Algorithms for the topology of arithmetic groups and Hecke actions II: Higher skeleta

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Previous work computing modular forms in homology

- Trace formula methods. (Chenevier, Renard, Taïbi, Cohen, Skoruppa, Zagier . . .)
- Modular symbols methods for $GL_2/\mathbb{Q}$. (Cremona, Manin, Stein, . . .)
- Voronoi and sharbly methods. (Ash, Doud, Gunnells, McConnell, Pollack, Top, van Geemen, Voronoi, Yasaki, . . .)
- Fundamental domains. (Greenberg, Voight, Page, Rahm, Sengün, . . .)
- Algebraic modular forms. (Gross, Savin, Lansky, Pollack, Greenberg, Voight, Dembélé, Donnelly, Loeffler, Cunningham, Chenevier, Lannes, Mégarbané, . . .)

Main results

Theorem (L, Page)

There exists a deterministic algorithm, given a congruence arithmetic group Γ *for which* $M = \Gamma\backslash X$ *is compact, which calculates*

- *A simplicial complex* S *having the same homotopy type as* M *and having at most* $O_{\mathrm{dim}}(\mathrm{Vol}(M))$ *simplices*
- *An explicit isomorphism* $\pi_1(S) \to \Gamma$

This algorithm terminates in time $O_{\mathrm{dim}}(\mathrm{Vol}(M)^2)$.

In addition, there is an algorithm which, given a cycle σ *in* $C^\bullet(S)$ *and a Hecke operator* T, *calculates a cycle in* $C^\bullet(S)$ *homologous to* $T\sigma$ *in time* $O_{\mathrm{dim}}(\mathrm{Vol}(M) \cdot \deg(T))$.

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We do not expect compactness to be an essential condition.

One idea

(1) Build a grid of points $F \subset M = \Gamma \backslash X$ for which $B_r(x), x \in F$, cover M.

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Theorem (Borsuk, Cech)

If the balls $B_r(x), x \in F$ are convex and their union covers M, then the nerve of the covering has the homotopy type of M.

The nerve in one interesting example

The arithmetic group Γ:

- $A =$ quaternion algebra over $Q(\sqrt{-2})$ of discriminant $\mathfrak{p}_2\mathfrak{p}_3$, $O =$ maximal order in A.
- $q =$ reduced norm on A.
- $L :=$ trace 0 elements of O.
- $\Gamma :=$ the principal congruence level $\mathfrak{p}_3'$ subgroup of $SO(Q, L)$.

The nerve in one interesting example

Covering and nerve information for $\mathrm{Cech}_{\Gamma\backslash X}(0.65)$:

- 1172s to compute a cover.
- 285s to compute the 1-skeleton of the nerve, i.e. when pairs of balls of radius $0.65/2$ centered at our grid points intersect.
- 2-skeleton of the nerve (3-fold intersections): started earlier this afternoon. Now $\frac{9}{10}$-finished.
- Simplices of each degree:

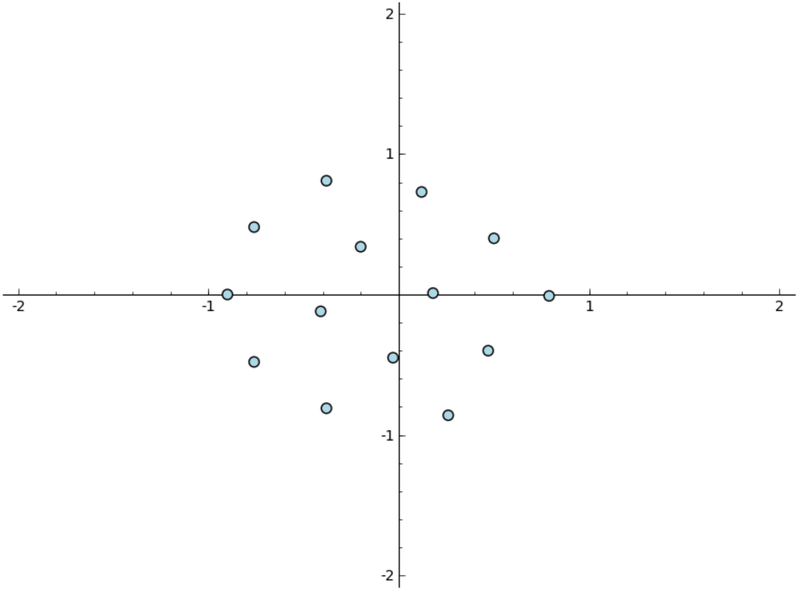
$$(N_0, N_1, N_2, \ldots) = (176, 3135, ?\text{ so far}, \ldots)$$

The nerve in one interesting example

Homology:

- abelianization of presentation for Γ: $\mathbb{Z}/3 \oplus \mathbb{Z}/6 \oplus \mathbb{Z}/6$.
- External consistency: confimed to agree with output from Aurel's Kleinian groups package.
- simplicial computation: not yet finished because computation of 2-skeleton of nerve not yet finished.

One iteration of expansion



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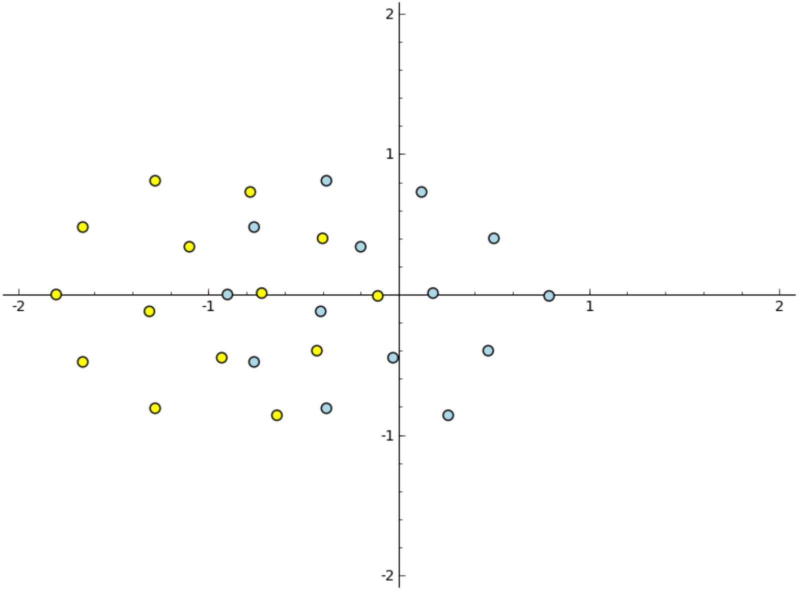

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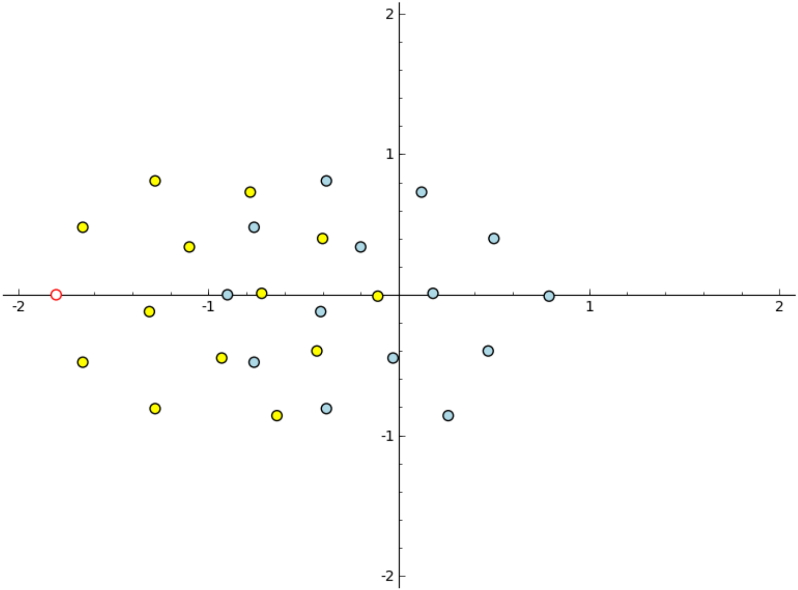

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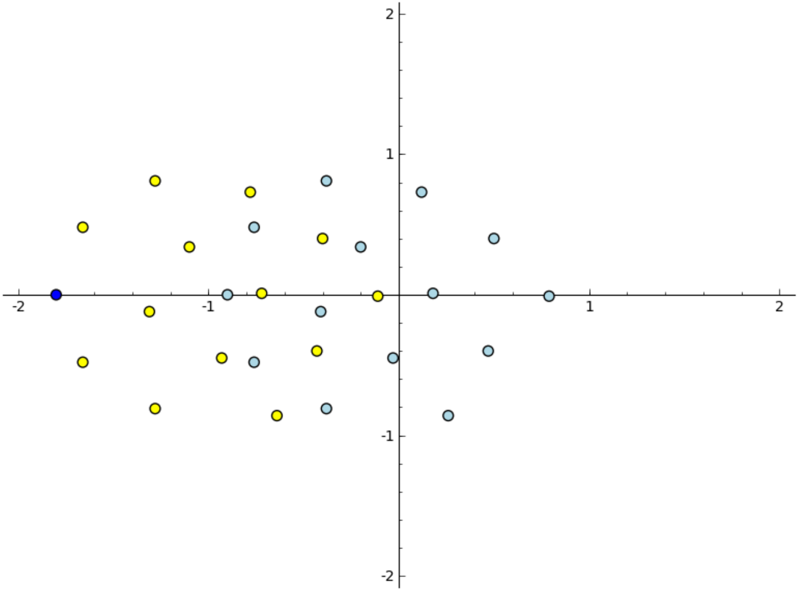

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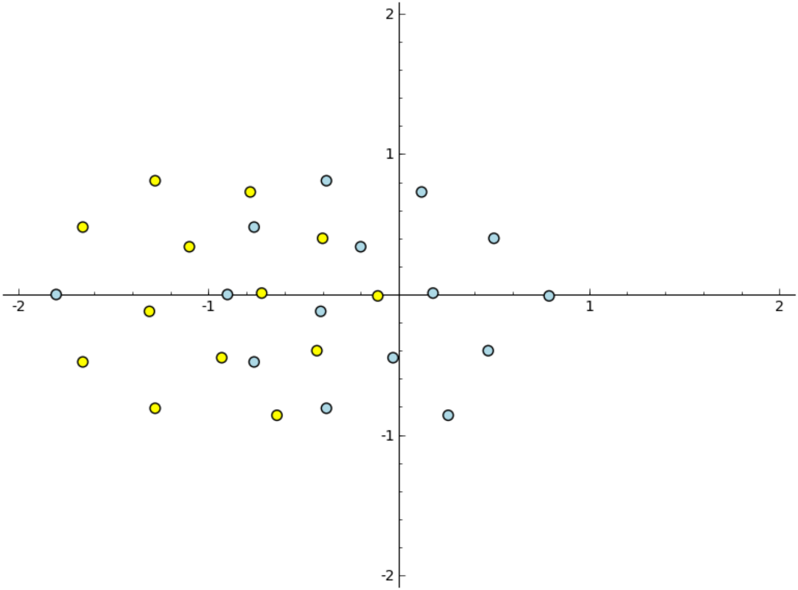

