Central Limit Theorems for linear statistics for Biorthogonal Ensembles

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Based on joint work with Jonathan Breuer (HUJI)

Princeton, April 2, 2014

Linear statistic

Given a function $f : \mathbb{R} \to \mathbb{R}$ and random points $x_1, \ldots, x_n \in \mathbb{R}$ the linear statistic $X_n(f)$ is defined as

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In this talk we will be discussing asymptotic behavior as $n \to \infty$ of the fluctuations of $X_n(f)$ where

1. $x_1, \ldots, x_n$ are random from a biorthogonal ensembles with a recurrence (so $\beta = 2$)
2. $f \in C^1(\mathbb{R})$ with at most polynomial growth at $\pm\infty$.

Central Limit Theorems

- Typically, there exists a limiting distribution ν of points as $n \to \infty$, and we have, almost surely,

$$\frac{1}{n} X_n(f) = \frac{1}{n} \sum_{j=1}^{n} f(x_j) \to \int f(x) \mathrm{d}\nu(x)$$

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- In several important examples it has been proved that the fluctuations $X_n(f) - \mathbb{E} X_n(f)$ satisfy a Central Limit Theorem

$$X_n(f) - \mathbb{E} X_n(f) \to N(0, \sigma(f)^2), \qquad \text{as } n \to \infty,$$

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$$X_n(f) - \mathbb{E}X_n(f) \to N(0, \sigma(f)^2), \qquad \text{as } n \to \infty,$$

for some $\sigma(f)^2$.
- No normalization, since $f \in C^1(\mathbb{R})$. In particular, the variance has a limit.

Example: Unitary ensembles

- Consider the probability measure on $\mathbb{R}^n$ defined by

$$\frac{1}{Z_n} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \mathrm{e}^{-n \sum_{j=1}^{n} V(x_j)} \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

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- As $n \to \infty$ there is a limiting density of points μ_V, which is the unique minimizer of

$$\iint \log \frac{1}{|x-y|} \mathrm{d}\nu(x) \mathrm{d}\nu(y) + \int V(x) \mathrm{d}\nu(x)$$

among all probability measures ν on $\mathbb{R}$. Hence

$$\frac{1}{n} X_n(f) \to \int f(x) \mathrm{d}\mu_V(x),$$

almost surely as $n \to \infty$.

Example: Unitary ensembles

Theorem (Johansson '98)

Under certain assumptions on V, in particular $S(\mu_V) = [\gamma, \delta]$, we have that for sufficiently smooth f

$$X_n(f) - \mathbb{E}X_n(f) \to N(0, \sum_{k=1}^{\infty} k|f_k|^2)$$

in distribution as $n \to \infty$, where

$$f_k = \frac{1}{2\pi i}\int_0^{2\pi} f\left(\frac{\delta-\gamma}{2}\cos\theta + \frac{\delta+\gamma}{2}\right) e^{-ik\theta} d\theta.$$

Example: Unitary ensembles

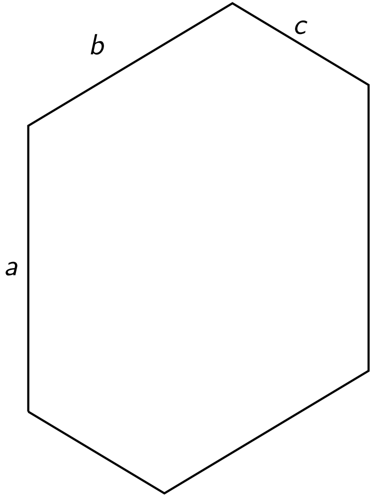
Some history

- Polynomial V, one-cut case and $f \in H^{2+\varepsilon}$ for some $\varepsilon > 0$ (+additional assumptions) Johansson '98
- Techniques were extended in Shcherbina-Kriecherbauer '10 Borot-Guionnet '12 to allow real analytic V.
- The one-cut assumption is essential. In multi-cut case there is not necessarily a limit to a Gaussian Pastur '06. In the multicut case, full asymptotic expansions of the partition function were recently obtained in Shcherbina '13, Borot-Guionnet '13.

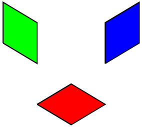
These results extend (with some appropriate modifications) to β-ensembles, but in this talk we assume $\beta = 2$.

- Other classical ensembles have been studied: e.g.β-Jacobi ensembles Dumitriu-Paquette '12,Borodin-Gorin '13.

