Constant Mean Curvature Tori in $\mathbb{R}^3$ and S^3

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Compact constant mean curvature surfaces (soap bubbles) are critical points for the area functional under variations which preserve the enclosed volume.

Question (Hopf)

Is an immersed compact constant mean curvature surface in $\mathbb{R}^3$ necessarily a round sphere?

Theorem (Hopf, 1951)

If $f : S^2 \to \mathbb{R}^3$ is an immersion with constant mean curvature, then $f(S^2)$ is a round sphere.

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These were generally thought to be the only soap bubbles until 1984, when Wente constructed

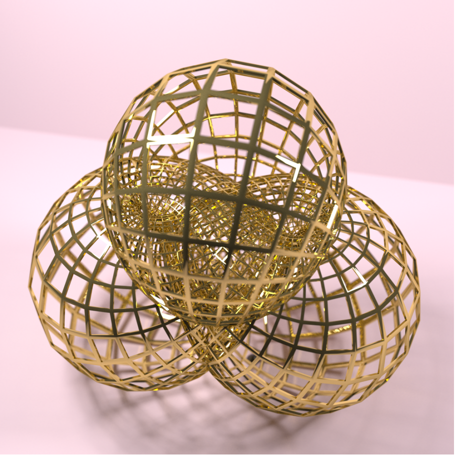

Figure : Wente torus

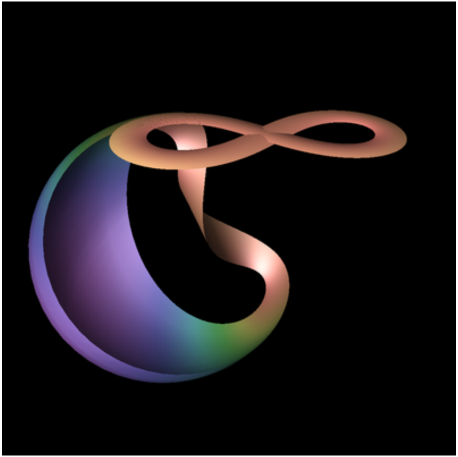

Figure : generating loops: i.e. why this is a torus

In fact,

Theorem (Kapouleas, 1987)

There are immersed compact mean curvature surfaces (CMC) in $\mathbb{R}^3$ *of every genus.*

We will concentrate on tori/planes, as they have an interesting link with algebraic geometry.

The fundamental observation (Pohlmeyer, Uhlenbeck) that links CMC surfaces with integrable systems is that for a simply-connected coordinate neighbourhood $U \subset \Sigma^2$,

$$\text{CMC immersion } f : U \to \mathbb{R}^3$$
$$\leftrightarrow$$
$$C^\times\text{-family of flat } SL(2,\mathbb{C}) \text{ connections } d + \phi_\lambda \text{ on } U \times \mathbb{C}^2,$$
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Identifying $\mathbb{R}^3$ with $su(2)$ via

$$e_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

a moving frame $F : \mathbb{R}^2 \to SU(2)$ for $f : \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$\mathrm{Ad}_F e_1 = \frac{f_x}{|f_x|}, \qquad \mathrm{Ad}_F e_2 = \frac{f_y}{|f_y|}, \qquad \mathrm{Ad}_F e_3 = N.$$

Bonnet Theorem in terms of H

For a conformal immersion $f : \mathbb{R}^2 \to \mathbb{R}^3$, with metric $4e^{2u} dz d\bar{z}$, the first and second fundamental forms are

$$I = \begin{pmatrix} 4e^{2u} & 0 \\ 0 & 4e^{2u} \end{pmatrix}, \qquad II = \begin{pmatrix} 4He^{2u} + Q + \bar{Q} & i(Q - \bar{Q}) \\ i(Q - \bar{Q}) & 4He^{2u} - (Q + \bar{Q}) \end{pmatrix}$$

where $H =$ mean curvature and $Q = \langle f_{zz}, N \rangle$.

$$Qdz^2 = \text{Hopf differential} = \text{trace-less part of (complexified) } II.$$

Theorem (Bonnet)

Given $4e^{2u} dz d\bar{z}$, Qdz^2 and H on $\mathbb{R}^2$ satisfying the Gauss-Codazzi equations, there is a conformal immersion $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that these are the metric, Hopf differential and mean curvature. This immersion is unique up to Euclidean motions.

