Growth Rate of Eigenfunctions

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Eigenvalue and eigenfunctions

For a given compact smooth n-manifold M, we have the following (smooth) eigenfunctions

$$-\Delta_g e_\lambda(x) = \lambda^2 e_\lambda(x).$$

And we have a discrete set of eigenvalues

$$\{0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \cdots < \infty\}$$

Basic questions

Two basic questions in spectral analysis

1. How the eigenvalues $\{\lambda_j\}$ are distributed on $[0, \infty)$?
2. How "large" the eigenfunctions are?

Weyl law

Eigenvalue: (classical Weyl formula, or Weyl law)

$$N(\lambda) = c_n \lambda^n + O(\lambda^{n-1}).$$

You can also recover $N(\lambda)$ by taking inverse Fourier transform on the truncated trace of half-wave operator:

$$N(\lambda) = \frac{1}{2\pi} \int \widehat{\chi_{[0,\lambda]}}(t) \left\{ \text{trace of } e^{it\sqrt{-\Delta}} \right\} dt.$$

Half-wave

The half-wave $e^{it\sqrt{-\Delta}}$ is the Fourier transform of the spectral measure

$$\hat{E}(t) = \int e^{-it\lambda} dE_\lambda = e^{-it\sqrt{-\Delta}}$$

Here

$$e^{-it\sqrt{-\Delta}} = \sum_j e^{-it\lambda_j} e_{\lambda_j} \otimes e_{\lambda_j} = \sum_j \widehat{\delta(\cdot - \lambda_j)}(t) e_{\lambda_j} \otimes e_{\lambda_j}$$

and $e^{it\sqrt{-\Delta}} f(x)$ is also the solution to the (half)-wave Cauchy problem:

$$\begin{cases} (i\partial_t + \sqrt{-\Delta})u(t,x) = 0 \\ u(0,x) = f(x). \end{cases}$$

More about Weyl law

Two things should be mentioned about Weyl law

1. It is sharp on spheres S^n
2. It is rarely sharp on other manifolds. For example: (Hlawka, 1950) On flat torus $\mathbb{T}^n$

$$N(\lambda) = c\lambda^n + O(\lambda^{n-1-\frac{n-1}{n+1}}).$$

   This is because on torus $e_\lambda(x)$ is a trigonometric polynomial as follows

$$e_\lambda(x) = \sum_{|m|^2=\lambda^2} \widehat{e_\lambda}(m)e^{im\cdot x}.$$

Duistermaat-Guillemin wave-trace

A key progress was made by Duistermaat and Guillemin in 1975, who proved that

1. The trace of $e^{it\sqrt{-\Delta}}$ is smooth when t is not equal to the period of any periodic geodesics (if any), or 0.
2. If the periodic geodesics are of zero Liouville measure, then

$$N(\lambda) = c_n\lambda^n + o(\lambda^{n-1}).$$

Improvement on non-positive manifolds

So to improve the remainder term in Weyl formula, a good candidate would be a manifold with very few periodic geodesics. In fact:
(Bérard, 1977) On manifolds without conjugate points, such as non-positive curvature manifolds, we have

$$N(\lambda) = c\lambda^n + O(\lambda^{n-1}/\log\lambda).$$

General L^p estimates

Weyl formula is in fact an L^∞ type estimates for the eigenfunctions:

$$||e_\lambda||_{L^\infty} \leq C\lambda^{\frac{n-1}{2}}||e_\lambda||_{L^2}.$$

So we may ask, how about an L^p estimates with $p < +\infty$? (of course, except for those from simple interpolation)

Sogge's general result

L^p estimates of the eigenfunctions on general manifolds (Sogge, 1987), using oscillatory integral techniques:

$$||e_\lambda||_{L^p(M)} \le C\lambda^{n(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}||e_\lambda||_{L^2(M)}, \quad \frac{2(n+1)}{n-1} \le p \le +\infty$$
$$||e_\lambda||_{L^p(M)} \le C\lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{p})}||e_\lambda||_{L^2(M)}, \quad 2 \le p \le \frac{2(n+1)}{n-1}.$$

These estimates are sharp on spheres. But lacking of explicit information the eigenfunctions makes improvement on other manifolds generally hard.