Example: Lozenge tilings



Tilings of the *abc*-hexagon with lozenges



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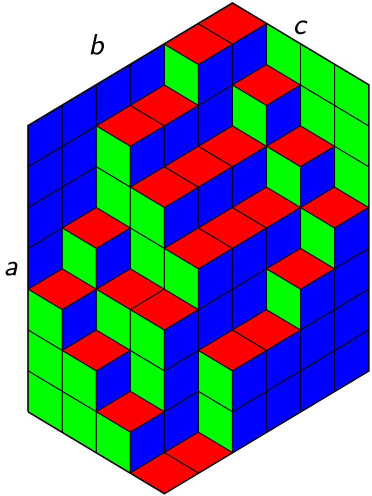
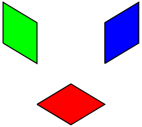


Tilings of the abc-hexagon with lozenges



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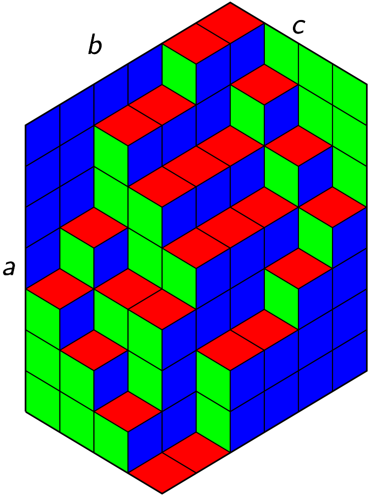

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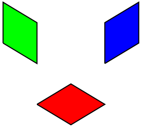

Random tilings: take uniform measure on all possible tilings

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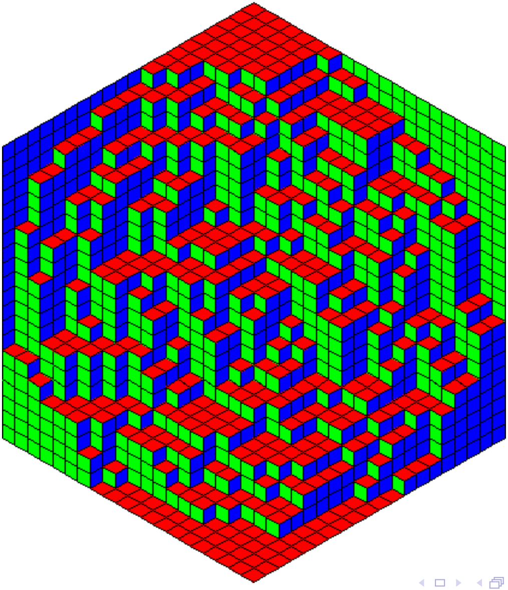

Gaussian Free Field

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- The example of lozenge tilings of a hexagon has been a special case in Petrov '13.
- For the purpose of this talk, the main interest is in the *1-d fluctuations*.....

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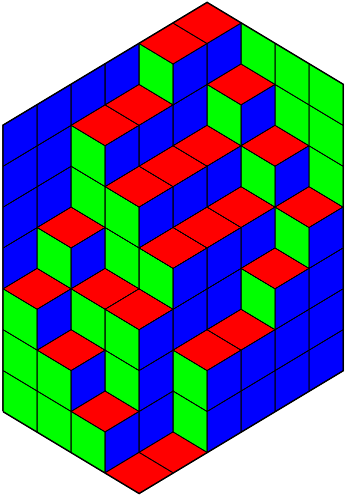

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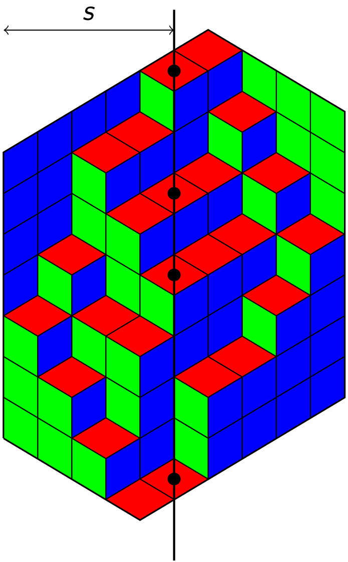

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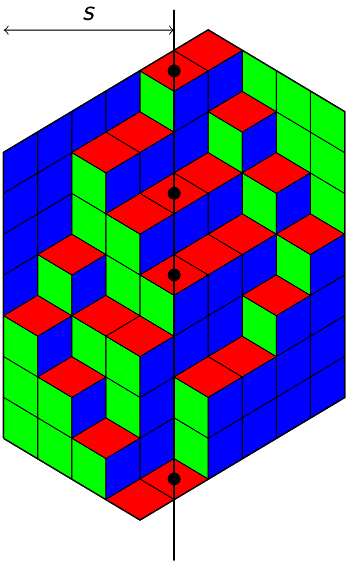

The centers $x_1 \ldots, x_n$ of the red tiles at a vertical line at distance s of the left side, have the jpdf on $\mathbb{N}^n$

$$\prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{j=1}^{n} w(x_j),$$

where w is the Hahn weight (with an appropriate choice of parameters). Johansson '02.