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When H is constant, the Gauss-Codazzi equations are unchanged by $Q \mapsto \lambda Q$ for $\lambda \in S^1$, giving a one-parameter family of CMC surfaces. Explicitly

$$\phi_\lambda = \frac{1}{2}\begin{pmatrix} u_z & -2He^u\lambda^{-1} \\ Qe^{-u}\lambda^{-1} & -u_z \end{pmatrix} dz + \frac{1}{2}\begin{pmatrix} -u_{\bar z} & -\bar{Q}e^{-u}\lambda \\ 2He^u\lambda & u_{\bar z} \end{pmatrix} d\bar z.$$

Allowing $\lambda \in \mathbb{C}^\times$, $\phi_\lambda = F_\lambda^{-1} dF_\lambda$ if and only if

$$d\phi_\lambda + [\phi_\lambda, \phi_\lambda] = 0 \quad \forall \lambda \in \mathbb{C}^\times \text{ (Maurer-Cartan equation).}$$

The Maurer-Cartan equation states that the connections $d_\lambda = d + \phi_\lambda$ (in the trivial bundle) are all **flat**.

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We have described a CMC immersion $f : \mathbb{R}^2 \to \mathbb{R}^3$ in terms of a family of flat $SL(2, \mathbb{C})$ connections $d + \phi_\lambda$ on $\mathbb{R}^2 \times \mathbb{C}^2$.
What can you do with a family of flat connections?

- can always study parallel sections

$$dA_\lambda(z) = [A_\lambda(z), \phi_\lambda(z)]$$

- if had $f : T^2 = \mathbb{R}^2/\Lambda \to \mathbb{R}^3$, can consider holonomy

$$H_\lambda^z : \pi_1(T^2, z) \to SL(2, \mathbb{C}).$$

$\pi_1(T^2, z)$ is abelian so the holonomy representation has common eigenspaces E_z.

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Theorem (Hitchin' 90, Pinkall-Sterling' 89)

Define a polynomial $a(\lambda)$ *by*
λ_0 *is a zero of* a *of order* $n \Leftrightarrow$ *the two eigenlines of* H_{λ_0} *agree to order* n.
The curve

$$y^2 = \lambda a(\lambda)$$

completes to a (finite genus) algebraic curve X_a, *called the spectral curve of* f.

For each $z \in T^2$ there is a line bundle E_z on X_a given by the eigenlines of H^z_λ and the map

$$z \mapsto E_z \otimes E_0^{-1} : T^2 \to \mathrm{Jac}(X_a)$$

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Theorem (Hitchin, Pinkall-Sterling, Bobenko)

There is an explicit 1-1 correspondence

$$\text{CMC tori } f : T^2 \to \mathbb{R}^3$$
$$\updownarrow$$
$$\text{spectral curve data, consisting of}$$

- *a hyperelliptic curve* X
- *marked points* $P_0, P_\infty \in X$
- *a line bundle* E_0 *on* X *of degree* $g+1$

satisfying certain symmetries and periodicity conditions.

This provides a linearisation of the equations for a constant mean curvature torus.

Omitting the periodicity conditions yields CMC planes; instead use parallel sections $A_\lambda(z)$ which, in the case of tori, commute with the holonomy and hence give the same curve.

Not all CMC planes arise from algebraic (finite genus) spectral curves those which do are called finite-type.

In particular, the above theorem says that all CMC tori in $\mathbb{R}^3$ are of finite-type.

There is very similar spectral curve correspondence for CMC immersions $\mathbb{R}^2 \to S^3$ of finite-type.
Again all the CMC tori are of finite-type and they are characterised by their spectral data satisfying periodicity conditions.
We would like to understand the moduli spaces of CMC tori in $\mathbb{R}^3$ and S^3, in particular:

Question

1. *Can one deform these tori, at least infinitesimally, and if so what is the dimension of the space of deformations?*
2. *How common are the CMC tori amongst CMC planes of finite-type?*

