Time, Space and Monotone Circuits

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Sept 29 2014

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- Many fundamental results in complexity theory concern the relationship between these two
- For even very simple problems, it is sometimes possible to dramatically reduce the space without increasing time much. Other times this doesn't appear to be the case.

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  - If ETH holds, then there is no $O(2^{(1-\epsilon)n})$ time algorithm, even with exponential space.

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- Which problems have true time-space trade-offs? Can we explain why they do or don't?
- Besides concrete problems, the important meta-algorithm "Dynamic Programming" always trades space for time.
- Basic question: What are the limits of the strategy?

Known Tradeoffs in Concrete Models

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  - Extension to inapproximiability for randomized branching programs. [Beame, Saks, Sun, Vee '02].

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    - ⋆ (Alon, Seymour, Thomas '90) Any DAG of degree $O(1)$ free of K_h-minors can be pebbled with $h^{3/2}n^{1/2}$ pebbles.

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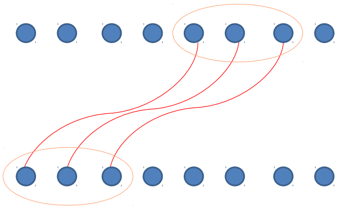
  - Valiant ['76] shows that any algebraic circuit over a finite field computing a linear transformation whose matrix has the property that, any square submatrix is full rank, has the graph-theoretic property of being a 'superconcentrator'.

Superconcentrator



Definition

A *superconcentrator* of capacity n is a DAG $G = (V, E)$ with two disjoint sets of vertices $I, O \subset V$, $|I| = |O| = n$ such that for all subsets $I' \subseteq I, O' \subseteq O$ with $|I'| = |O'|$, there exist $|I'|$ vertex-disjoint paths connecting I' and O'.

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- For (restricted) circuits (AC^0, monotone, algebraic...)
  - Valiant ['76] shows that for any algebraic circuit over a finite field (or any) computing a linear transformation, whose matrix has the property that any square submatrix is full rank, has the graph-theoretic property of being a 'superconcentrator'.
  - Savage ['77], Tompa ['81], others use such arguments to show that the Fourier Transform and similar e.g. cannot be computed with space $n^{1-\epsilon}$ without using time $n^{1+\epsilon}$.

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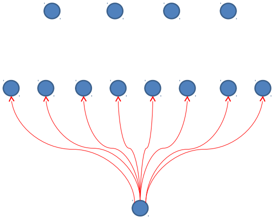
- (B., Beame, Impagliazzo '12) Tradeoffs even up to exponential space, with superpolynomial blowups in time.
- This result is different from previous time-space tradeoff results in that it technically extends the previously known (tight) lower bounds for time. It is a purely combinatorial argument and doesn't reduce to pebbling.

Bottleneck Counting Argument



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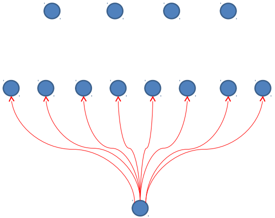

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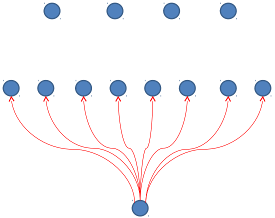

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  - Map *input assignments* to *gates* of the circuit and prove that the probability that any assignment goes to a particular gate is small, e.g. $< \epsilon$.
  - Conclude that there are at least ϵ^{-1} gates.

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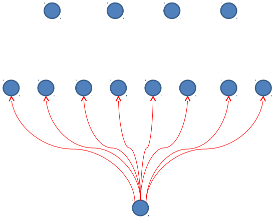

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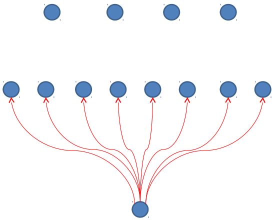

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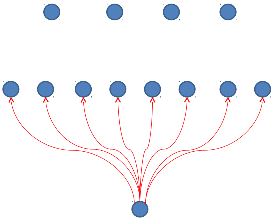

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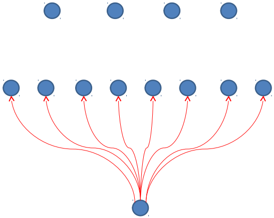

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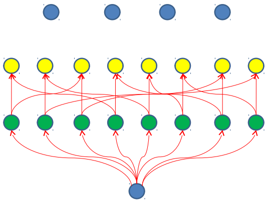
- Janos Simon ['97, et.al '13] points out the commonalities of these.

Bottlenecks and Tradeoffs



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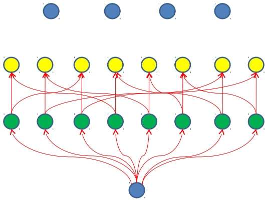

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- Consider a short proof, and consider the map from the size lower bound. By varying the parameters to define it, obtain a second map. We have $\Pr_{\vec{x}}[f_1(\vec{x}) = g] \leq \epsilon$ and $\Pr_{\vec{x}}[f_2(\vec{x}) = g] \leq \epsilon$ for all gates g.

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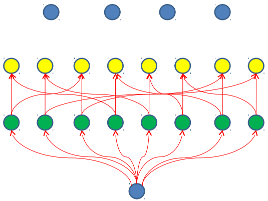

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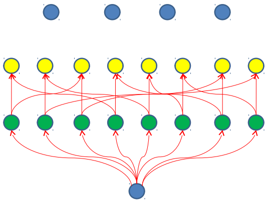

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- However, now prove also that for any g_1, g_2 that $\Pr_{\vec{x}}[f_1(\vec{x}) = g_1 \wedge f_2(\vec{x}) = g_2] \leq \epsilon^2$.
- If the size of the DAG is $\approx \epsilon^{-1}$, we have a weak form of expansion, and morally it implies a space lower bound.

Open Questions

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