fact that v is a solution to a quasi-linear parabolic partial differential system, from which (20) may be deduced.

We have the following enhancement of Huang [34, Theorem 3.4.8].

Global existence and convergence for gradient systems IX

Theorem 4.8 (Convergence rate under a Łojasiewicz-Simon gradient inequality)

Let $\mathscr{U}$ be an open subset of a real Banach space, $\mathscr{X}$, that is continuously embedded and dense in a Hilbert space, $\mathscr{H}$. Let $\mathscr{E} : \mathscr{U} \subset \mathscr{X} \to \mathbb{R}$ be an analytic function with gradient map $\mathscr{E}' : \mathscr{U} \subset \mathscr{X} \to \mathscr{H}$. Let $u : [0,\infty) \to \mathscr{X}$ be a smooth solution to the gradient system (19) and assume that $O(u) \subset \mathscr{U}_\sigma \subset \mathscr{U}$, where (c,σ,θ) are the Łojasiewicz-Simon constants for $(\mathscr{E},\varphi)$ and $\mathscr{U}_\sigma := \{x \in \mathscr{X} : \|x-\varphi\|_{\mathscr{X}} < \sigma\}$. Then there exists $u_\infty \in \mathscr{H}$ such that

$$\|u(t) - u_\infty\|_{\mathscr{H}} \leq \Psi(t), \quad t \geq 0, \tag{22}$$

where

$$\Psi(t) := \begin{cases} \dfrac{1}{c(1-\theta)} \left(c^2(2\theta-1)t + (\gamma - a)^{1-2\theta}\right)^{-(1-\theta)/(2\theta-1)}, & 1/2 < \theta < 1, \\ \dfrac{2}{c}\sqrt{\gamma - a}\exp(-c^2 t/2), & \theta = 1/2, \end{cases}$$

Global existence and convergence for gradient systems X

Theorem 4.8 (Convergence rate under a Łojasiewicz-Simon gradient inequality)

and a, γ are constants such that $\gamma > a$ and

$$a \leq \mathscr{E}(v) \leq \gamma, \quad \forall\, v \in \mathscr{U}.$$

If in addition u obeys Hypothesis 4.5, then $u_\infty \in \mathscr{X}$ and

$$\|u(t+1) - u_\infty\|_{\mathscr{X}} \leq 2C_1\Psi(t), \quad t \geq 0, \tag{23}$$

where $C_1 \in [1, \infty)$ is the constant in Hypothesis 4.5 for $\delta = 1$.

One calls a critical point $\varphi \in \mathscr{U}$ of $\mathscr{E}$ a *ground state* if $\mathscr{E}$ attains its minimum value on $\mathscr{U}$ at this point, that is,

$$\mathscr{E}(\varphi) = \inf_{u \in \mathscr{U}} \mathscr{E}(u).$$

Global existence and convergence for gradient systems XI

We have the following analogue of Huang [34, Theorem 5.1.1].

Theorem 4.9 (Existence and convergence of a global solution to a gradient system near a local minimum)

Let $\mathscr{U}$ be an open subset of a real Banach space, $\mathscr{X}$, that is continuously embedded and dense in a Hilbert space, $\mathscr{H}$. Let $\mathscr{E} : \mathscr{U} \subset \mathscr{X} \to \mathbb{R}$ be an analytic function with gradient map $\mathscr{E}' : \mathscr{U} \subset \mathscr{X} \to \mathscr{H}$. Let $\varphi \in \mathscr{U}$ be a ground state of $\mathscr{E}$ on $\mathscr{U}$ and suppose that (c, σ, θ) are the Łojasiewicz-Simon constants for $(\mathscr{E}, \varphi)$. Assume that

1. *For each $u_0 \in \mathscr{U}$, there exists a unique smooth solution to the Cauchy problem (19) with $u(0) = u_0$, on a time interval $[0, \tau)$ for some positive constant, τ;*
2. *Hypothesis 4.5 holds for smooth solutions to the gradient system (19);*

Global existence and convergence for gradient systems XII

Theorem 4.9 (Existence and convergence of a global solution to a gradient system near a local minimum)

3. *Given positive constants b and η, there is a constant $\delta = \delta(\eta, \tau, b) \in (0, \tau]$ such that if v is a smooth solution to the gradient system* (19) *on* $[0, \tau)$ *with* $\|v(0)\|_{\mathscr{X}} \leq b$, *then*

$$\sup_{t \in [0,\delta]} \|v(t) - v(0)\|_{\mathscr{X}} < \eta. \tag{24}$$

Then there is a constant $\varepsilon = \varepsilon(c, C_1, \delta, \theta, \rho, \sigma, \tau, \varphi) \in (0, \sigma/4)$ with the following significance. For each $u_0 \in \mathscr{U}_\varepsilon$, the Cauchy problem (19) *with $u(0) = u_0$ admits a global smooth solution, $u : [0, \infty) \to \mathscr{U}_{\sigma/2}$, that converges to a limit $u_\infty \in \mathscr{X}$ as $t \to \infty$ with respect to the $\mathscr{X}$ norm in the sense that*

$$\lim_{t \to \infty} \|u(t) - u_\infty\|_{\mathscr{X}} = 0 \quad \text{and} \quad \int_1^\infty \|\dot{u}(t)\|_{\mathscr{X}}\, dt < \infty.$$

Global existence and convergence for gradient systems XIII

Finally, we have the following analogue of Huang [34, Theorem 5.1.2].

Theorem 4.10 (Convergence to a critical point and stability of a ground state)

Assume the hypotheses of Theorem 4.9. Then the following hold:

1. *For each $u_0 \in \mathscr{U}_\varepsilon$, the Cauchy problem (19) with $u(0) = u_0$ admits a global smooth solution $u : [0, \infty) \to \mathscr{U}_{\sigma/2}$ that converges in $\mathscr{X}$ as $t \to \infty$ to some critical point $u_\infty \in \mathscr{U}_\sigma$;*
2. *The critical point, u_∞, satisfies $\mathscr{E}(u_\infty) = \mathscr{E}(\varphi)$;*
3. *As an equilibrium of (19), the point φ is Lyapunov stable;*
4. *If φ is isolated or a cluster point of the orbit $O(u)$, then φ is uniformly asymptotically stable.*

Thank you for your attention!

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