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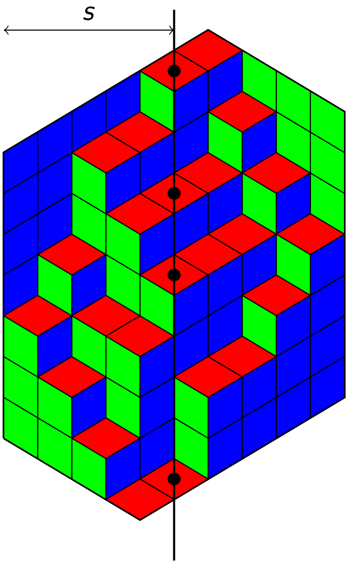

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Fluctuations:

$$X_n(f) - \mathbb{E}X_n(f) \to N(0, \sigma_f^2)$$

in distribution as $n \to \infty$.
(1-d section of the GFF+complex structure)

The previous two models (the Unitary Enembles and vertical sections of lozenge tilings of a hexagon) are both examples of orthogonal polynomial ensembles. We will show that the CLT's for these models are special cases of a more general CLT for such ensembles.

Orthogonal Polynomial Ensembles

Orthogonal polynomial ensembles

Let μ be a Borel measure on $\mathbb{R}$ with finite moments $\int |x|^k \mathrm{d}\mu(x) < \infty$. Then the *Orthogonal Polynomial Ensemble* of size n associated to μ is the probability measure on $\mathbb{R}^n$ proportional to

$$\sim \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \mathrm{d}\mu(x_1) \cdots \mathrm{d}\mu(x_n).$$

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We want to find minimal conditions on the measure μ such that we have

$$X_n(f) - \mathbb{E}X_n(f) \to N(0, \sigma(f)^2)) \qquad \text{as } n \to \infty,$$

for $f \in C^1(\mathbb{R})$.

Orthogonal polynomials

- We let $p_{k,n}$ be the orthogonal polynomial of degree k with respect to $\mu(= \mu_n)$, i.e.
$$\int p_{k,n}(x) p_{l,n}(x) \mathrm{d}\mu(x) = \delta_{kl}$$
- Three term recurrence
$$x p_{k,n}(x) = a_{k,n} p_{k+1,n}(x) + b_{k,n} p_{k,n}(x) + a_{k-1,n} p_{k-1,n}(x)$$
- Let J be the Jacobi operator corresponding to μ.
$$J = J^{(n)} = \begin{pmatrix} b_{0,n} & a_{0,n} & & \\ a_{0,n} & b_{1,n} & a_{1,n} & \\ & a_{1,n} & \ddots & \ddots \\ & & \ddots & \end{pmatrix}$$

Reproducing kernel

- The reproducing kernel is defined as

$$K_n(x, y) = \sum_{k=0}^{n-1} p_{k,n}(x) p_{k,n}(y)$$

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In other words, K_n as a operator on $\mathbb{L}_2(\mu)$ is an orthogonal projection, i.e. $K_n^* = K_n$ and $K_n^2 = K_n$, of rank n.

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- Christoffel-Darboux formula:

$$K_n(x,y) = a_{n,n} \frac{p_{n,n}(x) p_{n-1,n}(y) - p_{n,n}(y) p_{n-1,n}(x)}{x-y}.$$

Determinantal point process

The Orthogonal Polynomial Ensemble of size n associated to a Borel measure μ is a determinantal point process on $\mathbb{R}$ with kernel K_n and reference measure μ.

This means that

$$\mathbb{E}\left[\exp tX_n(f)\right] = \det\left(I + (\mathrm{e}^{tf} - 1)K_n\right)_{\mathbb{L}_2(\mu)}$$

where the right-hand side is the Fredholm determinant for operator with integral kernel

$$(\mathrm{e}^{tf(x)} - 1)K_n(x,y)$$

acting on $\mathbb{L}_2(\mu)$.

A first question

We recall that we ask whether we have a Central Limit Theorem

$$X_n(f) - \mathbb{E}X_n(f) \xrightarrow{?} N(0, \sigma(f)^2)$$

where

$$X_n(f) = \sum_{j=1}^{n} f(x_j),$$

where f is sufficiently smooth and $x_1, \ldots, x_n$ are from the OPE of size n for a Borel measure $\mu(= \mu_n)$ on $\mathbb{R}$.

Question: can we expect this to hold for general measures μ? Do we have the correct normalization or does the normalization depend on regularity properties of μ?