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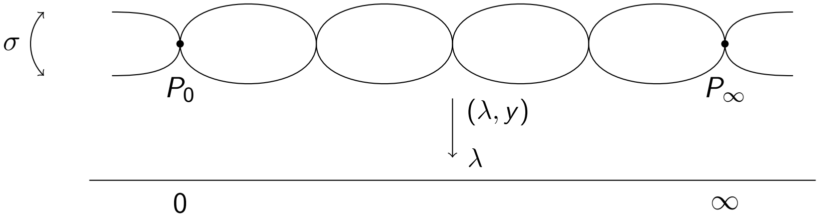
The line bundle E_0 is chosen from a real g-dimensional space, giving at least g deformation parameters. So it is better to ask:

Question

1. *Can one deform spectral curves of tori, at least infinitesimally, and if so what is the dimension of the space of deformations?*
2. *How common are the spectral curves of CMC tori amongst spectral curves of CMC planes of finite-type?*

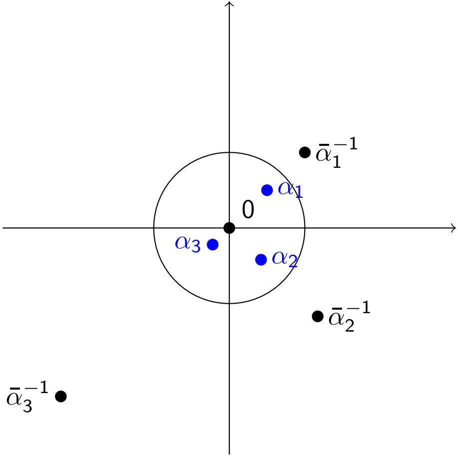
Writing the hyperelliptic curve X_a as $y^2 = \lambda a(\lambda)$, we have

- the hyperelliptic involution $\sigma : (\lambda, y) \mapsto (\lambda, -y)$



- an anti-holomorphic involution ρ without fixed points covering $\lambda \mapsto \bar{\lambda}^{-1}$

$$\rho : (\lambda, y) \mapsto (\bar{\lambda}^{-1}, \bar{y}\bar{\lambda}^{-g-1}).$$



Writing $\rho^* a$ to mean $\lambda^{\deg a} a(\bar{\lambda}^{-1})$,

$$\overline{\rho^* a} = a \qquad\qquad \text{reality condition.}$$

We consider the space $\mathcal{H}^g$ of smooth spectral curve data (X_a, λ) of genus g as an open subset of $\mathbb{R}^{2g}$, given by $(\alpha_1, \ldots, \alpha_g)$, where

$$X_a : \quad y^2 = \lambda a(\lambda) = \lambda \prod_{i=1}^{g} \frac{(\lambda - \alpha_i)(1 - \overline{\alpha_i}\lambda)}{|\alpha_i|}.$$

Periodicity

In the case of tori, let $\mu_1(\lambda, y)$, $\mu_2(\lambda, y)$ be the eigenvalues of the holonomy H^z_λ about the 2 generators of $\pi_1(T^2, \mathbb{Z})$.

Then $\Theta_1 = d\log\mu_1$ and $\Theta_2 = d\log\mu_2$ are meromorphic differentials with

- no residues
- double poles at P_0, P_∞
- $\sigma^*\Theta = -\Theta$, $\rho^*\Theta = -\overline{\Theta}$
- periods in $2\pi\sqrt{-1}\mathbb{Z}$.

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In general, let

$$\mathcal{B}_a = \left\{ \begin{array}{c} \text{meromorphic differentials } \Theta \text{ with} \\ \text{no residues, double poles at } P_0, P_\infty, \\ \sigma^*\Theta = -\Theta,\ \rho^*\Theta = -\overline{\Theta} \text{ and having} \\ \text{purely imaginary periods} \end{array} \right\}.$$

$\mathcal{B}_a$ is a real 2-plane.

We obtain a real-analytic rank two vector bundle

$$\mathcal{B} \to \mathcal{H}^g$$

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Periodicity Conditions for CMC $T^2 \to \mathbb{R}^3$

$X_a \in \mathcal{H}^g$ gives a doubly-periodic CMC immersion into $\mathbb{R}^3$

$\Leftrightarrow$

there exists a frame (Θ_1, Θ_2) of $\mathcal{B}_a$ such that

1. their periods lie in $2\pi\sqrt{-1}\mathbb{Z}$ ($\Leftrightarrow \Theta_1 = d\log\mu_1, \Theta_2 = d\log\mu_2$)
2. for some $\lambda_0 \in S^1$, called the Sym point
   (a) for γ a curve in X connecting the two points in $\lambda^{-1}(\lambda_0)$,
   $$\int_\gamma \Theta_1, \int_\gamma \Theta_2 \in 2\pi\sqrt{-1}\mathbb{Z}$$
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Deformations of CMC Tori

For spectral curves of CMC tori in $\mathbb{R}^3$,

$$\# \text{ free parameters} = \# \text{ periodicity conditions.}$$

A CMC torus has $\mathbb{R}^g$ deformations, all isospectral.