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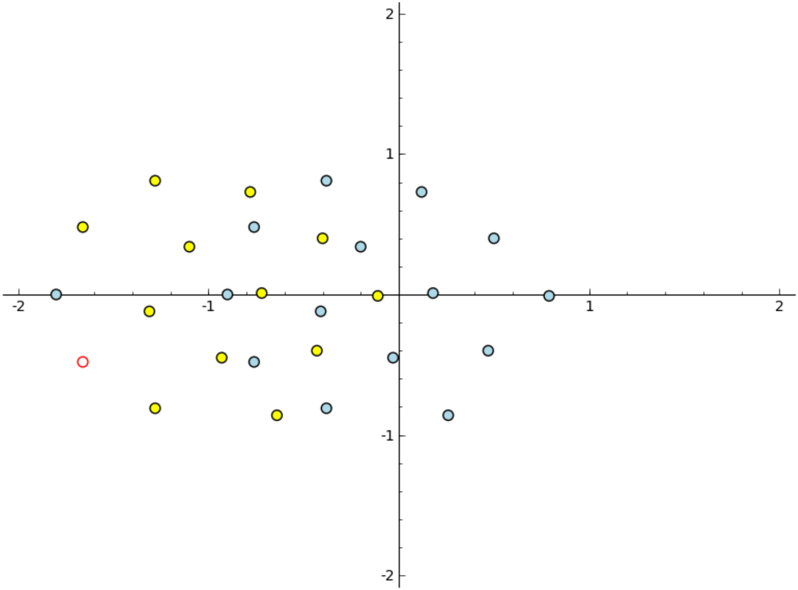

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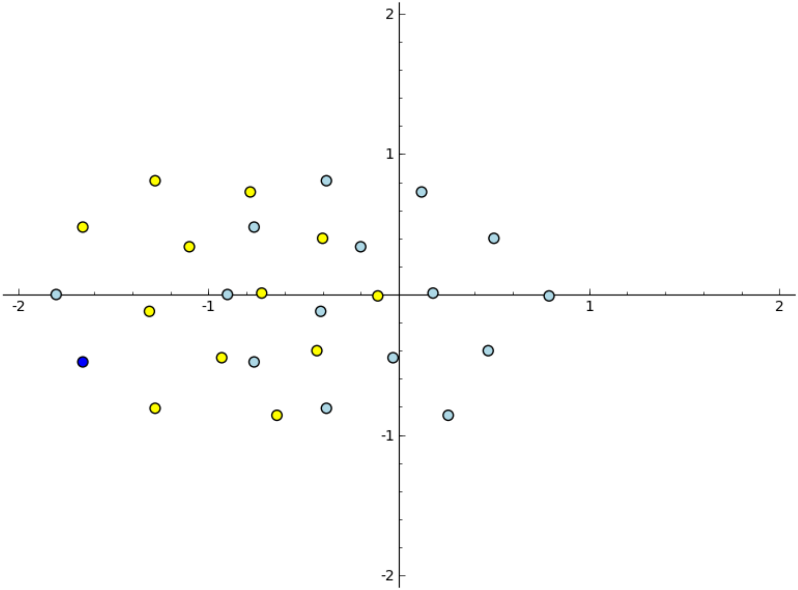

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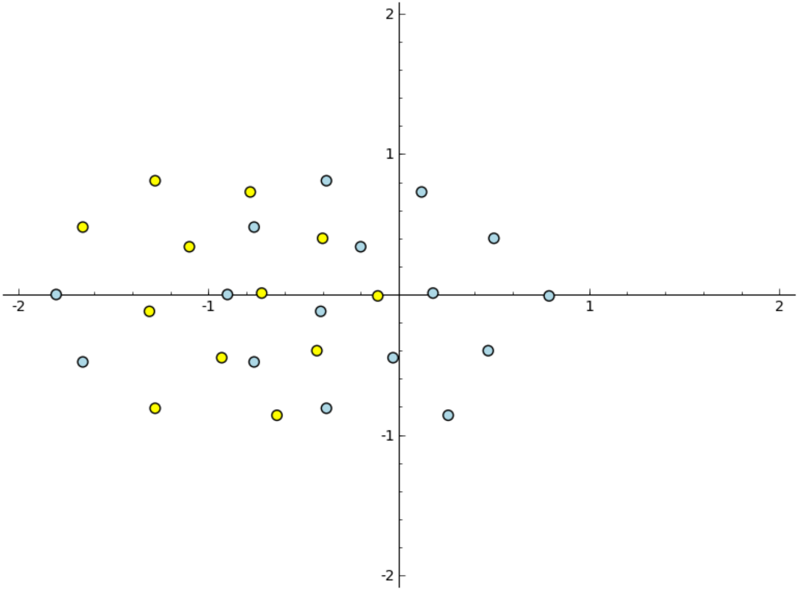

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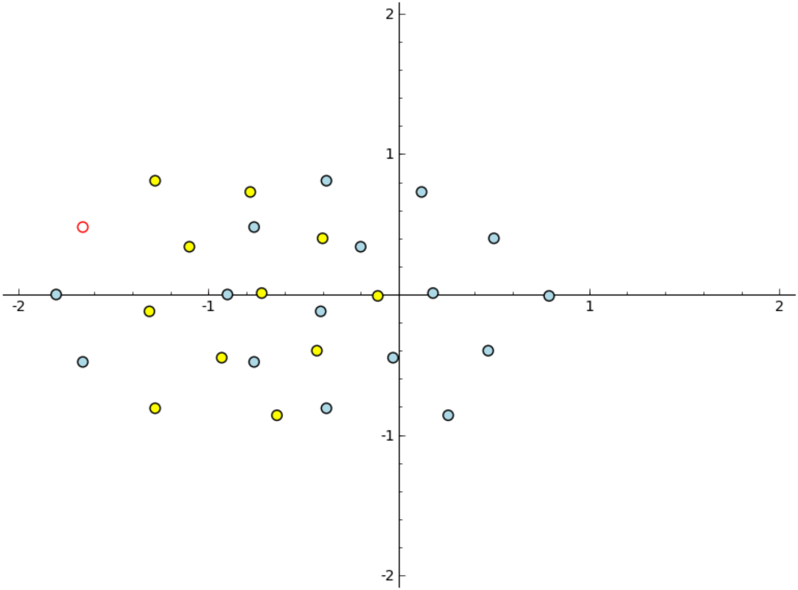

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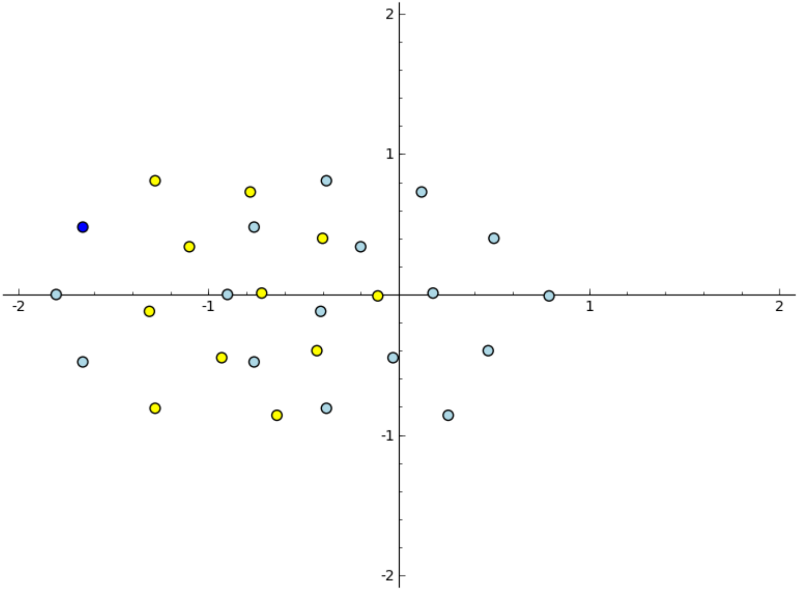

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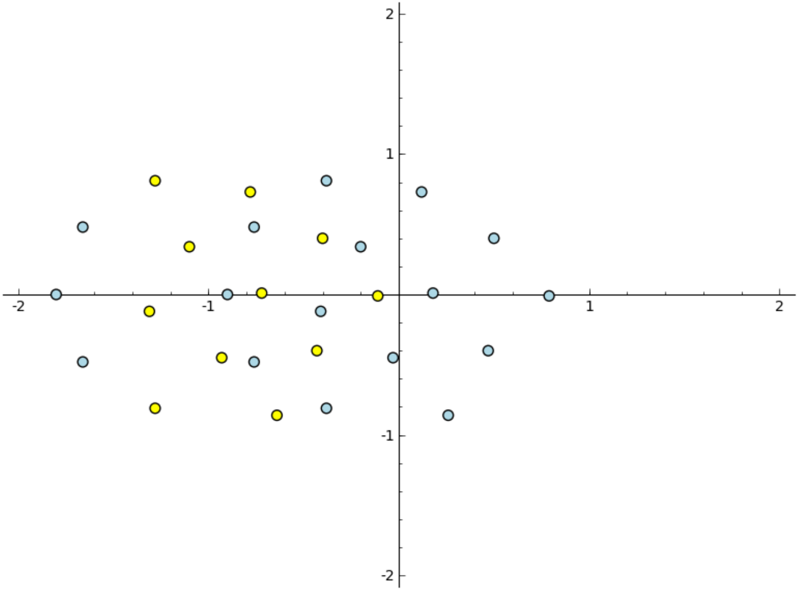

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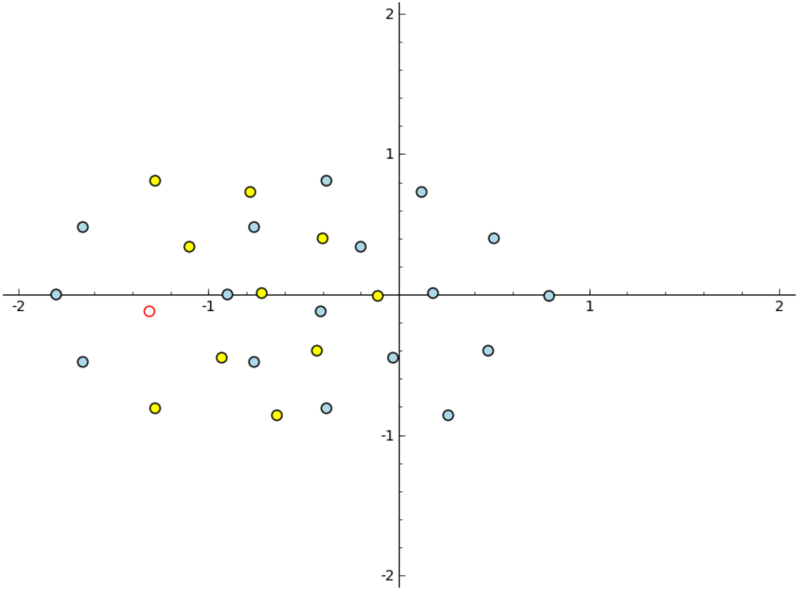