Concentration inequality

Theorem (Breuer-D '13)

For a bounded function f we have

$$\left|\mathbb{E}\left[\mathrm{e}^{t(X_n(f)-\mathbb{E}X_n(f))}\right]\right| \leq \exp(A|t|^2 \operatorname{Var} X_n(f)), \quad |t| \leq \frac{1}{3\|f\|_\infty}$$

where A is a universal constant. And hence

$$\mathbb{P}\left(|X_n(f) - \mathbb{E}X_n(f)| \geq \varepsilon\right) \leq 2\exp\left(-\min\left(\frac{\varepsilon^2}{4A \operatorname{Var} X_n(f)}, \frac{\varepsilon}{6\|f\|_\infty}\right)\right)$$

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The theorem holds for any determinantal process that has a kernel K_n that is an orthogonal projection, i.e. $K_n = K_n^2$ and $K_n^* = K_n$.

Concentration inequality

- There is always the crude bound $\mathrm{Var} X_n(f) = n\|f\|_\infty^2$. With the concentration inequality this implies

$$\frac{1}{n^{1/2+\varepsilon}} \left(X_n(f) - \mathbb{E} X_n(f) \right) \to 0,$$

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- No conditions on μ! This holds for general DPP with orthogonal projection as a kernel.
- The bound on the variance can be improved significantly for C^1 functions.

Computing the variance

Compute the variance of a linear statistic

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\begin{aligned}
&\mathrm{Var} X_n(f) \\
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&= \frac{1}{2}\iint \left(f(x)-f(y)\right)^2 K_n(x,y)^2 \mathrm{d}\mu(x)\mathrm{d}\mu(y)\\
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$$
= \frac{a_{n,n}^2}{2} \iint \left(\frac{f(x) - f(y)}{x-y}\right)^2 \left(p_n(x) p_{n-1}(y) - p_n(y) p_{n-1}(x)\right)^2 \mathrm{d}\mu(x) \mathrm{d}\mu(y)
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Lipschitz functions

If f is a Lipschitz function, i.e. $|f|_{\mathcal{L}} = \sup_{x,y} \left| \frac{f(x)-f(y)}{x-y} \right| < \infty$, then

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Hence if we have $a_{n,n} = \mathcal{O}(1)$ as $n \to \infty$, then the variance remains bounded. Pastur '06

Conclusion: if f is Lipschitz and $a_{n,n} = \mathcal{O}(1)$ as $n \to \infty$, then

$$X_n(f) - \mathbb{E} X_n(f) \sim 1,$$

and we have exponential tails uniformly in n.

Polynomial test functions

Recall the formula for the variance:

$$\mathrm{Var} X_n(f)$$
$$= \frac{a_{n,n}^2}{2} \iint \left(\frac{f(x) - f(y)}{x - y} \right)^2 (p_n(x) p_{n-1}(y) - p_n(y) p_{n-1}(x))^2 \mathrm{d}\mu(x) \mathrm{d}\mu(y)$$

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If is a polynomial, the variance can be written in terms of *finitely many* recurrence coefficients $a_{n+k,n}$ and $b_{n+k,n}$.

Polynomial test functions

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If is a polynomial, the variance can be written in terms of *finitely many* recurrence coefficients $a_{n+k,n}$ and $b_{n+k,n}$.

In particular,

For $k \in \mathbb{Z}$ we have that $a_{n+k,n}$ and $b_{n+k,n}$ have a limit as $n \to \infty$

$$\Longrightarrow$$

$VarX_n(f)$ has a limit for any polynomial f

CLT for Orthogonal Polynomial Ensembles

Theorem (Breuer-D '13)

Assume that for any $k \in \mathbb{Z}$ we have

$$a_{n+k,n} \to a \qquad b_{n+k,n} \to b$$

as $n \to \infty$. Then, if f is a polynomial with real coefficients,

$$X_n(f) - \mathbb{E}X_n(f) \to N\left(0, \sum_{k=1}^{\infty} k|\hat{f}_k|^2\right)$$

where

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(2a\cos\theta + b)\mathrm{e}^{-\mathrm{i}\theta k}\mathrm{d}\theta.$$

CLT for Orthogonal Polynomial Ensembles

Theorem (Breuer-D '13)

Assume that there exists a subsequence $\{n_j\}_j$ such that for any $k \in \mathbb{Z}$ we have

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Central Limit Theorems for C^1 functions

Theorem (Breuer-D' 13)

Suppose there is a compact $E \subset \mathbb{R}$ such that for $k \in \mathbb{N}$ we have

$$\int_{\mathbb{R}\setminus E} |x|^k K_n(x,x)\mathrm{d}\mu(x) = o(1/n),$$

as $n \to \infty$. Then the CLT holds for any $f \in C^1(\mathbb{R})$ such that $|f(x)| \leq C(1+|x|^k)$ for some $C > 0$ and $k \in \mathbb{N}$.