L^p improvement on non-positive manifolds

For large p, we can interpolate with Bérard's result. And by $(\log\lambda)^\alpha \ll \lambda^{+0}$ we may obtain a slightly better improvement with the price of losing endpoint case.
(Hassell-Tacey, 2013)

$$||e_\lambda||_{L^p} \leq C\frac{\lambda^{\frac{n-1}{2}-\frac{n}{p}}}{(\log\lambda)^{\frac{1}{2}}}||e_\lambda||_{L^2}, \quad p > \frac{2(n+1)}{(n-1)}.$$

Notice the improvement over:

$$||e_\lambda||_{L^p} \leq C\frac{\lambda^{\frac{n-1}{2}-\frac{n}{p}}}{(\log\lambda)^{\frac{1}{2}-\frac{n+1}{p(n-1)}}}||e_\lambda||_{L^2}, \quad p \geq \frac{2(n+1)}{(n-1)}.$$

But it seems very *hard* to break the log barrier.

Zygmund's L^4 estimates on torus

There may be some hope in establishing much-better-than-sphere estimates on torus due to the explicit construction of eigenfunctions.

(Zygmund, 1974) On 2-dimensional torus we have

$$||e_\lambda||_{L^4(\mathbb{T}^2)} \leq C||e_\lambda||_{L^2(\mathbb{T}^2)}.$$

The proof is very simple and based on the simplicity of S^1, so no hope to generalize to higher-dimensions.

Higher dimensional L^p estimates on torus

In higher dimensions, we may have

1. $n = 3$, $||e_\lambda||_{L^4} \leq C_\varepsilon \lambda^\varepsilon ||e_\lambda||_{L^2}$, due to arithmetic observation (relatively simple).
2. $n \geq 4$ (Bourgain 2011), $||e_\lambda||_{L^p} \leq C_\varepsilon \lambda^\varepsilon \|e_\lambda\|_{L^2}$, $p \leq 2n/(n-1)$. This is based on an earlier work (Bourgain-Guth, 2011), which improved Tomas-Stein restriction conjecture using multilinear oscillatory integral techniques.

L^p conjecture on torus

So a conjecture:
Conjecture: Is it possible on higher-dimensional torus we have

$$||e_\lambda||_{L^p} \leq C||e_\lambda||_{L^2}, \quad \text{for some } p > 2?$$

So far we have no idea how to break the λ^ε barrier (compared with neg-curved manifold case).

A relatively new approach

If the growth rate of eigenfunctions reflects largeness of singularity of $e^{it\sqrt{-\Delta}}$ which propagates along the geodesics, then in ideal case the global L^p estimates may be dominated by the restriction of eigenfunctions on geodesics.

Spherical harmonics

Spherical harmonics are the spherical part of the solution of Laplace equation, also the eigenfunctions of $\sqrt{-\Delta_{S^n}}$. In 2-d, if we use $\theta \in [0, \pi]$ to denote the latitude, $\phi \in [0, 2\pi]$ the longitude, then the eigenfunctions associated with $\sqrt{\ell(\ell+1)}$ is

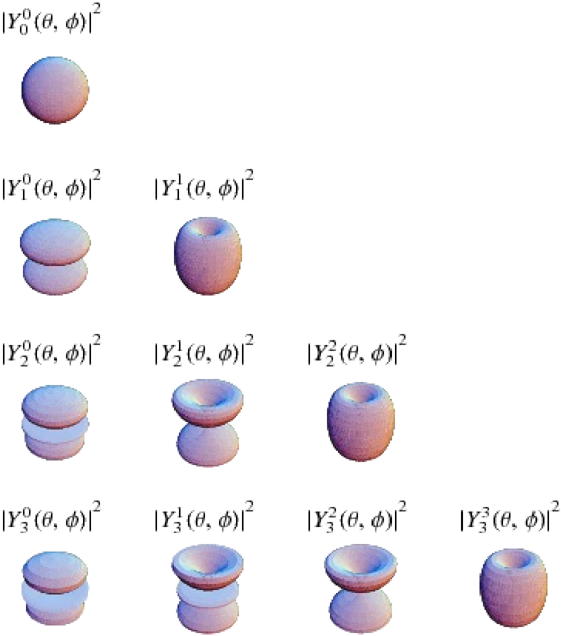
$$Y_\ell^m(\theta, \phi) = c_{m,\ell} P_\ell^m(\cos\theta) e^{im\phi}, \quad -\ell \leq m \leq \ell$$

in which P_ℓ^m is the associated Legendre polynomials of degree ℓ and order m.

