An Isoperimetric Inequality for the Hamming Cube[1]

and applications to Integrality Gaps in Bounded-degree Graphs

Siavosh Benabbas

Department of Computer Science
University of Toronto

[1]Based on joint work with H. Hatami and A. Magen

Outline

Outline

The Frankl-Rödl Theorem

\$250 Question of P. Erdös 1970s

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.

The Frankl-Rödl Theorem

\$250 Question of P. Erdös 1970s

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0,1\}^n$,

The Frankl-Rödl Theorem

\$250 Question of P. Erdös 1970s

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0, 1\}^n$,
- no two element of U are different in exactly d coordinates,

The Frankl-Rödl Theorem

\$250 Question of P. Erdös 1970s

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0,1\}^n$,
- no two element of U are different in exactly d coordinates,

How big can U be?

The Frankl-Rödl Theorem

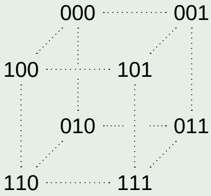
\$250 Question of P. Erdös 1970s

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0,1\}^n$,
- no two element of U are different in exactly d coordinates,

How big can U be?

Example

$n = 3$



The Frankl-Rödl Theorem

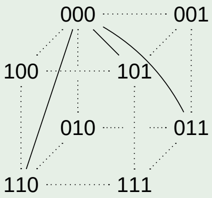
\$250 Question of P. Erdös 1970s

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0,1\}^n$,
- no two element of U are different in exactly d coordinates,

How big can U be?

Example

$n = 3 \qquad \delta = 2/3 \qquad d = 2$



The Frankl-Rödl Theorem

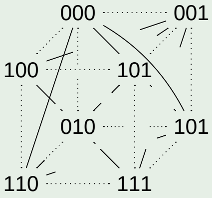
\$250 Question of P. Erdös 1970s

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0,1\}^n$,
- no two element of U are different in exactly d coordinates,

How big can U be?

Example

$n = 3 \qquad \delta = 2/3 \qquad d = 2$



The Frankl-Rödl Theorem

\$250 Question of P. Erdös 1970s

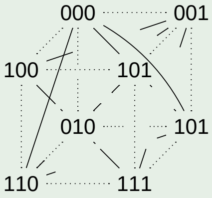
- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0,1\}^n$,
- no two element of U are different in exactly d coordinates,

How big can U be?

Example

$$n = 3 \qquad \delta = 2/3 \qquad d = 2$$

How many vertices can you select without selecting both end points of an edge?



The Frankl-Rödl Theorem

\$250 Question of P. Erdös 1970s

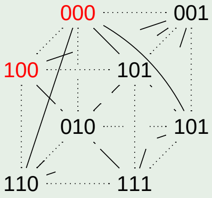
- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0, 1\}^n$,
- no two element of U are different in exactly d coordinates,

How big can U be?

Example

$$n = 3 \qquad \delta = 2/3 \qquad d = 2$$

How many vertices can you select without selecting both end points of an edge?



The Frankl-Rödl Theorem

$250 Question of P. Erdös 1970s

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0, 1\}^n$,
- no two element of U are different in exactly d coordinates,

How big can U be?

Theorem ([Frankl and Rodl, 1987])

U is exponentially small.

$$\mu = \frac{|U|}{2^n} \leq \xi^n \qquad \xi = \xi(\delta) < 1$$

“Density” Frankl-Rödl

$250 Question of P. Erdös 1970s

"Density" Frankl-Rödl

$250 Question of P. Erdös 1970s

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0,1\}^n$,
- no two element of U are different in exactly d coordinates,

How big can U be?

Theorem ([Frankl and Rodl, 1987])

U is exponentially small.

$$\mu = \frac{|U|}{2^n} \leq \xi^n \qquad \xi = \xi(\delta) < 1$$

“Density” Frankl-Rödl

“Density” Frankl-Rödl

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0, 1\}^n$, $|U| = \mu 2^n$
- no two element of U are different in exactly d coordinates,

Theorem ([Frankl and Rodl, 1987])

“Density” Frankl-Rödl

“Density” Frankl-Rödl

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0, 1\}^n$, $|U| = \mu 2^n$
- How many pairs of elements of U are different in exactly d coordinates?