Corollary

If μ is non-varying and has compact support $S(\mu)$, then the CLT holds for any $f \in C^1(S(\mu))$.

Examples 1a and 1b

- Unitary Ensembles. Absolutely continuous measure

$$\mathrm{d}\mu_n(x) = \mathrm{e}^{-nV(x)}\mathrm{d}x.$$

  with $S(\mu_V) = [\gamma, \delta]$.

- Lozenge tiling of the hexagon. Discrete measure:

$$\mathrm{d}\mu = \sum_{x=0}^{N} \binom{\alpha + x}{x} \binom{\beta + N - x}{N - x} \delta_x.$$

  This called the *Hahn* weight and the orthogonal polynomials are the Hahn polynomials.

In both cases one can prove that, with given parameters, the recurrence coefficients have limits and the CLT follows from the general CLT.

Example 2

The following result for the non-varying situation follows by combining the CLT with a theorem by Denissov and Rakhmanov.

Theorem

Fix

$$d\mu = w(x)dx + d\mu_{sing},$$

and assume that $S_{ess}(\mu) = [\gamma, \delta]$ *and* $w(x) > 0$ *Lebesgue a.e. on* $[\gamma, \delta]$. *Then for any* $f \in C^1(\mathbb{R})$ *we have*

$$X_n(f) - \mathbb{E}X_n(f) \to N(0, \sum_{k=1}^{\infty} k|f_k|^2)$$

where

$$\hat{f}_k = \frac{1}{2\pi i}\int_0^{2\pi} f\left(\frac{\delta-\gamma}{2}\cos\theta + \frac{\delta+\gamma}{2}\right)e^{-ik\theta}d\theta.$$

Example 3: different CLT's for different subsequences!

Partition $\mathbb{N}$ into blocks $A_1, B_2, A_2, B_2, \ldots$ such that

$$|A_j| = 2^{j^2}, \qquad |B_j| = 3^{j^2}.$$

and set $b_n = 0$ and

$$a_n = \begin{cases} 1, & n \in A_j \\ \frac{1}{2}, & n \in B_j \end{cases}$$

Then the spectral measure μ for J is supported on $[-2, 2]$ and is singular with respect to the Lebesgue measure.

For the OPE for μ one can show that

$$\frac{1}{n} X_n(f) \to \int f(x) \mathrm{d}\mu_{eq}(x)$$

where μ_{eq} is the equilibrium measure for the interval $[-2, 2]$.

Example 3: different CLT's for different subsequences!

Hence the exists subsequence n_j such that

$$X_{n_j}(f) - \mathbb{E}X_{n_j}(f) \to N(0, \sum_{k=1}^{\infty} k|\hat{f}_k|^2)$$

where

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(\cos\theta)\mathrm{e}^{-\mathrm{i}k\theta}\mathrm{d}\theta.$$

There also exists subsequences n_j' such that

$$X_{n_j'}(f) - \mathbb{E}X_{n_j'}(f) \to N(0, \sum_{k=1}^{\infty} k|\tilde{f}_k|^2)$$

where

$$\tilde{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(2\cos\theta)\mathrm{e}^{-\mathrm{i}k\theta}\mathrm{d}\theta.$$

Example 4

Example 3 can be modified such that the spectral measure is absolutely continuous on $[-1,1]$ and purely singular on $[-2,2] \setminus (-1,1)$ and $b_n = 0$ and the a_n are such that for each $a \in [\frac{1}{2}, 1]$ there exists a subsequence $\{n_j(a)\}_j$ such that

$$\forall k \in \mathbb{Z} \quad a_{n_j(a)+k} \to a, \qquad \text{as } j \to \infty.$$

Hence for each a there is an CLT for a special subsequence and each CLT is different.

We stil have

$$\lim_{n\to\infty} \frac{1}{n} X_n(f) = \int f(x) \mathrm{d}\mu_{eq}(x)$$

where μ_{eq} is the equilibrium measure for the interval $[-2,2]$.

Right Limits

Some notation

- Let J be the Jacobi operator corresponding to μ.

$$J = J^{(n)} = \begin{pmatrix} b_{0,n} & a_{0,n} & & \\ a_{0,n} & b_{1,n} & a_{1,n} & \\ & a_{1,n} & \ddots & \ddots \\ & & \ddots & \end{pmatrix}$$

Tri-diagonal semi-infinite. Since we allow varying measures, we allow J to depend on n and write $J = J^{(n)}$.