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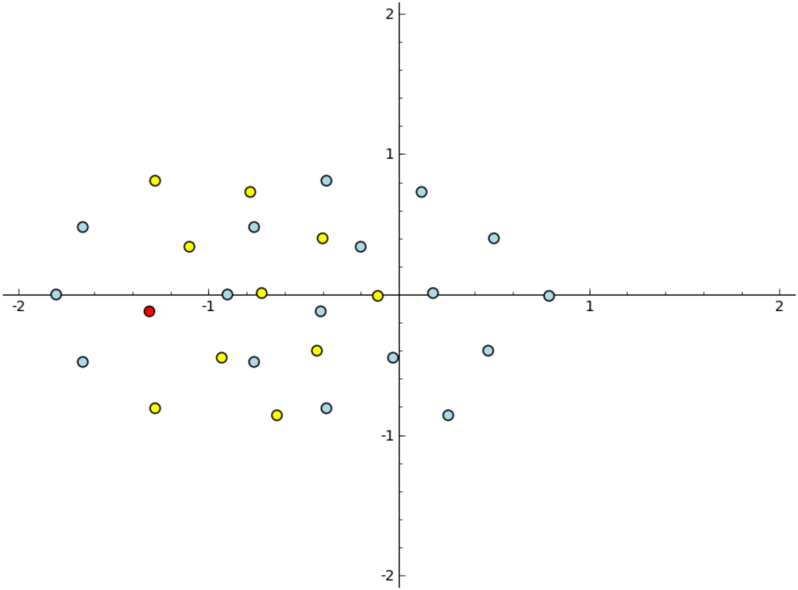

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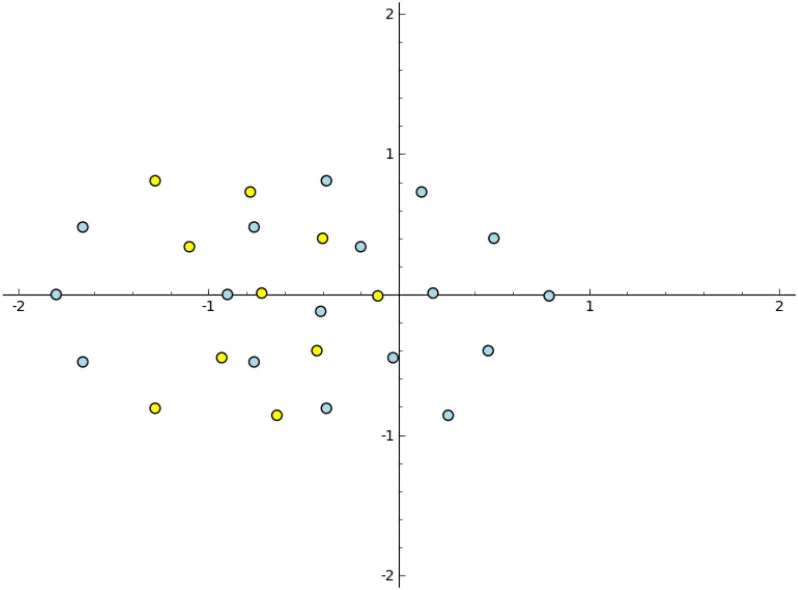

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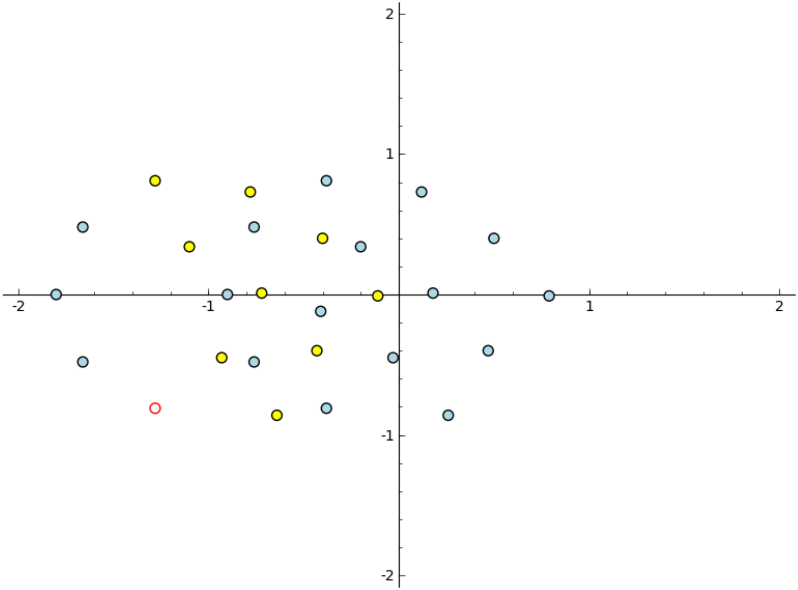

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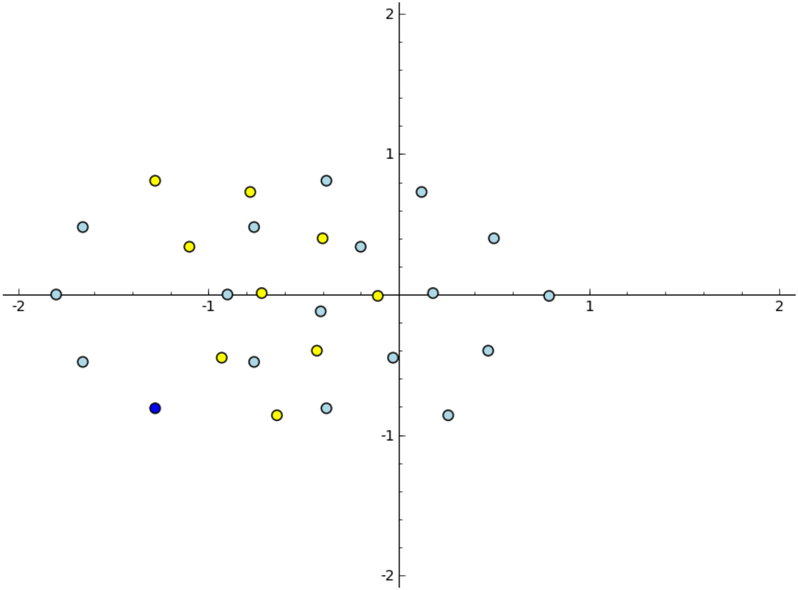

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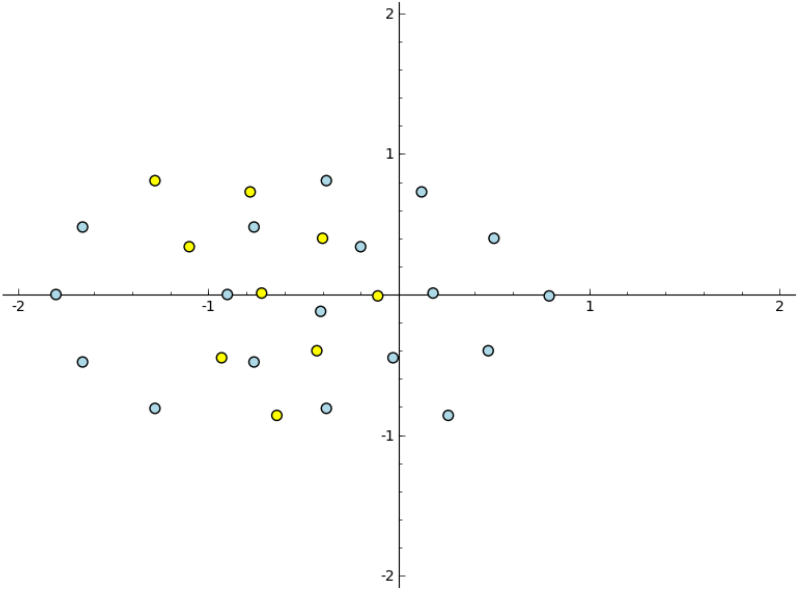

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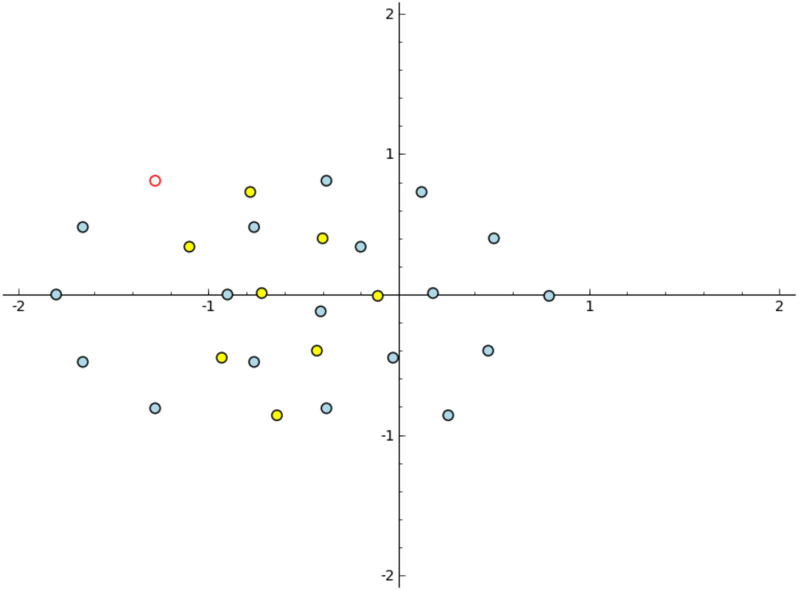

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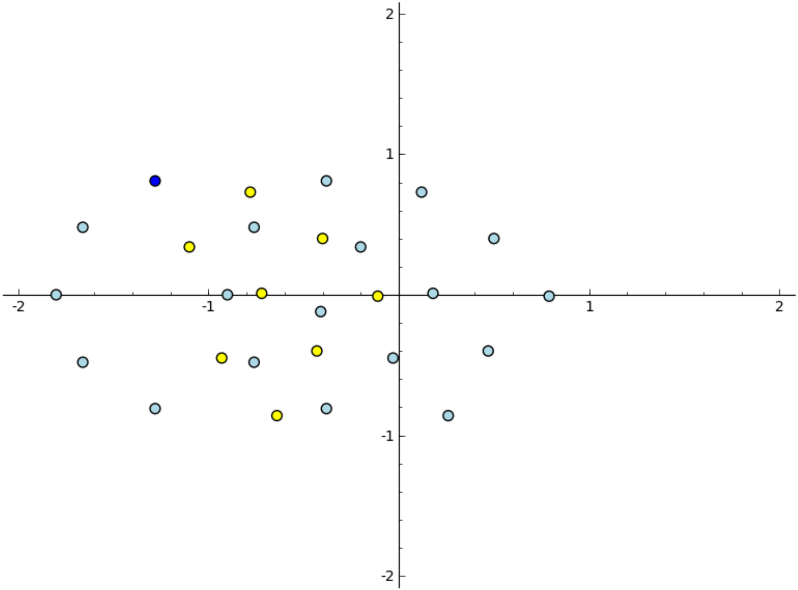

