Quadratic families of elliptic curves and degree 1 conic bundles

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joint with **Massimiliano Mella**

Elliptic curves

$E := \left(y^2 = a_3x^3 + a_2x^2 + a_1x + a_0\right) \subset \mathbb{A}^2_{xy}$

Major problem 1: Find all solutions.
(over $\mathbf{Q}$ or number fields or ...)

Major problem 2: Are there infinitely many solutions?

Weak variant: There are "many" $(a_3, \ldots, a_0) \in K^4$ for which E has infinitely many solutions.

Families of elliptic curves

- $a_i(t) \in K[t]$ polynomials
- family of elliptic curves

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– *nontrivial family:*
at least two of the curves E_t are smooth, elliptic and not isomorphic to each other over K.

– Equivalent to:
discriminant is not identically 0 and
not all the $a_i(t)$ are multiples of the same square.

Main Corollary

– K *a number field,*
– $a_i(t) \in K[t]$ *polynomials of degree 2,*
– *nontrivial family. Then*

$$E_t := (y^2 = a_3(t)x^3 + a_2(t)x^2 + a_1(t)x + a_0(t))$$

has infinitely many solutions for about
$\sqrt{\text{all possible } t \in K}$ *(arranged by height).*

Previous work

1. If a_i have degree 1: easy (linear equation for t)
2. Degree 2 case: R. Munshi studied the case when the a_i are multiples of each other (+ CM case)
3. If a_i have degree ≥ 3: probably not true (K3 surfaces)

Geometry enters

We focus on the **algebraic surface**

$$T := \left(y^2 = a_3(t)x^3 + a_2(t)x^2 + a_1(t)x + a_0(t)\right) \subset \mathbb{A}^3_{xyt}$$

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Theorem

– *K field of characteristic $\neq 2$*
– *$a_i(t) \in K[t]$ polynomials of degree 2*
– *giving a nontrivial family.*
$\Rightarrow$ T is unirational over K.

unirational: there is a dominant map $\phi : \mathbb{P}^2 \dashrightarrow T$, so $\phi(\mathbb{P}^2(K))$ gives "many" K-points on T.

Conic bundles

Projection to x-axis (using $\deg a_i = 2$)

$$T := \left(y^2 = b_2(x)t^2 + b_1(x)t + b_0(x)\right) \subset \mathbb{A}^3_{xyt}$$

where the $b_i(x) \in K[x]$ are cubics.

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Compactify: $S \to \mathbb{P}^1$, generic fibers are smooth, rational.

Max Noether (1870): birational to $\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{C}$.

Corollary: S is rational after a finite degree
field extension K'/K (In our case, degree $\mid 2^7 \cdot 7!$)

Minimal conic bundles

$S \to \mathbb{P}^1$ such that every fiber is a conic;

general fibers: smooth conics,

special fibers: conjugate pairs of lines.

Main invariant: $\delta(S) =$ number of singular fibers

Degree: $(K_S^2) = 8 - \delta(S)$

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Arithmetic gets harder as $\delta(S)$ increases

Del Pezzo cases: $0 \leq \delta(S) \leq 7$
Boundary case: $\delta(S) = 8$ (families of $g = 1$ curves)
Hard cases: $\delta(S) \geq 9$ (families of $g \geq 2$ hyperelliptic curves)

Our surface T

$T = \left(y^2 = b_2(x)t^2 + b_1(x)t + b_0(x)\right) \subset \mathbb{A}^3_{xyt}$
where the $b_i(x) \in K[x]$ are cubics.

Singular fibers:
– at roots of $b_1^2 - 4b_2b_0$: 6 singular fibers
– at infinity: need to blow up/down to get that singular iff $a_3(t)$ is not a square.

$\Rightarrow \delta(T) = 7$ so T is a degree 1 conic bundle.