- Given a Laurent polynomial $s(z) = \sum_{j=-q}^{p} s_j z^j$, the Laurent matrix/operator $L(s)$ and Toeplitz matrix/operator $T(s)$ are defined as

$$(L(s))_{k,l} = s_{k-l}, \quad \forall k, l \in \mathbb{Z}.$$
$$(T(s))_{k,l} = s_{k-l}, \quad \forall k, l \in \mathbb{N}.$$

Right limits

Definition

A two-sided infinte matrix J^R is call a right limit of $\{J^{(n)}\}_n$ iff there exists a subsequence $\{n_j\}_j$ such that

$$\left(J^R\right)_{k,l} = \lim_{j\to\infty} J^{(n_j)}_{n_j+k,n_j+l}, \quad \forall k,l \in \mathbb{Z}.$$

Example: there exists a subsequence n_j such that for all k we have $a_{n_j+k,n_j} \to a$ and $b_{n_j+k,n_j} \to b$ if and only if $L(az+b+a/z)$ is a right limit of $\{J^{(n)}\}_n$.

Right limits have proved to be very useful when studying the spectra of Jacobi operators Last-Simon '99, Last-Simon '06, Remling '11

Laurent matrices as right limits

In this way, we can reformulate the CLT for OPE:

Theorem (Breuer-D '13)

Suppose that $L(s)$ is a right limit of $J^{(n)}$ with subsequence $\{n_j\}_j$. Then, for any polynomial f with real coefficients,

$$X_{n_j}(f) - \mathbb{E}X_{n_j}(f) \to N\left(0, \sum_{k=1}^{\infty} k|\hat{f}_k|^2\right),$$

where

$$\hat{f}_k = \frac{1}{2\pi i}\oint_{|z|=1} f(s(z))\frac{dz}{z^{k+1}}.$$

A general limit theorem

Theorem

Let J^R *be a right limit of* J *with subsequence* $\{n_j\}_j$*. Define*

$$\left(J_M^R\right)_{kl} = \begin{cases} (J^R)_{kl}, & k,l = -M,\ldots,M, \\ 0, \ \textit{otherwise} \end{cases}.$$

Let P_- *be the projection on the negative coefficients* $(P_- x)_k = \begin{cases} x_k & k < 0 \\ 0 & k \geq 0. \end{cases}$*. Then for any polynomial* f *we have*

$$\lim_{j\to\infty} \mathbb{E}\left[\exp t(X_{n_j}(f) - \mathbb{E}X_{n_j}(f))\right]$$
$$= \lim_{M\to\infty} \mathrm{e}^{t \operatorname{Tr} P_- f(J_M^R)} \det\left(I + P_-(\mathrm{e}^{tf(J_M^R)} - I)P_-\right)$$

In particular, both limits exist.

A general limit theorem

If J^R is a Laurent matrix, the right-hand side can be computed giving the previous CLT.

It is an interesting open problem to find the limiting value at the right-hand side if J^R is, for example, two-periodic. This should of course match with the formulas for the multi-cut case by Shcherbina '13, Borot-Guionnet '13

Biorthogonal ensembles

Biorthogonal ensembles

Let μ be a Borel measure on $\mathbb{R}$ and $\{\phi_j\}$,$\{\psi_j\}$ two families of linearly indendent functions. Then the corresponding *biorthogonal ensemble of size n* is defined as the probability measure on $\mathbb{R}^n$ proportional to

$$\sim \det\left(\phi_{k-1}(x_j)\right)_{j,k=1}^n \det\left(\psi_{k-1}(x_j)\right)_{j,k=1}^n \mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n)$$

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By taking linear combination of the rows we can assume that

$$\int_{\mathbb{R}} \phi_k(x)\psi_l(x)\mathrm{d}\mu(x) = \delta_{kl}.$$

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The biorthogonal ensembles is a determinantal point process on $\mathbb{R}$ with kernel

$$K_n(x,y) = \sum_{k=0}^{n-1} \phi_j(x)\psi_j(y)$$

and reference measure μ.

Biorthogonal ensembles with a recurrence

- We will assume that there exists a banded one-sided infinite matrix J such that

$$x \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \vdots \end{pmatrix} = J \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \vdots \end{pmatrix}$$

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- We will allow that J varies with n, i.e. $J = J^{(n)}$, in which case we assume that the number of bands is independent of n.

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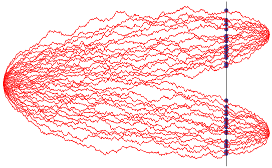
  Since J is banded this is a recurrence.
- We will allow that J varies with n, i.e. $J = J^{(n)}$, in which case we assume that the number of bands is independent of n.
- There many interesting biorthogonal ensembles that admit such a recurrence. In particular when we are concerned Multiple Orthogonal Polynomials Ensembles.