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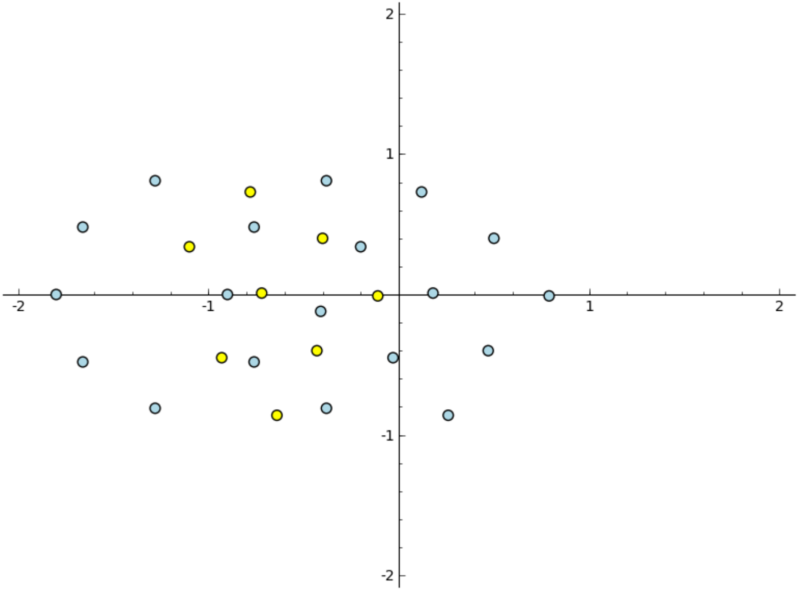

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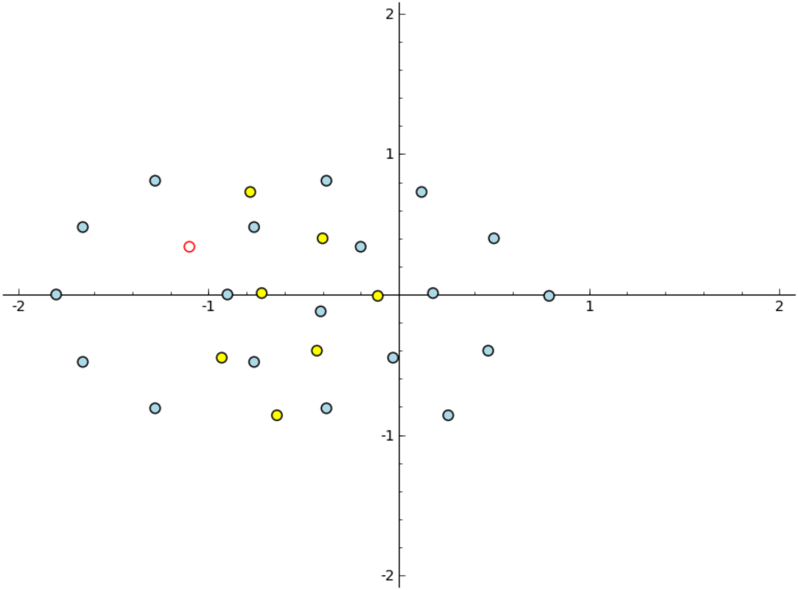

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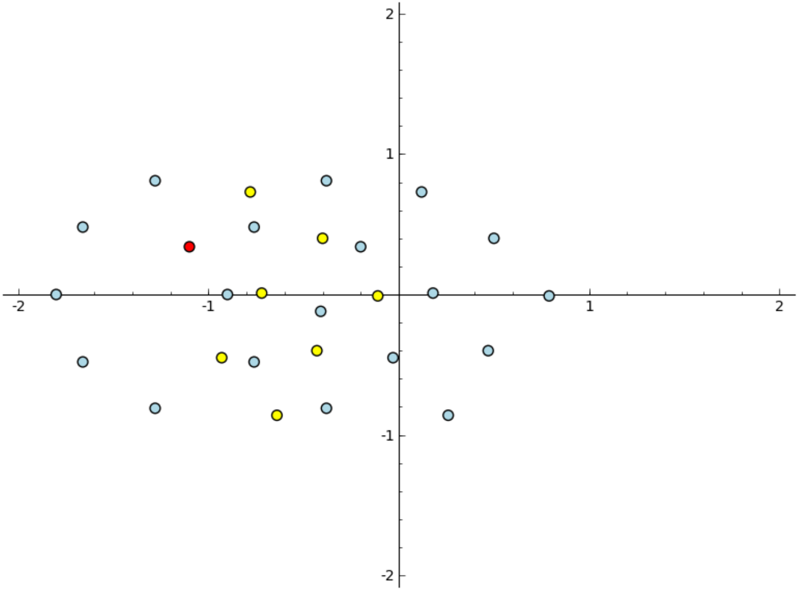

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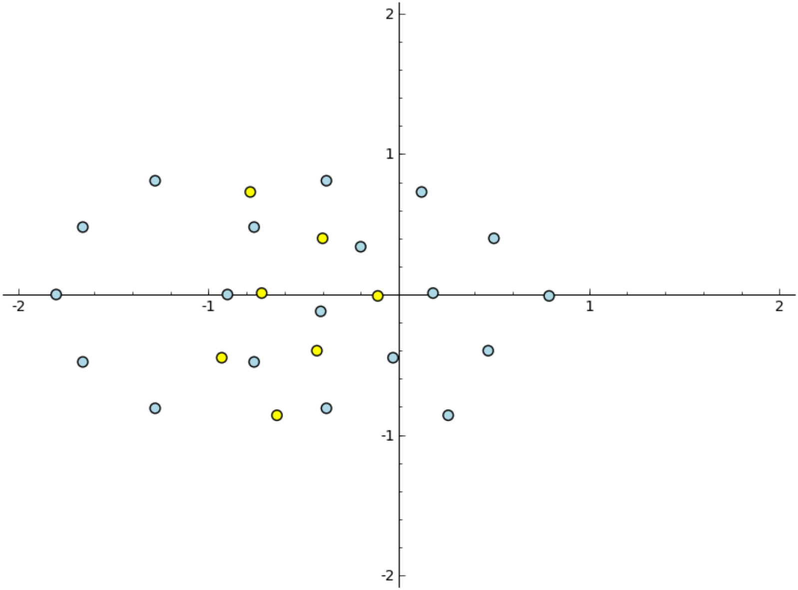

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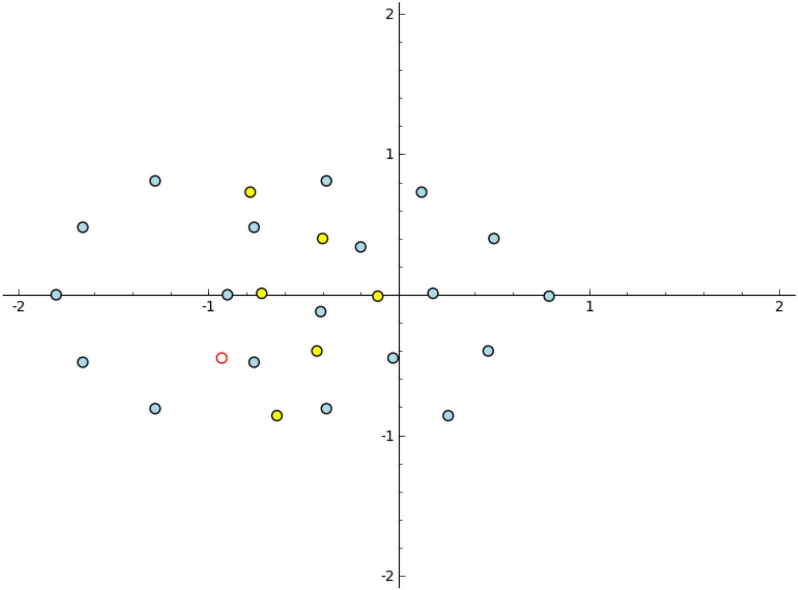

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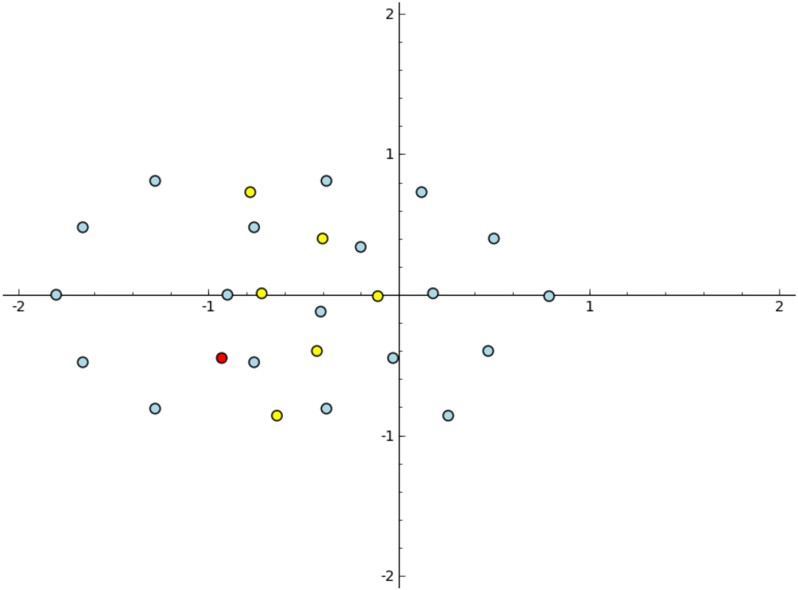

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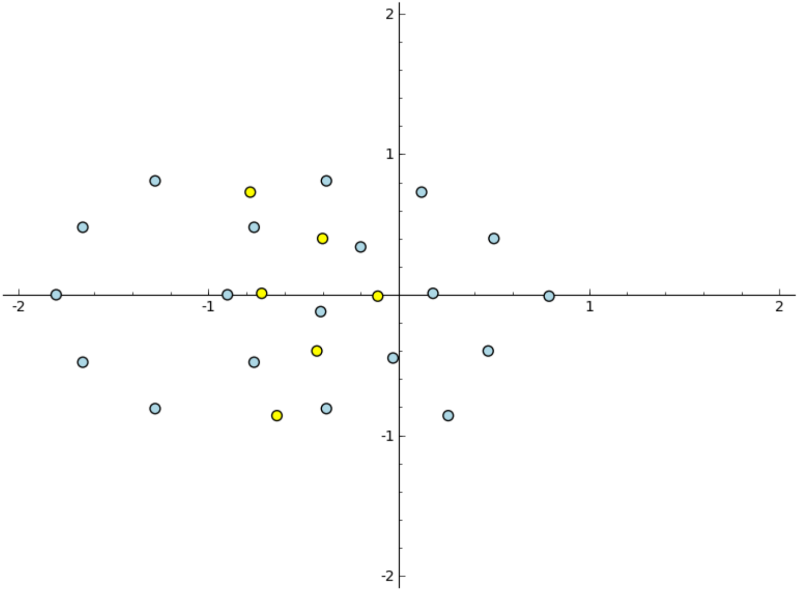