Zonal and sectoral spherical harmonics

Two extremal cases:

1. $m = 0$, zonal spherical harmonics, $Y_\ell^0 = c_{m,\ell} P_\ell(\cos\theta)$.
2. $m = \pm\ell$, sectoral spherical harmonics (some call it highest weight spherical harmonics),
   $|Y_\ell^\ell| = |c_\ell| \cdot |\sin^\ell \theta| = |c_\ell| \cdot |x_1 + ix_2|^\ell$.


$|Y_0^0(\theta, \phi)|^2$
$|Y_1^0(\theta, \phi)|^2$
$|Y_1^1(\theta, \phi)|^2$
$|Y_2^0(\theta, \phi)|^2$
$|Y_2^1(\theta, \phi)|^2$
$|Y_2^2(\theta, \phi)|^2$
$|Y_3^0(\theta, \phi)|^2$
$|Y_3^1(\theta, \phi)|^2$
$|Y_3^2(\theta, \phi)|^2$
$|Y_3^3(\theta, \phi)|^2$

B-G-T theorem

(Burq-Gérard-Tzvetkov, 2006), inspired by (Reznikov, 2004)
If γ is a geodesic arc on surface M

$$\|e_\lambda\|_{L^p(\gamma)} \leq C\lambda^{\frac{1}{4}} \|e_\lambda\|_{L^2(M)}, \quad 2 \leq p \leq 4$$
$$\|e_\lambda\|_{L^p(\gamma)} \leq C\lambda^{\frac{1}{2}-\frac{1}{p}} \|e_\lambda\|_{L^2(M)}, \quad 4 \leq p \leq \infty.$$

The first estimate is sharp on $M = S^2$ by sectoral spherical harmonics, and the latter is sharp by zonal spherical harmonics.

Equivalence between global and restricted L^p estimates

(Bourgain, 2009) and (Sogge, 2009): On surface M, the following are equivalent for L^2 normalized eigenfunctions:

$$\|e_\lambda\|_{L^4(M)} \in o(\lambda^{\frac{1}{4}})$$

and

$$\sup_{\gamma} \|e_\lambda\|_{L^2(\gamma)} \in o(\lambda^{\frac{1}{4}})$$

Here $\gamma \subset M$ is a unit-length geodesic arc. And the latter (therefore both) is proved to be true by (Sogge-Zelditch,2011) on surface with non-positve curvature.

Improvement on torus

It is classical that (see Hardy-Wright)

$$||e_\lambda||_{L^2(\gamma)} \lesssim ||e_\lambda||_{L^\infty(\mathbb{T}^2)} \leq C_\varepsilon \lambda^\varepsilon ||e_\lambda||_{L^2(\mathbb{T}^2)}$$

So it leaves us a question, whether it is possible (for any smooth curve or with curvature)

$$\|e_\lambda\|_{L^2(\gamma)} \leq C \, \|e_\lambda\|_{L^2(\mathbb{T}^2)} \, .$$

(Bourgain-Rudnick, 2011): If γ is an analytic curved arc in $\mathbb{T}^2$ (also in $\mathbb{T}^3$), then

$$||e_\lambda||_{L^2(\gamma)} \approx ||e_\lambda||_{L^2(\mathbb{T}^2)},$$

when λ is large.

Conjecture

For a real analytic curved arc γ in $\mathbb{T}^n$, $n \geq 4$, do we still have

$$||e_\lambda||_{L^2(\gamma)} \approx ||e_\lambda||_{L^2(\mathbb{T}^n)}?$$

Thank you!