Can The Continuum Problem be Solved?

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A talk in the IAS Computer Science Seminar

outline

The Continuum Hypothesis

Question (Cantor-1878)

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Answer (Cantor's answer-The weak Continuum Hypothesis)

Every infinite set of reals is either countable or equinumerous with the whole real line

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The Continuum Hypothesis is equivalent to

Hypothesis

If F is a function from the reals onto an ordinal α then the cardinality of α is at most $\aleph_1$.

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Dieudonné-1976: *Beyond classical analysis there is an infinity of different mathematics and for the time being no definitive reason compels us to chose one rather than another*

The Gödelean conviction

Gödel-1947 *Cantor's conjecture must be either true or false and its undecidability from the axioms can only mean that these axioms do not contain a complete description of this reality and such a belief is by no means chimerical , since it is possible to point out ways in which a decision of the question might nevertheless be obtained*

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There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field and yielding such powerful methods for solving problems . . . that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory

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Theorem (Levy-Solovay 1967)

The continuum hypothesis is independent even if one adds to the axioms of Set Theory any of the accepted axioms of strong infinity.

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As usual we get π^0_1 statements about the natural numbers that we believe to be true but they are not provable from ZFC. (say the statement that some statement Φ is consistent with ZFC).On the other hand appropriate large cardinals axioms imply them.To some extent we can consider the truth of π^0_1 statements as "verifiable" or at least "falsifiable".

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As argued by Woodin one can almost practically construct a Quantum system that the falsity of Ψ is equivalent to the system having some positive probability (admittedly very small) of ataining a particular state.So there is a physical fact which is very likely to be true and the only natural way to derive it requires large cardinal .

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Or the impact could be that one will be able to derive some experimentally testable consequences form the scientific theory based some one set of axioms for Set Theory that can not be derived form another set of Axioms.
Is this an outrageous speculation?

A Physical Example

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Resolution of the Einstein-Podolsky-Rosen and Bell Paradoxes

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A model of spin-$\frac{1}{2}$ statistics that explains the observed frequencies on the basis of the validity of the principle of locality is proposed. The model is based on the observation that certain density conditions on the unit sphere correspond with the observed frequencies while the resulting expectation values violate Bell's inequality.

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Bell[1] has observed that no hidden-variable theory satisfying a principle of locality can reproduce the quantum statistics of electron pairs in the singlet spin state. Bell's argument was simplified by Wigner[2] and put in its most general testable form by Clauser and Horne.[3] Various experiments[4] designed to test the locality principle have shown the observed frequencies to conform with quantum mechanics (i.e., to violate Bell's

that includes complete proofs and generalizations to other spin (angular momentum) states, as well as some predictions, will be published shortly.

Let $S^{(2)}$ be the (surface of a) unit sphere in three-dimensional Euclidean space: $S^{(2)}=\{x \in E^{(3)} \mid |x|=1\}$. Define a spin function as any function, $s: S^{(2)} \to \{-\frac{1}{2}, \frac{1}{2}\}$, which satisfies $s(-x) = -s(x)$. The purpose of the first part of this paper is to develop some *mathematical* constraints

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$\cap\, c(y,\theta)]$ is the (average) density of $\{x \mid s(x)=\frac{1}{2}\}$ in $c(y,\theta)$. We have the following:

Existence theorem.—There exists a spin function s such that for all $y \in S^{(2)}$ and all $0<\theta<\pi$ the set $\{x \mid s(x)=\frac{1}{2}\} \cap c(y,\theta)$ is m_θ measurable and

$$\frac{m_\theta[\{x \mid s(x)=\frac{1}{2}\} \cap c(y,\theta)]}{2\pi \sin\theta} = \begin{cases} \cos^2(\frac{1}{2}\theta) & \text{if } s(y)=\frac{1}{2}, \\ \sin^2(\frac{1}{2}\theta) & \text{if } s(y)=-\frac{1}{2}. \end{cases} \qquad (1)$$