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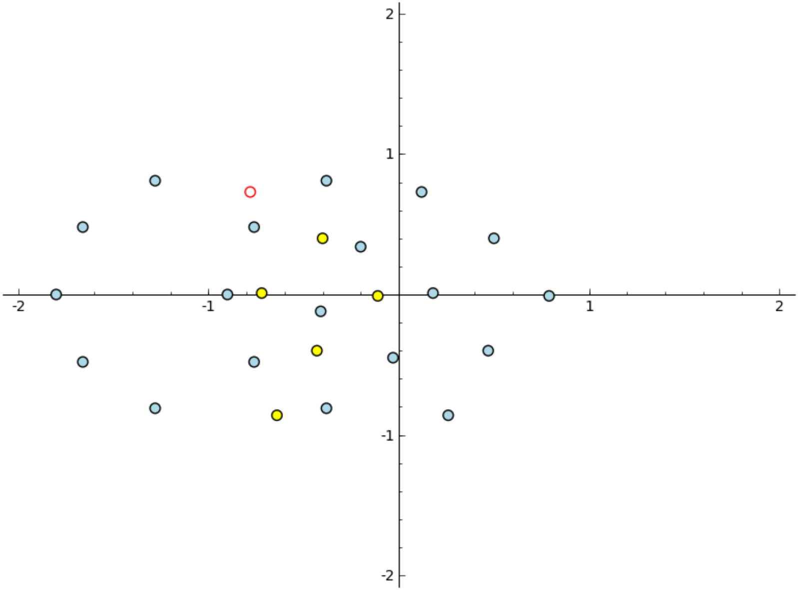

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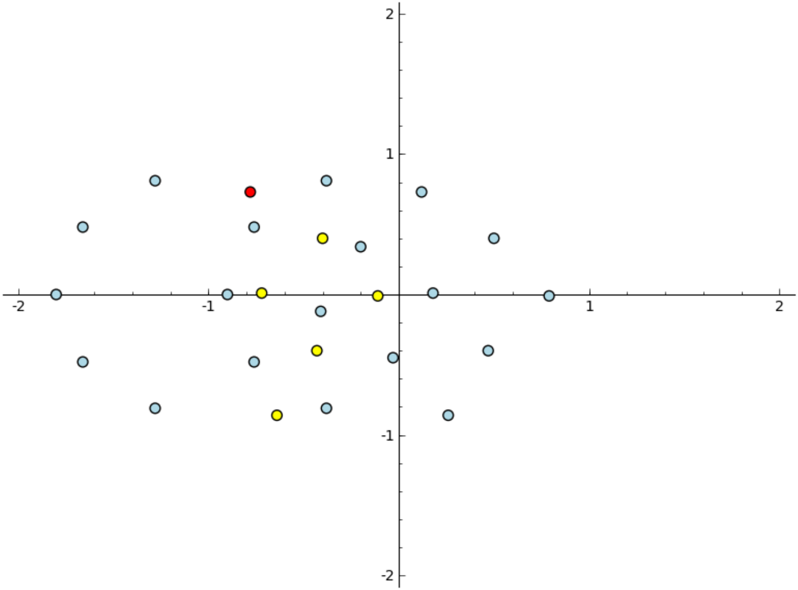

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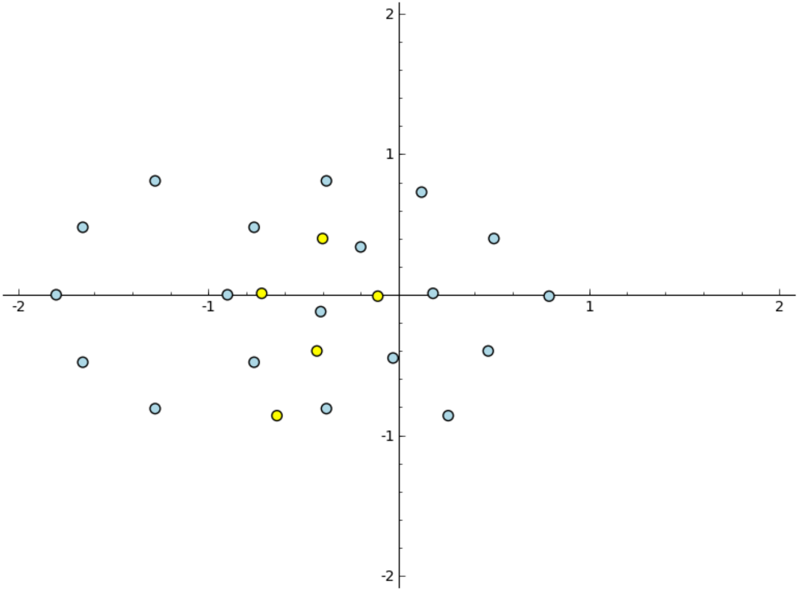

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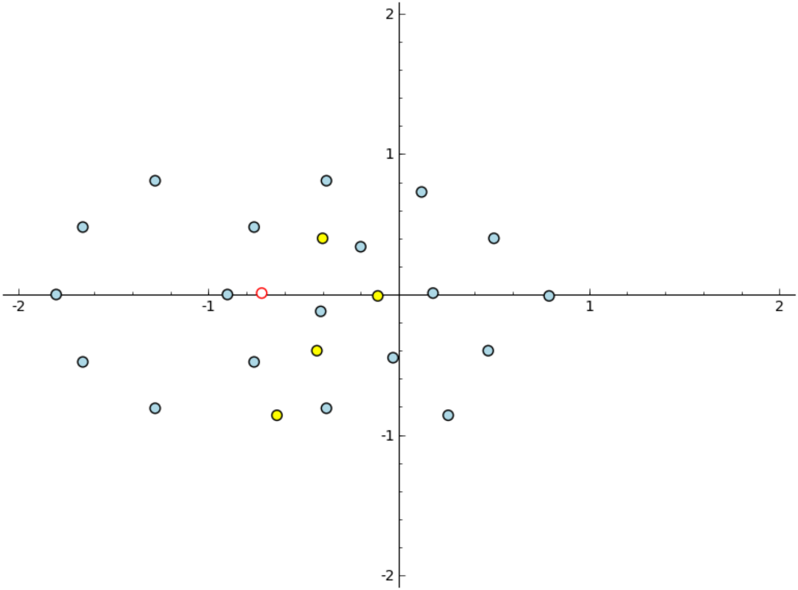

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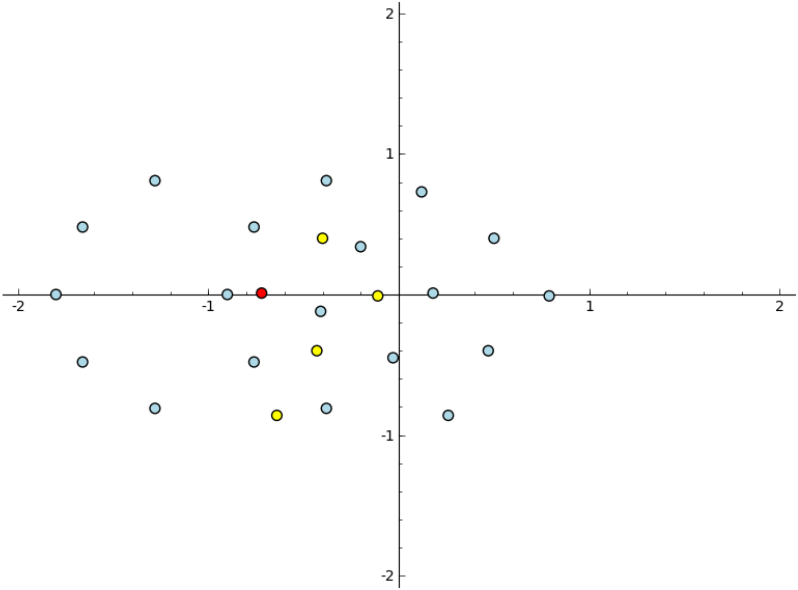

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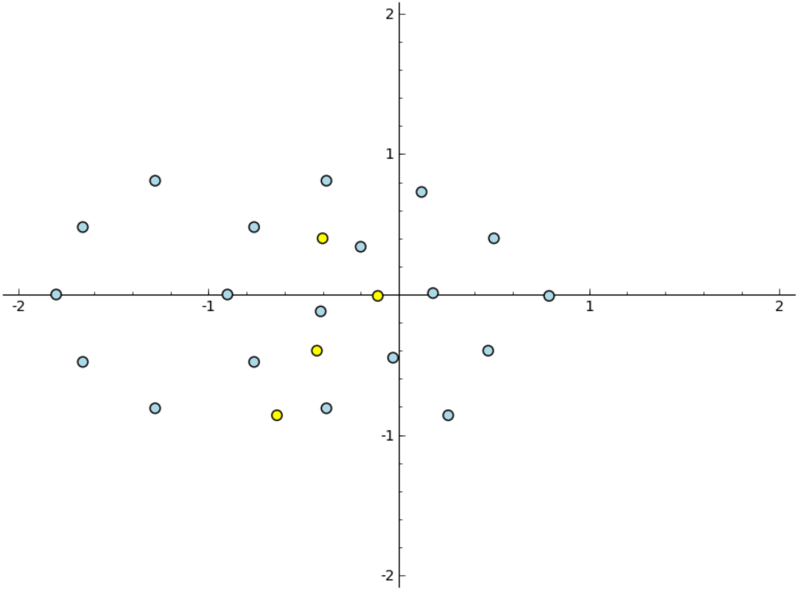

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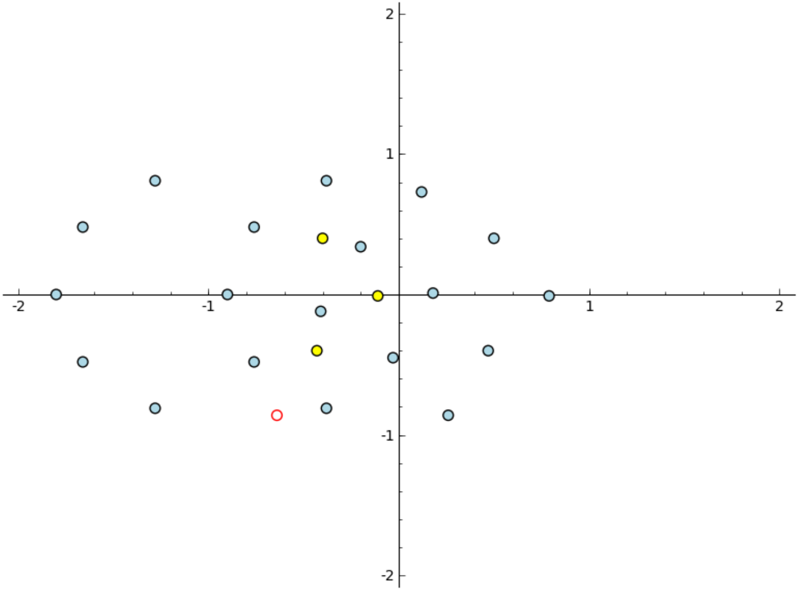

