Gan-Gross-Prasad conjecture and local relative trace formulas

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G locally compact group and $H \leqslant G$ closed. Assume both are unimodular. Then $X = H \backslash G$ has a G-invt measure.

$$R : G \to U\big(L^2(X)\big) \text{ by right translation}$$

Program :

- Decompose as precisely as possible $L^2(X)$ in terms of $\hat{G} = \{\text{irreducible unitary reps of } G\}/isom$
- Write down a trace formula

Assume G is compact then $L^2(X) \simeq \hat{\bigoplus}_{\pi\in\hat{G}} m(\pi)\pi$, $m(\pi)\in\mathbb{N}$. Let act $f\in C(G)$ on $L^2(X)$ by $R(f)=\int_G f(g)R(g)dg$. Then $R(f)$ is a kernel operator :

$$(R(f)\varphi)(x)=\int_X K_f(x,y)\varphi(y)dy$$

$$K_f(x,y)=\int_H f(x^{-1}hy)dh$$

In many cases (e.g. G is a Lie group and f is smooth), $R(f)$ is of trace class and

$$Trace\ R(f)=\int_X K_f(x,x)dx=\int_{H/conj} I(h,f)dh$$

$I(h,f)$: orbital integral. Using the decomposition of $L^2(X)$ we also get

$$Trace\ R(f)=\sum_{\pi\in\hat{G}} m(\pi)tr\pi(f)$$

Trace formula :

$$\int_{H/conj} I(h,f)dh=\sum_{\pi\in\hat{G}} m(\pi)tr\pi(f)$$

For G noncompact, in general $R(f)$ is not trace class> Arthur's idea is to consider "truncated" trace :

$$J^T(f) = \int_X K_f(x,x)u(x,T)dx$$

where $u(.,T)$ is the characteristic function of a compact set $\Omega_T \subset X$ covering X as $T \to \infty$. Then try to evaluate $J^T(f)$ as $T \to \infty$ in two different ways : geometric and spectral.

Arthur's local trace formula

$F/\mathbb{Q}_p$ fin. ext. and $\mathbb{H}/F$ connected semisimple (e.g. $\mathbb{H} = SL_n, PGL_n, SO(V)\ldots$). Let $H = \mathbb{H}(F)$, it's a loc profinite group.

$$H \hookrightarrow H \times H = G$$

Harish-Chandra : $L^2(H) \simeq \int_{Temp(H)} \tau \otimes \tau^\vee d\mu(\tau)$ where $Temp(H)$ is the subset of tempered representations.

Theorem (Arthur)

For $f \in C_c^\infty(G)$ (loc. constant compactly supported) which is cuspidal, we have

$$\int_{H_{ell}/conj} I(h,f)dh = \sum_{\pi \in T_{ell}(G)} \frac{m(\pi) tr\, \pi(f)}{d(\pi)}$$

Gan-Gross-Prasad case

$[E:F]=2$, W_{n+1} a hermitian space of dim $n+1$ and $W_n \subset W_{n+1}$ a nondegenerate hyperplane.

$$H = U(W_n) \hookrightarrow U(W_n) \times U(W_{n+1}) = G$$

From Harish-Chandra decomposition, we get $L^2(X) \simeq \int_{Temp(G)} m(\pi)\pi d\mu(\pi)$ where $m(\pi) = dim\ Hom_H(\pi, 1)$.

Theorem

For $f \in C_c^\infty(G)$ cuspidal, we have

$$\lim_{s\to 0^+} \int_{\Xi} I(x,f)\Delta(x)^s dx = \sum_{\pi\in T_{ell}(G)} \frac{m(\pi) tr\, \pi(f)}{d(\pi)}$$

Application : The Gan-Gross-Prasad conjecture describing representations $\pi \in Temp(G)$ s.t. $m(\pi) \neq 0$ in terms of their Langlands parameter.