The complete proof of the theorem will be published separately. The existence theorem belongs to a family of "strange" or seemingly "paradoxical" results that one can prove in set theory. The proof involves transfinite induction on circles and is based on two observations. Firstly, that the intersection of two nonidentical circles contains at most two points and, secondly, that any subset of $c(y,\theta)$ whose coordinality is strictly less than the continuum is m_θ measurable and has m_θ measure zero. To ensure that the second premise is true, we have to assume the validity of the continuum hypothesis, or at least the validity of the (strictly) weaker Martin's axiom.[5] It is important to note that there exists no analytic expression or algorithm by which one can calculate the values of a spin function that satisfy Eq. (1) for the different directions. In fact, the set $\{x \mid s(x)=\frac{1}{2}\}$ turns out to be nonmeasurable in terms of the Lebesgue measure *on the sphere* and the existence theorem may turn out to be independent of the usual axioms of set theory. The proof of the theorem actually establishes the existence of infinite-

definite values everywhere on the sphere—our use of probabilities reflects our ignorance of these values.

I have interpreted formula (1) as an expression for conditional *probabilities*. A natural question to ask is whether we can find a probability space from which we get the values of (1) by conditionalization. In other words we are looking for a probability space such that for all $y \in S^{(2)}$ the event "spin up in the y direction" is defined and has probability $\frac{1}{2}$. Also we want that for all x and y the probability of the joint event "spin up in the x direction and spin up in the y direction" will be $\frac{1}{2}\cos^2(\frac{1}{2}\theta)$, where θ is the angle between x and y. With use of Bell's inequality one can prove that no such probability space exists.[6] [Roughly speaking the values $\frac{1}{2}\cos^2(\frac{1}{2}\theta)$ are incompatible with the additivity axiom for probability.] My way out of this problem is to interpret $\cos^2(\frac{1}{2}\theta)$ as the conditional expectation for "spin up" on a circle, given that the spin is up in the center of the circle. From this perspective Bell's theorem shows that

Crash Course in Physics

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Given a particle whose spin is 1. (=a boson). The spin can be measured along any axis x and the possible values are $S_x = 1, 0, -1$.
If x, y, z are mutually orthogonal then one can not measure simultaneously any two of S_x, S_y, S_z. (The corresponding operators do not commute.) But the squares S_x^2, S_y^2, S_z^2 do commute and hence can be measured simultaneously . It is a result of QM that always

$$S_x^2 + S_y^2 + S_z^2 = 2$$

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The hidden variable assumption claims that the particle carry some predetermined values of S_x^2, S_y^2, S_y^2 such that this values are what we measure. The Kochen-Specker Theorem claims that this is impossible.

Theorem (Kochen-Specker)

There is no function S defined on the unit sphere of the 3-dimensional space S_2 such that for every $x \in S_2$ $S(x) = 1, 0$ and such that for every $x, y, z \in S_2$ which are mutually orthogonal

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Theorem (Pitowsky)

Assume the Continuum Hypothesis . Then there is a function S defined on S_2 getting only the values $0, 1$ and such that for every $x \in S_2$ the set of pairs (y, z) such that x, y, z are mutually orthogonal and such that

$$S(x) + S(y) + S(z) \neq 2$$

is countable.

While being a wild shot it is impossible that Scientific Theories will prefer one Set Theory over others because it makes the scientific theory simpler and more elegant. It may even be possible that in order to derive certain experimentally testable results one would have to prefer one Set Theory over others.

Another Potential Example

Definition

H is a separable Hilbert space. $B(H)$ is the algebra of bounded operators on H. $K(H)$ the ideal of compact operators. The Calkin Algebra of H is the quotient algebra $B(H)/K(H)$.

Theorem (Philips,Weaver,Farah)

The problem whether all automorphisms of the Calikin Algebra are inner is independent of ZFC.

Whether all automorphisms of the Calkin Algebra are inner could have a Physical meaning.

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If the Continuum Hypothesis fails then there should be a "simple","definable" evidence for this failure.

A counter example could be a counter example to the weak continuum hypothesis , namely a definable set of reals which is neither countable nor of the cardinality of the continuum.or it could be a counter example to the full continuum hypothesis i.e. a definable "simple map of the reals on a ordinal of cardinality $\geq \aleph_2$.