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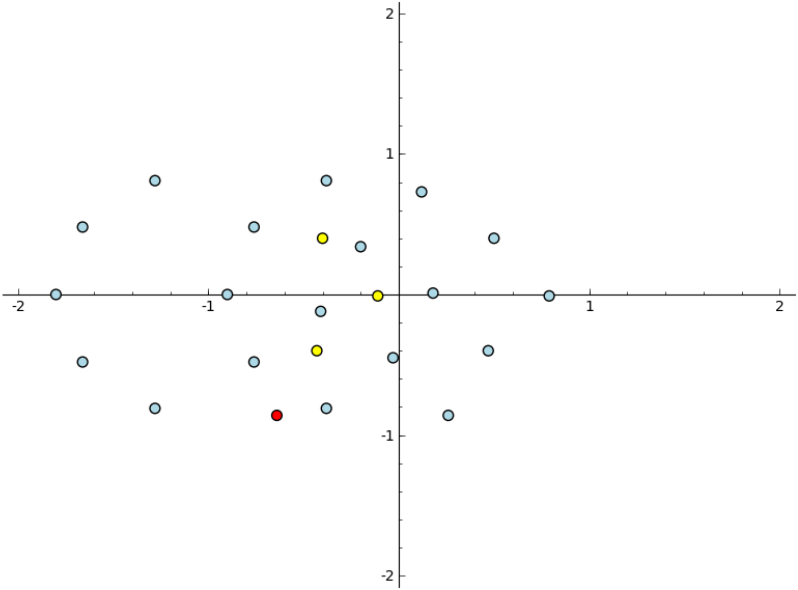

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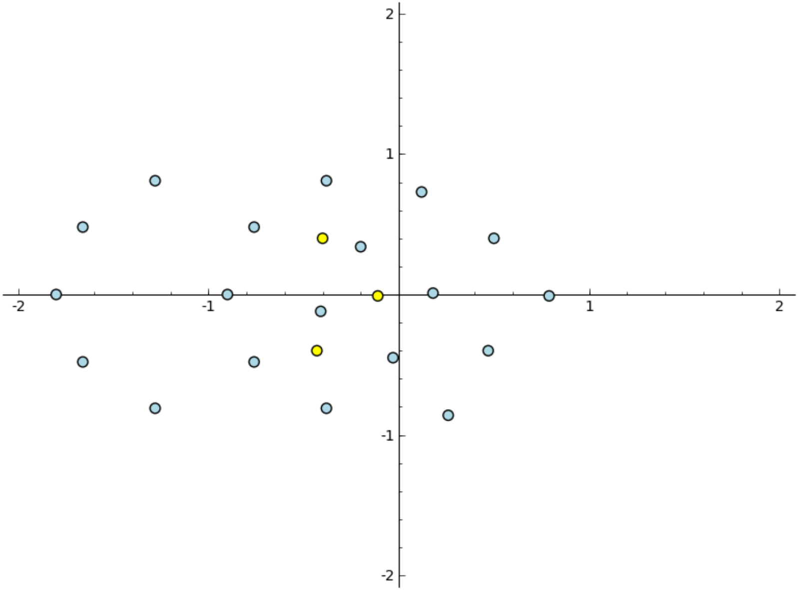

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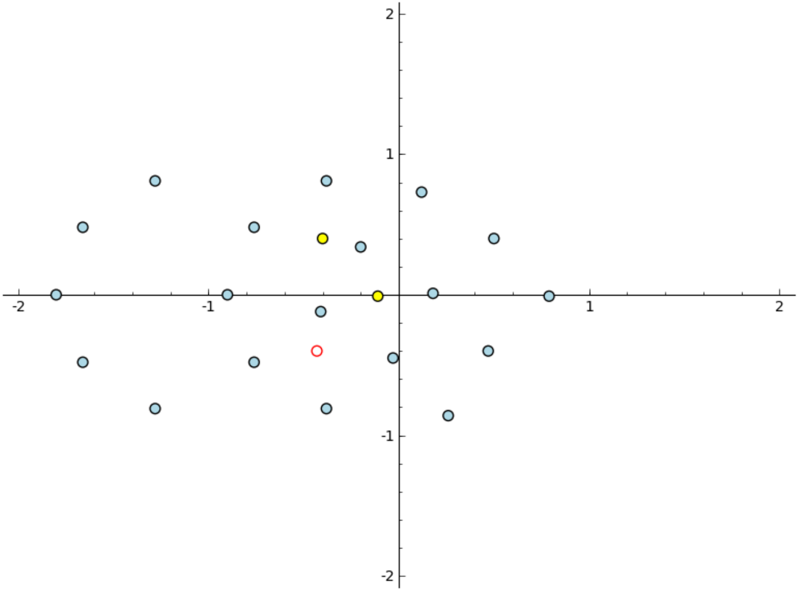

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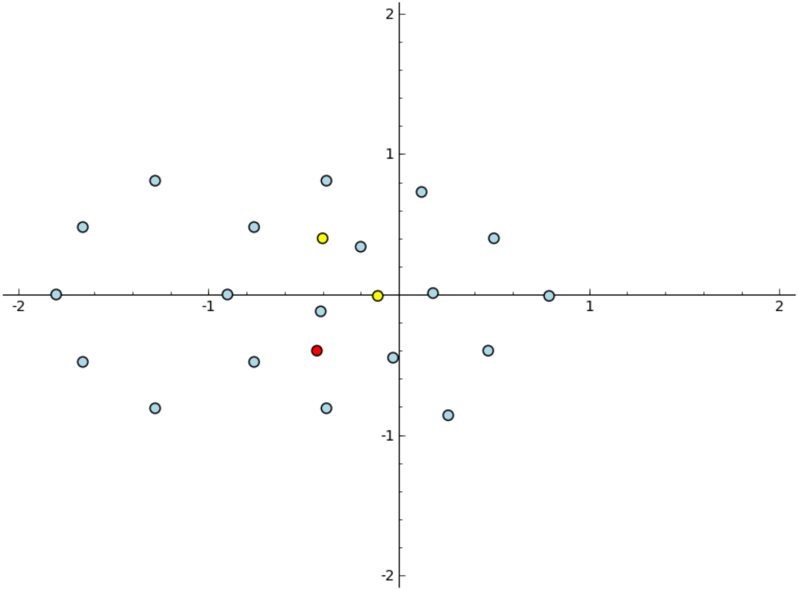

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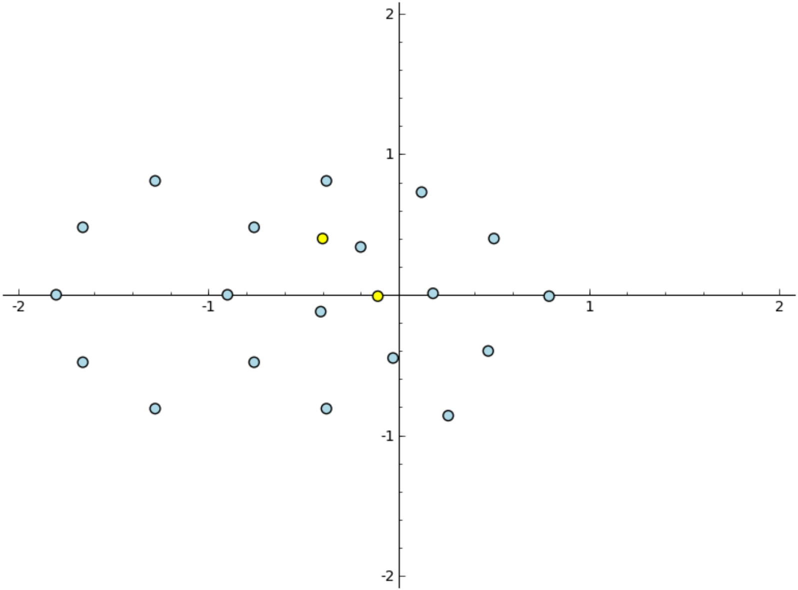

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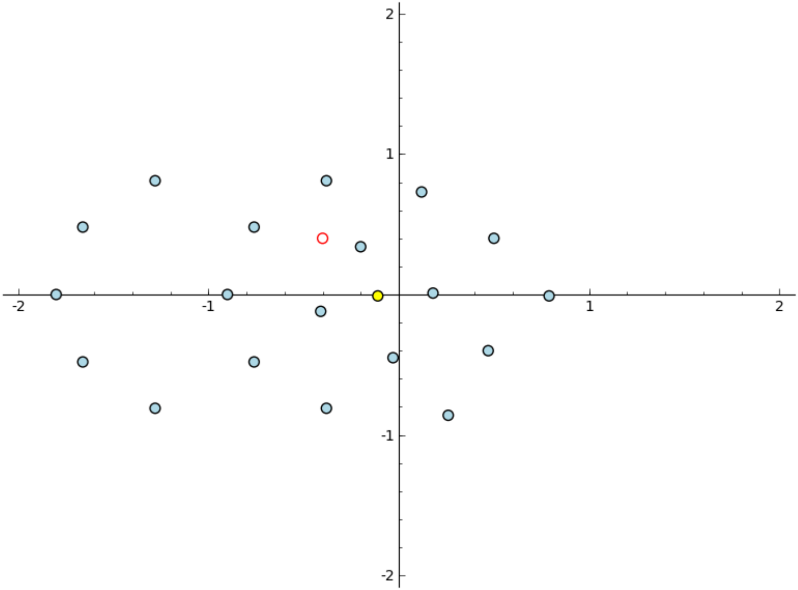

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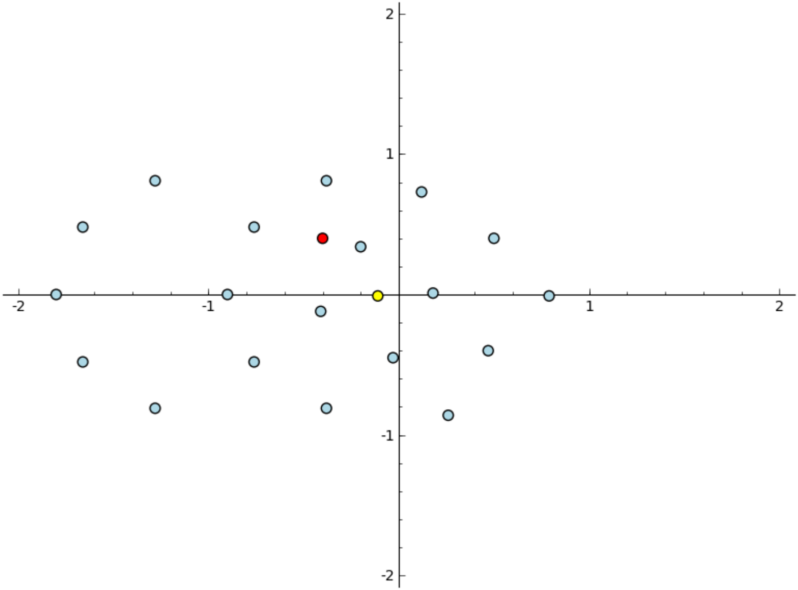