Example: unitary ensembles with external source

- Consider the eigenvalues of a $n \times n$ Hermitian matrix randomly chosen from
$$\frac{1}{Z_n} \mathrm{e}^{-c\,\mathrm{Tr}(M^2 - AM)} \mathrm{d}M$$
where $c > 0$ and A is a diagonal matrix with eigenvalues $a_1, \ldots, a_m$ of multiplicities $k_1, \ldots, k_m$. Brezin-Hikami '97.
- Alternatively, consider n non-intersecting brownian paths, with k_j paths conditioned to start at 0 a time $t = 0$ and end at b_j at time $t = 1$.



Location of paths at time t has the same distribution as the eigenvalues of M with $a_j = 2tb_j$ and $c = 1/t(1-t)$.

The eigenvalues form a biorthogonal ensemble constructed out of multiple Hermite polynomials with m weights. These polynomials satisfy an $m+2$-term recurrence. Bleher-Kuijlaars '04

Example: two matrix model

- Consider the eigenvalues of two $n \times n$ Hermitian matrices (M_1, M_2) randomly chosen from

$$\frac{1}{Z_n} \mathrm{e}^{-\operatorname{Tr}(V(M_1)+W(M_2)-\tau M_1 M_2)} \mathrm{d}M_1 \mathrm{d}M_2$$

where V and W are polynomials such that the integral converges.

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  where V and W are polynomials such that the integral converges.
- In Eynard-Mehta '98 it was shown that the eigenvalue correlations have determinantal struture with kernels constructed out of the biorthogonal polynomials p_j and q_k of degree j and k such that

$$\iint p_j(x) q_k(y) \mathrm{e}^{-V(x)-W(y)+\tau xy} \mathrm{d}x \mathrm{d}y = \delta_{jk}.$$

Example: two matrix model

- The eigenvalues of M_1, when averaged over M_2, form a biorthogonal ensembles with

$$\phi_j(x) = p_{j-1}(x) \qquad \psi_j(x) = \int q_{j-1}(y) \mathrm{e}^{-(V(x)+W(y)-\tau xy))} \mathrm{d}y.$$

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- The polynomials p_j satisfy a $d_W + 1$-term recurrence, where d_W is the degree of W. Bertola-Eynard-Harnad '02

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- In D-Kuijlaars '08, D-Kuijlaars-Mo '12, D-Geudens-Kuijlaars '11, D-Geudens '13 the asymptotics have been computed for

$$W(y) = y^4/4 + \alpha y^2/2 \text{ and } V \text{ even,}$$

using Riemann-Hilbert methods, from which we can compute the recurrence coefficients.

CLT for biorthogonal ensembles

Theorem (Breuer-D '13)

Assume that $L(s)$ is a right limit of $J^{(n)}$ with subsequence $\{n_j\}_j$ for some $s(z) = \sum_{j=-q}^{p} s_j z^j$. Then, if f is a polynomial with real coefficients,

$$X_{n_j}(f) - \mathbb{E}X_{n_j}(f) \to N\left(0, \sum_{k=1}^{\infty} k|\hat{f}_k|^2\right),$$

as $j \to \infty$, where

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Note: for orthogonal polynomial ensembles $p = q = 1$.

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Note: for orthogonal polynomial ensembles $p = q = 1$.

In case $K_n^* \neq K_n$ then we have no concentration inequality and hence extension to $f \in C^1(\mathbb{R})$, without more a priori knowledge.

A general limit theorem

Theorem

Let J^R be a right limit of J with subsequence $\{n_j\}_j$. Define

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In particular, both limits exist.

Some words about the proofs

Cumulant expansion

Expand the moment-generating function

$$\begin{aligned}\mathbb{E}\left[\exp tX_n(f)\right]) &= \det(1+(\mathrm{e}^{tf}-1)K_n) \\ &= \exp \operatorname{Tr}\log\Big(1+(\mathrm{e}^{tf}-1)K_n\Big) \\ &= \exp\left(\sum_{m=1}^{\infty} t^m C_m^{(n)}(f)\right)\end{aligned}$$

where

$$C_m^{(n)}(f) = \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i\geq 1} \frac{\operatorname{Tr} f^{l_1} K_n \cdots f^{l_j} K_n}{l_1! \cdots l_j!}$$

The are called the cumulants.

For a Central Limit Theorem one needs to show that

$$\lim_{n\to\infty} C_m^{(n)}(f) \to \begin{cases} 2\sigma(f)^2, & m = 2 \\ 0, & m \geq 3. \end{cases}$$

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A first problem: each term in the sum

$$C_m^{(n)}(f) = \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i\geq 1} \frac{\operatorname{Tr} f^{l_1} K_n \cdots f^{l_j} K_n}{l_1! \cdots l_j!}$$

grows linearly with n. Clearly, some very effective cancelation must come into play!

By using the identity $x = \log\left(1 + (\mathrm{e}^x - 1)\right)$ and expanding the right-hand side we obtain

$$\sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i \geq 1} \frac{1}{l_1! \cdots l_j!} = 0$$

if $m \geq 2$.