Theorem ([Frankl and Rodl, 1987])

“Density” Frankl-Rödl

“Density” Frankl-Rödl

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0,1\}^n$, $|U| = \mu 2^n$
- How many pairs of elements of U are different in exactly d coordinates?

Theorem ([Frankl and Rodl, 1987])

$$\Pr_{x,y}[x \in U, y \in U]$$

x, y chosen randomly so that $d_H(x,y) = d$.

“Density” Frankl-Rödl

“Density” Frankl-Rödl

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0,1\}^n$, $|U| = \mu 2^n$
- How many pairs of elements of U are different in exactly d coordinates?

Theorem (1)

By [Frankl and Rodl, 1987]:

$$\Pr_{x,y}[x \in U, y \in U] > 0,$$

x, y chosen randomly so that $d_H(x,y) = d$.

“Density” Frankl-Rödl

“Density” Frankl-Rödl

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0,1\}^n$, $|U| = \mu 2^n$
- How many pairs of elements of U are different in exactly d coordinates?

Theorem (1)

We show:

$$\Pr_{x,y}[x \in U, y \in U] > \epsilon,$$

x, y chosen randomly so that $d_H(x,y) = d$.

“Density” Frankl-Rödl

“Density” Frankl-Rödl

- Fix $0 < \delta < 1$. Let $n \in \mathbb{N}$, and $d \sim \delta n$ be an even integer.
- $U \subseteq \{0,1\}^n$, $|U| = \mu 2^n$
- How many pairs of elements of U are different in exactly d coordinates?

Theorem (1)

We show:

$$\Pr_{x,y}[x \in U, y \in U] > \epsilon, \qquad \epsilon = 2(\mu/2)^{\frac{2}{1-|1-2\delta|}} - o(1)$$

x, y chosen randomly so that $d_H(x,y) = d$.

Our results

Theorem

$\forall \delta \in (0,1)$, and large enough n, if $U \subseteq \{0,1\}^n$, $|U| = \mu 2^n$,

$$\Pr_{x,y}[x \in U, y \in U] > \epsilon \qquad \epsilon = 2(\mu/2)^{\frac{2}{1-|1-2\delta|}} - o(1)$$

x, y chosen randomly so that $d_H(x,y) \simeq \delta n$ is an even integer.

Our results

Theorem

$\forall \delta \in (0,1)$, and large enough n, if $U \subseteq \{0,1\}^n$, $|U| = \mu 2^n$,

$$\Pr_{x,y}[x \in U, y \in U] > \epsilon \qquad \epsilon = 2(\mu/2)^{\frac{2}{1-|1-2\delta|}} - o(1)$$

x, y chosen randomly so that $d_H(x,y) \simeq \delta n$ is an even integer.

Theorem (A new Isoperimetric Inequality)

$\forall \delta \in (0,1)$, and large enough n, if $U, W \subseteq \{0,1\}^n$, $|U|, |W| \geq \mu 2^n$,

$$\Pr_{x,y}[x \in U, y \in W] > \epsilon \qquad \epsilon = \mu^{\frac{2}{1-|1-2\delta|}} - o(1)$$

x, y chosen randomly so that $d_H(x,y) = d$ or $d+1$, $d = \lfloor \delta n \rfloor$.