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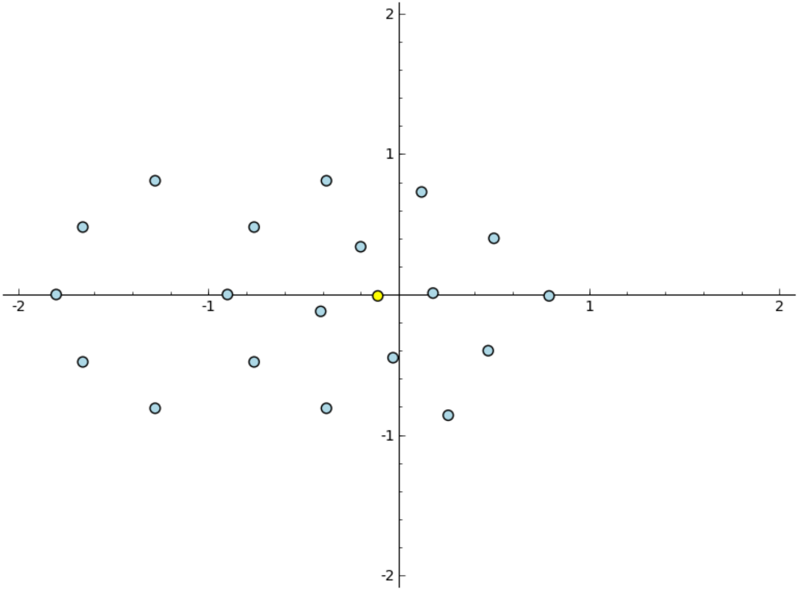

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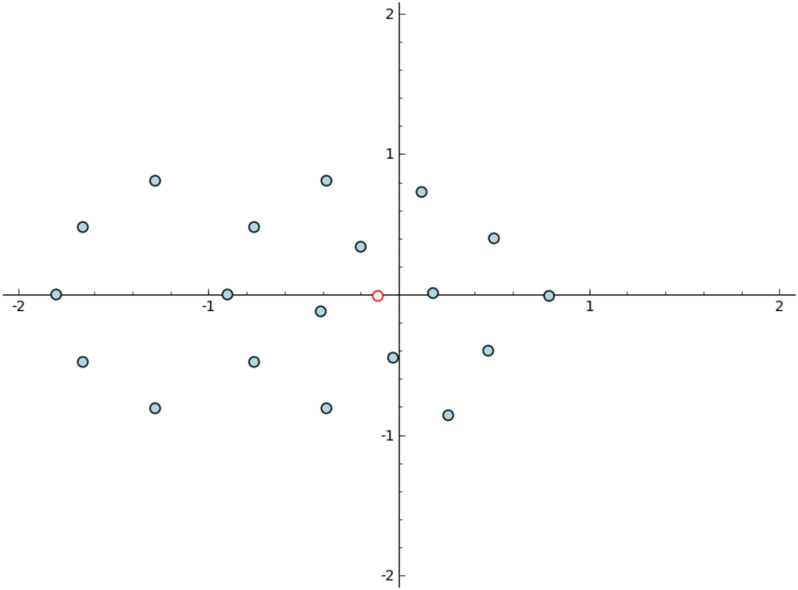

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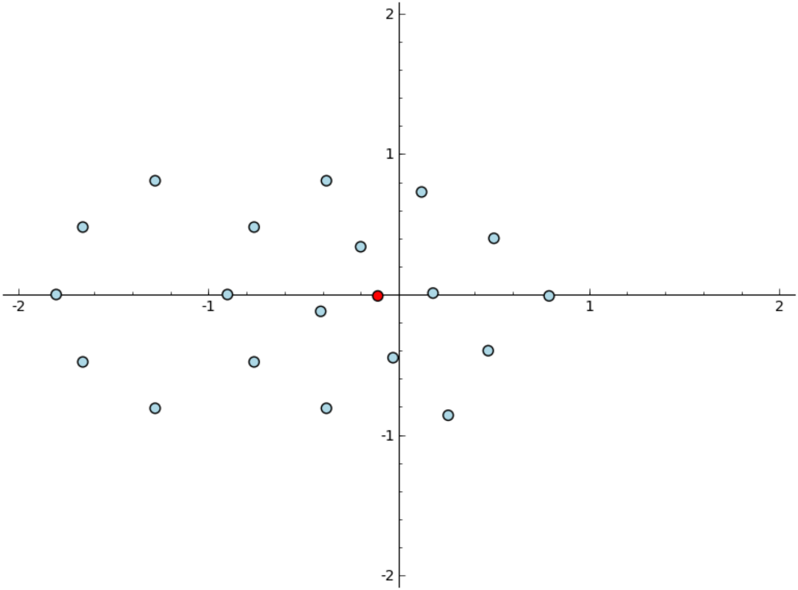

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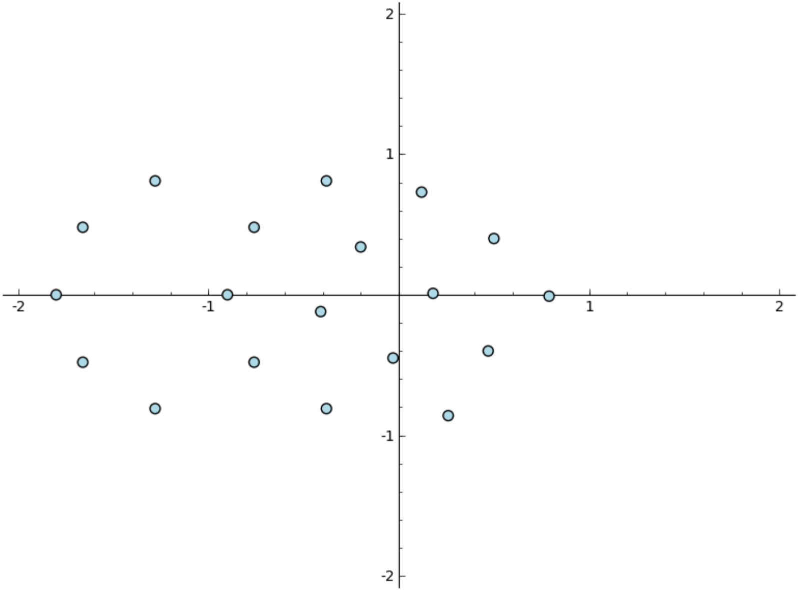

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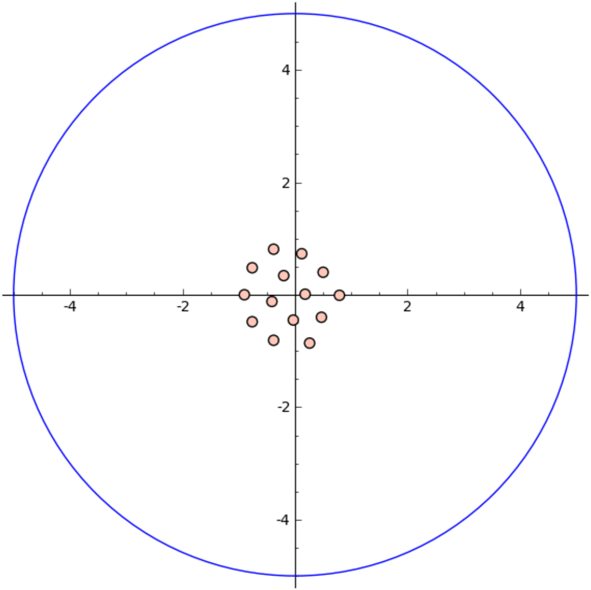

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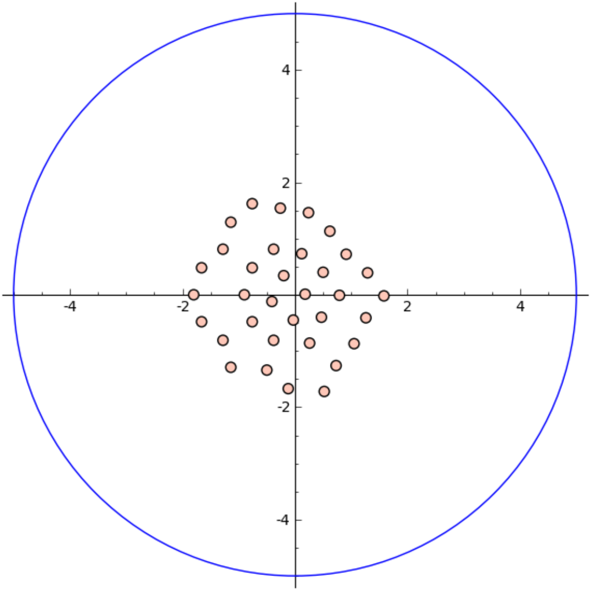

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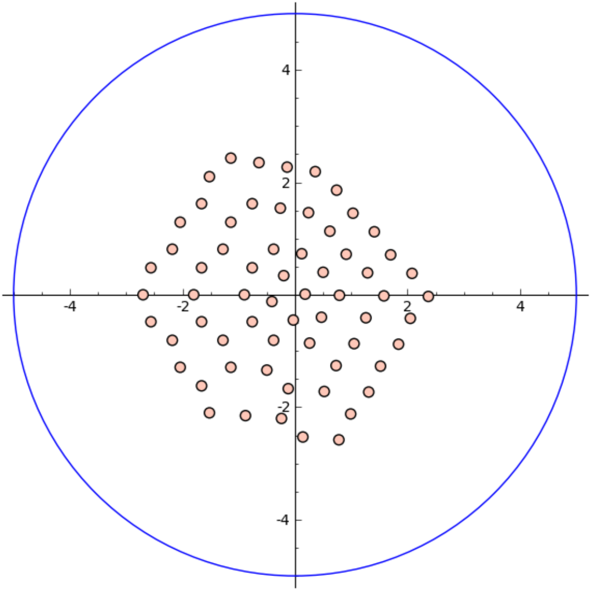

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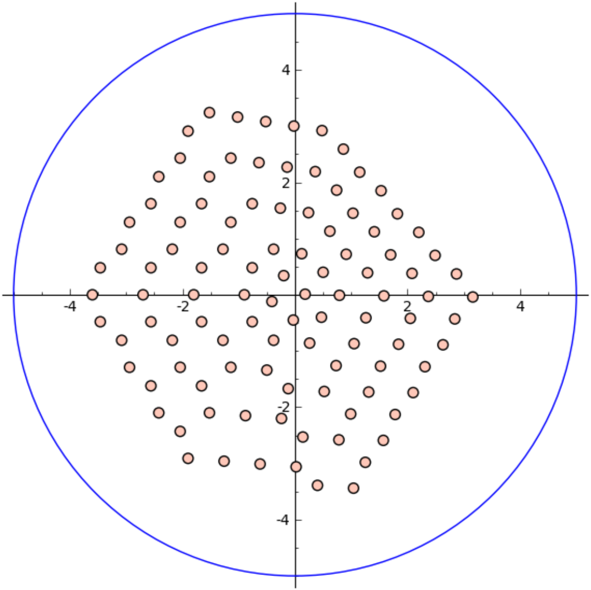