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Hence

$$C_m^{(n)}(f) = \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i \geq 1} \frac{\operatorname{Tr} f^{l_1} K_n \cdots f^{l_j} K_n - \operatorname{Tr} f^m K_n}{l_1! \cdots l_j!}$$

for $m \geq 2$.

Now each term in the double sum can be proved to be bounded and this captures a first cancelation.

Concentration inequality

Using $K_n^2 = K_n$ and $K_n^* = K_n$ one can show that

$$\left| \operatorname{Tr} f^{l_1} K_n \cdots f^{l_j} K_n - \operatorname{Tr} f^m K_n \right| \leq j m^2 \|f\|_\infty^{m-2} \|[f, K_n]\|_2^2, \qquad m \geq 2$$

where $\|[f, K_n]\|_2$ is the Hilbert-Schmidt norm for the commutator of K_n anf f (viewed as a multiplication operator).

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If $K_n^* = K_n$ then

$$\|[f, K_n]\|_2^2 = 2 \mathrm{Var} X_n(f)$$

and the mentioned concentration inequality Breuer-D' 13 follows after a further arranging of terms.

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and the mentioned concentration inequality Breuer-D' 13 follows after a further arranging of terms.

For a CLT we need to capture another cancelation.

Use orthogonality to rewrite

$$C_m^{(n)}(f) = \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i\geq 1} \frac{\operatorname{Tr} f^{l_1} K_n \cdots f^{l_j} K_n - \operatorname{Tr} f^m K_n}{l_1! \cdots l_j!}$$

$$= \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i\geq 1} \frac{\operatorname{Tr} f(J)^{l_1} P_n \cdots f(J)^{l_j} P_n - \operatorname{Tr} f(J)^m P_n}{l_1! \cdots l_j!}$$

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$$C_m^{(n)}(f) = \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i \geq 1} \frac{\operatorname{Tr} f^{l_1} K_n \cdots f^{l_j} K_n - \operatorname{Tr} f^m K_n}{l_1! \cdots l_j!}$$

$$= \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i \geq 1} \frac{\operatorname{Tr} f(J)^{l_1} P_n \cdots f(J)^{l_j} P_n - \operatorname{Tr} f(J)^m P_n}{l_1! \cdots l_j!}$$

By using the band structure of J (and hence of $f(J)$ for polynomial f) it is not hard to prove that

$$\operatorname{Tr} f(J)^{l_1} P_n \cdots f(J)^{l_j} P_n - \operatorname{Tr} f(J)^m P_n$$

only depends on a finite and fixed number of recurrence coefficients.

Use orthogonality to rewrite

$$C_m^{(n)}(f) = \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i \geq 1} \frac{\operatorname{Tr} f^{l_1} K_n \cdots f^{l_j} K_n - \operatorname{Tr} f^m K_n}{l_1! \cdots l_j!}$$

$$= \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i \geq 1} \frac{\operatorname{Tr} f(J)^{l_1} P_n \cdots f(J)^{l_j} P_n - \operatorname{Tr} f(J)^m P_n}{l_1! \cdots l_j!}$$

By using the band structure of J (and hence of $f(J)$ for polynomial f) it is not hard to prove that

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only depends on a finite and fixed number of recurrence coefficients.

This does not prove the desired asymptotic behavior, but it does show that only a relatively small part of J matters in the fluctuation.

Comparison/Universality principle

If J_1 and J_2 have the same right limit J^R along the same subsequence $\{n_j\}$, then the leading term in the asympotic behavior of the fluctuations for the linear statistics for polynomial f along that subsequence is the same.

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Hence if J has a Laurent matrix $L(s)$ as a right limit, then we can assume that $J = T(s)$ from the start! Here $T(s)$ is the Toeplitz operator with symbol s. Hence we only need to compute the CLT for the special Toeplitz case and the rest follows by the comparison principle.

Lemma (Breuer-D' 13)

Let $s(z)=\sum_{j=-q}^{p} s_j z^j$ be a Laurent polynomial. Then

$$\lim_{n\to\infty} \det\left(I+P_n(\mathrm{e}^{T(s)}-I)P_n\right)\mathrm{e}^{-\operatorname{Tr} P_n T(s)} = \exp\left(\frac{1}{2}\sum_{k=1}^{\infty} k s_k s_{-k}\right)$$

Proof is based on Ehrhardt's generalization of the Helton-Howe-Pincus formula: if $[A,B]$ is trace class, then

$$\det \mathrm{e}^{-A}\mathrm{e}^{A+B}\mathrm{e}^{-B} = \exp\frac{1}{2}\operatorname{Tr}[A,B].$$

(This beautiful formula essentially captures the final cancelations in the cumulant expansion)

Thank you for your attention!