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Gödel It is consistent to have an uncountable set which is the complement of an analytic set ("Co-Analytic set ") with no perfect subset.

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More detailed hierarchy :

A set is Σ^1_1 (=Analytic) if it is the continuous image of a Borel set.It is Π^1_1 if it is the complement of a Σ^1_1 set. In general a set is Σ^1_{n+1} if it is a continuous image of a Π^1_n set and it is a Π^1_{n+1} if it is the complement of a Σ^1_{n+1} set.

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Theorem (Solovay)

1. *If there exists a measurable cardinal then every Σ^1_2 set is either countable or contains a perfect subset.*
2. *There is a model with measurable cardinal with a Π^1_2 set violating CH.*

Theorem (Foreman,M. Shelah)

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The large cardinal required by the last theorem was substantially reduced by Woodin:

Theorem

If the exists infinitely many Woodin cardinals with a measurable cardinal above them then every set in $L[R]$ is either countable or contains a perfect subset.

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- *A UB set of reals is Lesbegue measurable, has the Baire property etc.*
- *A family of infinite subsets of natural numbers $\mathcal{F}$ (which can be considered to be a set of reals by identifying the characteristic function a set of natural numbers with a binary expansion of a real number) which is UB is Ramsey. (i.e. there is an infinite set of natural numbers A such that either for every infinite $B \subseteq A \quad B \in \mathcal{F}$ or for every infinite $B \subseteq A \quad B \notin \mathcal{F}$.*

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- *If there is a Woodin cardinal then every UB set is either countable or contains a perfect subset.*

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- *If there is a Woodin cardinal then every UB set is either countable or contains a perfect subset.*

Theorem (Woodin)

If there are unboundedly many Woodin cardinals then every set in $L[R]$ is UB.

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Conjecture

If A is UB and $|A| > \aleph_1$ then A contains a perfect subset.

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In a generic extension of L we can have a UB set of cardinality $\aleph_1 < 2^{\aleph_0}$. So the values we know are possible are $\leq \aleph_0, \aleph_1, 2^{\aleph_0}$

Conjecture

If A is UB and $|A| > \aleph_1$ then A contains a perfect subset.

If the conjecture is true then it could serve as an argument for the continuum being either $\aleph_1$ or $\aleph_2$.

Slogans for forcing Axioms

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Forcing Axioms

Definition

Let P be a partially ordered sets. λ a cardinal, the forcing axiom $FA_\lambda(P)$ is the statement that for every collection $\langle D_\alpha | \alpha < \lambda \rangle$ of dense subsets of P , there is a filter $G \subset P$ such that for all $\alpha < \lambda$, $G \cap D_\alpha \neq \emptyset$.

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For every $\mathcal{P}$ $FA_{\aleph_0}(\mathcal{P})$ is always true. For many very simple P $FA_{2^{\aleph_0}}(P)$ is false, so the first interesting case is $\lambda = \aleph_1$ when $\aleph_1 < 2^{\aleph_0}$.

For some partially ordered sets P there are obvious obstacles to having $FA_{\aleph_1}(p)$.

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Martin's Axiom (MA) is the statement that that $FA_\lambda(P)$ holds for every c.c.c. partially ordered set P and $\lambda < 2^{\aleph_0}$.

Theorem (Martin, Solovay)

MA is consistent with $\aleph_1 < 2^{\aleph_0}$. (It is consistent with many different values of the continuum.

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If P does not satisfy a technical condition (called "stationary preserving" or SP) then $FA_{\aleph_1}(P)$ fails.

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Fact

If P does not satisfy a technical condition (called "stationary preserving" or SP) then $FA_{\aleph_1}(P)$ fails.

Definition

MM ("Martin's Maximum") is the statement that $FA_\lambda(P)$ holds for every P which is SP and for every $\lambda < 2^{\aleph_0}$.

Theorem (Foreman,M., Shelah)

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Further results of Todorcevic, Velckovic, Moore etc. show that almost any strengthening of MA with the failure of CH implies that the continuum is $\aleph_2$.

conclution

Some possibilities for the continuum problem are more equal!