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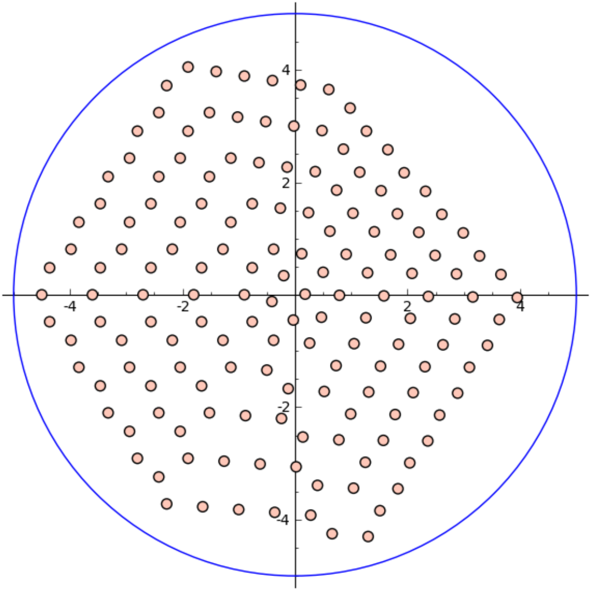

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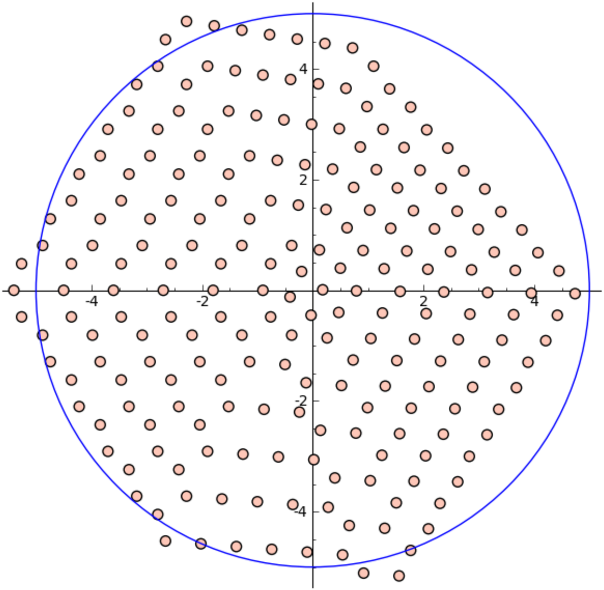

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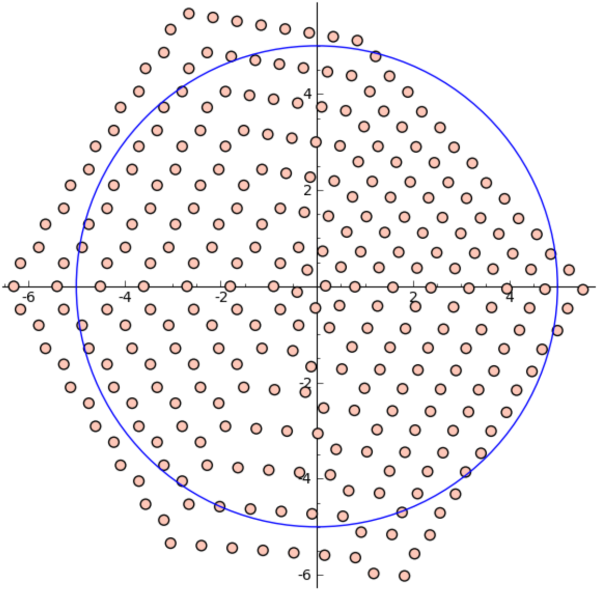

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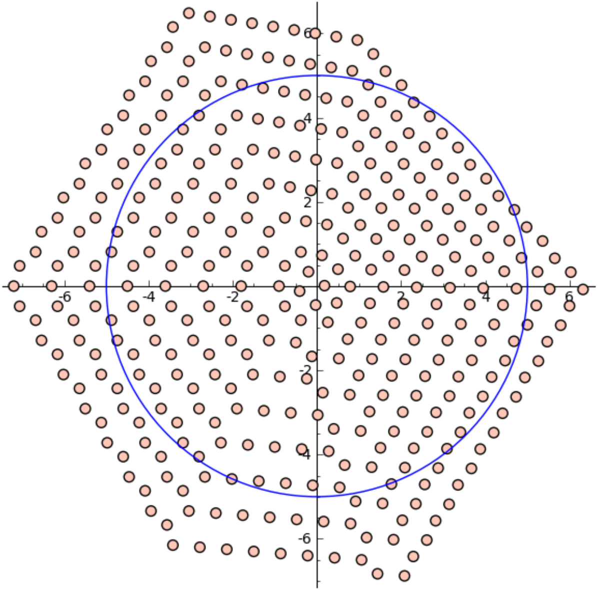

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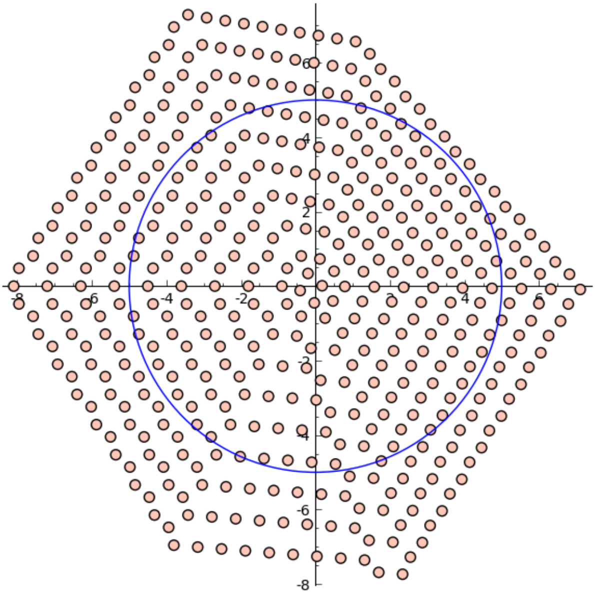

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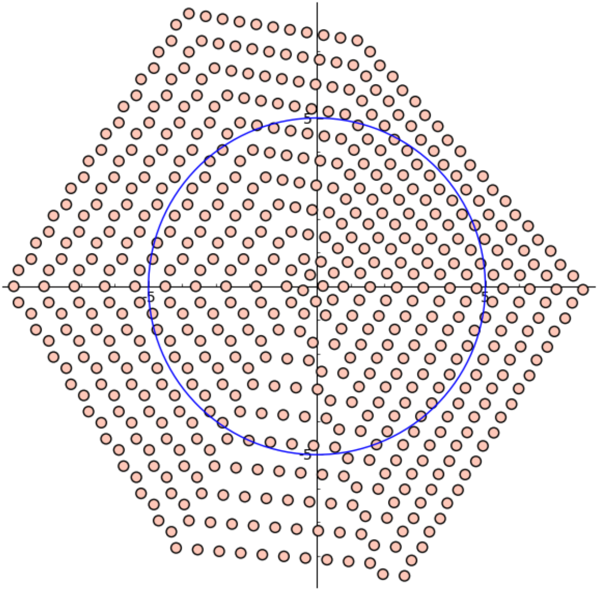

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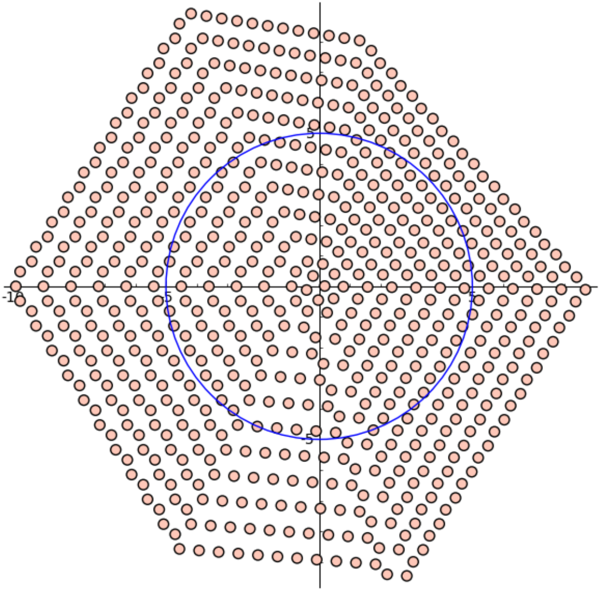

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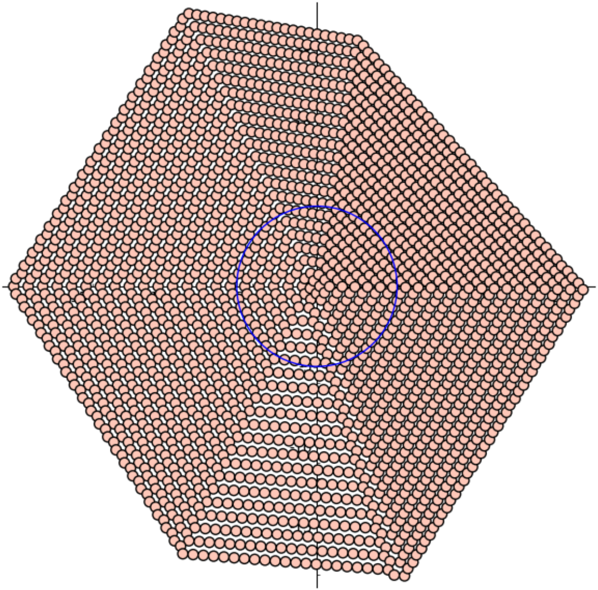

Algorithm to compute the nerve

Discussed at board.

Difficulties of computing with nerves

- Time-consuming to compute.
- Difficult to store.

Revisiting our 3-dimensional example using a Rips complex

Covering and nerve information for $\mathrm{Rips}_{\Gamma\backslash X}(0.65)$:

- 1172s to compute a cover.
- 285s to compute the 1-skeleton of the nerve, i.e. when pairs of balls of radius $0.65/2$ centered at our grid points intersect.
- The 1-skeleton completely determines the higher skeleta!
- Simplices of each degree:

$$(N_0, N_1, \ldots) = (176, 3135, 11836, 23159, 24484, 14915, 5268)$$

3-dimensional example using a Rips complex

Homology for $\mathrm{Rips}_{\Gamma\backslash X}(0.65)$:

- abelianization of presentation for Γ: $\mathbb{Z}/3 \oplus \mathbb{Z}/6 \oplus \mathbb{Z}/6$.
- simplicial computation: 103s over $\mathbb{F}_2$, 96s over $\mathbb{F}_3$, 84s over $\mathbb{F}_{11}$, 154s over $\mathbb{F}_{1009}$. Betti numbers:

$$\begin{aligned} \text{mod } 2 &: (1, 2, 2, 1, 0, 0, \ldots) \\ \text{mod } 3 &: (1, 3, 3, 1, 0, 0, \ldots) \\ \text{mod } 11 &: (1, 0, 0, 1, 0, \ldots) \\ \text{mod } 1009 &: (1, 0, 0, 1, \ldots). \end{aligned}$$

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Simplicial computation of mod p betti numbers succeeded here in reasonable time.

Rips complex challenge

Suppose balls of radius r are convex in M. Can you prove that

$\mathrm{Rips}_{\Gamma\backslash X}(r)$ is homotopy equivalent to M?