A CMC torus in S^3 has $\mathbb{R}^g$ isospectral deformations.

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CMC Tori in S^3

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Theorem (—–, Schmidt)

For each $\lambda_1 \neq \lambda_2 \in S^1$, $\mathcal{P}^g(\lambda_1, \lambda_2)$ is dense in $\mathcal{H}^g$.
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CMC tori in $\mathbb{R}^3$

Theorem (Ercolani–Knörrer–Trubowitz '93, Jaggy '94)

For every $g > 0$, there exist CMC tori of spectral genus g. There are at least countably many spectral curves of genus g satisfying the periodicity conditions.

In the Euclidean case, CMC tori are not dense amongst CMC planes of finite type.
Writing

$$\mathcal{P}^g_{\lambda_0} = \{X_a \in \mathcal{H}^g \mid X_a \text{ is a spectral curve of a CMC torus with Sym point } \lambda_0\},$$

the closure of $\mathcal{P}^g_{\lambda_0}$ is contained in the real subvariety

$$\mathcal{S}^g_{\lambda_0} = \{X_a \in \mathcal{H} \mid \text{ all } \Theta \in \mathcal{B}_a \text{ vanish at } \lambda_0\},$$

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$$\mathcal{S}^g_{\lambda_0} = \{X_a \in \mathcal{H} \mid \text{ all } \Theta \in \mathcal{B}_a \text{ vanish at } \lambda_0\},$$

which has codimension at least 2.

The set

$$\begin{aligned}\mathcal{S}^g &= \bigcup_{\lambda_0 \in S^1} \mathcal{S}^g_{\lambda_0} \\ &= \{X_a \in \mathcal{H} \mid \text{ all } \Theta \in \mathcal{B}_a \text{ have a common root on } S^1\},\end{aligned}$$

which contains the closure of spectral curves of CMC tori, is in general not a subvariety.

However it is contained in the subvariety

$$\mathcal{R}^g = \{X_a \in \mathcal{H}^g \mid \text{ all } \Theta \in \mathcal{B}_a \text{ have a common root }\}.$$

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Intuitive Picture

Recall that for real varieties we may have smooth points of different dimension within the same irreducible component

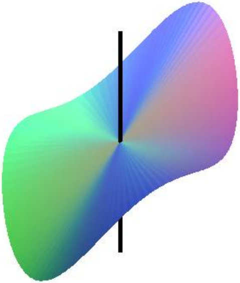

Figure : Cartan's Umbrella: $z(x^2 + y^2) = x^3$

$\mathcal{R}^g$ has smooth points of different dimensions, with those of the highest dimension (the "cloth") contained in $\mathcal{S}^g$.

Theorem (——, Schmidt)

For $X_a \in \mathcal{R}^g$, if

$$\dim_{X_a} \mathcal{R}^g = 2g - 1 \text{ (i.e. codimension 1 in } \mathcal{H}^g\text{)}$$

then X_a belongs to the closure of the spectral curves of constant mean curvature tori in $\mathbb{R}^3$. This closure is contained in $\mathcal{S}^g$.

The 2-plane $\mathcal{B}_a$ of differentials allows us to define new integer invariants of a CMC immersion $f : \mathbb{R}^2 \to \mathbb{R}^3$.

Taking a basis Θ_1, Θ_2 of $\mathcal{B}_a$, the function

$$h; = \frac{\Theta_1}{\Theta_2} : \mathbb{C}P^1 \to \mathbb{C}P^1$$

is well-defined up to Möbius transformations, and has degree

$$\deg(h) = g + 1 - \deg(\gcd(\mathcal{B}_a)).$$

On S^1, the function h is real-valued so

$$\tilde{h} = \frac{h+i}{h-i} : S^1 \to S^1$$

and we call the degree $\deg_{S^1} h$ of this map the real degree of h.