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– K *field of characteristic* $\neq 2$

– $\pi : S \to \mathbb{P}^1$ *conic bundle with* $\delta(S) \leq 7$, *then*

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– Segre (1951), Manin (1966): $\delta(S) \leq 5$
– $\delta(S) = 6$: Manin knew many cases
– $\delta(S) = 7$: there is always a K-point since $(K_S^2) = 1$.

Geometry for $\delta(S) = 5, 6, 7$: Weak del Pezzo

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$\delta(S) = 7$: $|-K_S|$ is a pencil but $|-2K_S|$ gives
$S \to Q \subset \mathbb{P}^3$ double cover,
Q: quadric cone, degree $2 \cdot 3$ ramification.

Bertini involution: $\tau_B : S \to S$ over Q

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Two pencils: double cover $\sigma : S \to \mathbb{P}^1 \times \mathbb{P}^1$

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$\sigma : S \to \mathbb{P}^1 \times \mathbb{P}^1$ double cover

(General case) In suitable coordinates:

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$\Rightarrow \tau_B(F^*) \subset S$ is a (degree 8) multi-section;

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Problems in the Special Case:
– F^* is not rational and
– even if we find a rational fiber F_0, then $\tau_B(F_0) = F_0$.

Multiplication maps (a side direction)

E elliptic curve: we have $m_E : x \mapsto m \cdot x$.

S with elliptic pencil: these glue to $m_S : S \dashrightarrow S$.

Compute: let $F \subset S$ be a smooth fiber. Then

either: F consists of 6-torsion points
or: $m_2(F)$ or $m_3(F)$ is a multi-section.

Corollary: If S has a K-point that is not 6-torsion
then S is unirational.

Main problem: Can p^* be the only K-point?

Historic note: Secant map for cubic surfaces

$F_K \subset \mathbb{P}^3$ cubic surface over K,
$p_1, p_2 \in F$: let ℓ be the line through them,
$\ell \cap F = \{p_1, p_2, q\}$, we get

$\phi : F \times F \dashrightarrow F$ given by $\phi(p_1, p_2) = q$;

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Working with conjugate point pairs gives the following:

Proposition. Let L/K be a degree 2 field extension. Weil restriction gives a dominant rational map

$$\phi_{L/K} : \mathfrak{R}_{L/K}(F_L) \dashrightarrow F_K.$$

Corollary. If we have $u : \mathbb{P}^2_L \dashrightarrow F_L$ then we get

$$\phi_{L/K}(u) : \mathbb{P}^4_K \dashrightarrow F_K.$$

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Concrete geometric question: Find
– class of surfaces X such that for $n \geq 1$ there are
– linear systems $|B_n|$ such that for general $\{p_1, \ldots, p_n\} \subset X$
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Meta Conjecture

Only for conic bundles with $\delta = 7$.

Multi-secant map for $\delta(S) = 7$

S: conic bundle with $\delta(S) = 7$
K_S: canonical class,
F: fiber class.

Theorem

For general $\{p_1, \ldots, p_n\} \subset S$
$|-K_S + nF|(-2p_1 - \cdots - 2p_n)$ *is a*
pencil with a unique base point.

Corollary

S: *conic bundle with* $\delta(S) = 7$. *Then*
– multi-secant map is defined on an open subset of S^n
– combining with the Weil restriction gives

$$\phi : \mathbb{P}^{1290240} \dashrightarrow S.$$

(exponent $= 2n$ *for* $n = 7! \cdot 2^7$*)*

More economical method?

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Birational preimage of the new F^* after elementary transformation at q, q' plus Bertini involution.

- there are polynomials f, g, F, G, H of degrees 8,6,17,17,26
- C^* is given by $x = f/g$, $t = F/G$, $y = H/(gG)$
- multiplicity 8 at q, q',
- multiplicity 1 at p^* and
- $\left(H/(gG)\right)^2 = a_3(F/G)(f/g)^3 + \cdots + a_0(F/G)$.