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Theorem (——, Schmidt)

Define $V_j = \{a \in \mathcal{H}^g \setminus \mathcal{S}^g \mid \deg_{S^1} \tilde{h} = j\}$. *Then for* $g \geq 1$*, the set* $\mathcal{H}^g \setminus \mathcal{S}^g$ *is the following union of non-empty, open and disjoint sets:*

$$\mathcal{H}^g \setminus \mathcal{S}^g = V_{1-g} \cup V_{3-g} \cup \ldots \cup V_{g-3} \cup V_{g-1}.$$

Theorem (——, Schmidt)

For $X_a \in \mathcal{R}^g$ the following statements are equivalent:

(1) $\dim_a \mathcal{R}^g = 2g - 1$ *(i.e. codimension 1 in $\mathcal{H}^g$)*

(2) *a belongs to the closure of at least two different V_j,* $j = 1-g, 3-g, \ldots, g-3, g-1$

(3) *a belongs to the closure of* $\{\tilde{a} \in \mathcal{H}^g \mid \deg(\gcd(\mathcal{B}_{\tilde{a}})) = 1\}$.

(4) *There exists $\lambda_0 \in S^1$ such that $a \in S^g_{\lambda_0}$ with* $\dim_a S^g_{\lambda_0} = 2g - 2$.

Moreover, if one of these equivalent conditions is satisfied than X_a belongs to the closure of the spectral curves of constant mean curvature tori in $\mathbb{R}^3$. This closure is contained in $\mathcal{S}^g$.

Period-Preserving Deformations

Let $\mathcal{F} \to \mathcal{H}^g$ denote the frame bundle of $\mathcal{B}$.

Elements of $\mathcal{F}$ are of the form (a, Θ_1, Θ_2).

Writing

$$\Theta_k := \frac{b_k(\lambda)d\lambda}{\lambda y}$$

we may represent the differentials Θ_k by polynomials b_k of degree $g+1$, real with respect to ρ.

Suppose we have a tangent vector $(\dot{a}, \dot{b}_1, \dot{b}_2) \in T_{(a,b_1,b_2)}\mathcal{F}$ which infinitesimally preserves the periods of the differentials Θ_1, Θ_2. Then

$$\dot{\Theta}_k = d\dot{q}_k \text{ for some meromorphic functions } \dot{q}_k \text{ on } X$$

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We may write

$$\dot{q}_k = \frac{ic_k(\lambda)}{y},$$

where c_k is a polynomial of degree $g+1$ which is real with respect to ρ. Equating partial derivatives

$$\frac{\partial}{\partial\lambda}\dot{q}_k d\lambda = \frac{\partial}{\partial t}\Theta_k$$

which expands to

$$(2\lambda a c_1' - a c_1 - \lambda a' c_1)i = 2a\dot{b}_1 - \dot{a}b_1, \tag{1}$$

$$(2\lambda a c_2' - a c_2 - \lambda a' c_2)i = 2a\dot{b}_2 - \dot{a}b_2, \tag{2}$$

where a dot denotes the derivative with respect to t, evaluated at $t=0$, whilst a prime means the derivative with respect to λ.

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where a dot denotes the derivative with respect to t, evaluated at $t=0$, whilst a prime means the derivative with respect to λ.

Computing c_2 (1) $- c_1$ (2) gives

$$2a\left(c_1'c_2\lambda - c_2'c_1\lambda + c_1\dot{b}_2 - c_1\dot{b}_1\right) = \dot{a}(c_1b_2 - c_2b_1).$$

so any roots of a at which $\dot{a}$ does not vanish are roots of $c_1b_2 - c_2b_1$. In fact from (1) and (2) the same is true at all roots of a so

$$c_1b_2 - c_2b_1 = Qa \tag{3}$$

with Q a polynomial of degree two, real with respect to ρ.

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A tangent vector to $\mathcal{F}$ at (a, Θ_1, Θ_2) which infinitesimally preserved periods defined

1. polynomials c_1, c_2 satisfying (1) and (2)
2. a quadratic polynomial $Q(\lambda)$ satisfying (3).

Conversely, given a quadratic polynomial $Q(\lambda)$ we try to

1. solve (3) for c_1, c_2
2. solve (1) and (2) for $(\dot{a}, \dot{b}_1, \dot{b}_2)$

Flowing along the resulting vector field preserves periods, a useful technique in proving the above results.

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