High level of the proof of Frankl-Rödl Theorem

Theorem ([Frankl and Rodl, 1987])

$$\forall U \subseteq \{0,1\}^n, |U| \geq \mu 2^n : \qquad (\mu = \xi^n)$$

$$\Pr_{x,y}[x \in U, y \in U] > 0 \qquad d_H(x,y) = \delta n$$

High level of the proof of Frankl-Rödl Theorem

Theorem ([Frankl and Rodl, 1987])

$$\forall U \subseteq \{0,1\}^n, |U| \geq \mu 2^n : \qquad (\mu = \xi^n)$$

$$\Pr_{x,y}[x \in U, y \in U] > 0 \qquad d_H(x,y) = \delta n$$

Reduces to,

Theorem ([Frankl and Rodl, 1987])

$$\forall U' \subseteq \{0,1\}^n, |U'| \geq \mu 2^n/n : \qquad (\mu = \xi^n)$$

$$\Pr_{x,y}\left[x \in U', y \in U'\right] > 0 \qquad \sum_i x_i y_i = \delta' n$$

High level of the proof of Frankl-Rödl Theorem

Theorem ([Frankl and Rodl, 1987])

$$\forall U \subseteq \{0,1\}^n, |U| \geq \mu 2^n : \qquad (\mu = \xi^n)$$

$$\Pr_{x,y}[x \in U, y \in U] > 0 \qquad d_H(x,y) = \delta n$$

Reduces to,

Theorem ([Frankl and Rodl, 1987])

$$\forall U' \subseteq \{0,1\}^n, |U'| \geq \mu 2^n/n : \qquad (\mu = \xi^n)$$

$$\Pr_{x,y}\left[x \in U', y \in U'\right] > 0 \qquad \sum_i x_i y_i = \delta' n$$

This first step fails for us!

High level of our proof

Theorem (1)

$$\forall U \subseteq \{0,1\}^n, |U| \geq \mu 2^n :$$

$$\Pr_{x,y}[x \in U, y \in U] > \epsilon = \epsilon(\delta, \mu) \qquad d_H(x,y) = \delta n$$

High level of our proof

Theorem (1)

$$\forall U \subseteq \{0,1\}^n, |U| \geq \mu 2^n :$$
$$\Pr_{x,y}[x \in U, y \in U] > \epsilon = \epsilon(\delta,\mu) \qquad d_H(x,y) = \delta n$$

We reduce it to,

Theorem ([Mossel et al., 2006])

$$\forall U \subseteq \{0,1\}^n, |U| \geq \mu 2^n :$$
$$\Pr_{x,y}[x \in U, y \in U] > \epsilon = \epsilon(\delta,\mu) \quad y_i = \begin{cases} 1 - x_i & w.p.\ \delta \\ x_i & w.p.\ 1-\delta \end{cases}$$

High level of our proof (cont.)

Fix $U \subseteq \{0,1\}^n, |U| \geq \mu 2^n$,

$$P_1 := \Pr_{x,y}[x \in U, y \in U] \qquad d_H(x,y) = \delta n$$

$$P_2 := \Pr_{x,y}[x \in U, y \in U] \qquad \mathbb{E}[d_H(x,y)] = \delta n$$

High level of our proof (cont.)

Fix $U \subseteq \{0,1\}^n, |U| \geq \mu 2^n$,

$$P_1 := \Pr_{x,y}[x \in U, y \in U] \qquad d_H(x,y) = \delta n$$

$$P_2 := \Pr_{x,y}[x \in U, y \in U] \qquad \mathbb{E}[d_H(x,y)] = \delta n$$

Show that: $|P_1 - P_2| = o(1).$

High level of our proof (cont.)

Fix $U \subseteq \{0,1\}^n, |U| \geq \mu 2^n$, define $\mathbf{1}_U(x) = \begin{cases} 1 & x \in U \\ 0 & \text{o.w} \end{cases}$

$$P_1 := \Pr_{x,y}[x \in U, y \in U] \qquad d_H(x,y) = \delta n$$

$$P_2 := \Pr_{x,y}[x \in U, y \in U] \qquad \mathbb{E}[d_H(x,y)] = \delta n$$

Show that: $|P_1 - P_2| = o(1).$

High level of our proof (cont.)

Fix $U \subseteq \{0,1\}^n, |U| \geq \mu 2^n$, define $\mathbf{1}_U(x) = \begin{cases} 1 & x \in U \\ 0 & \text{o.w} \end{cases}$

$$P_1 := \Pr_{x,y}[x \in U, y \in U] = \mathop{\mathbb{E}}_{x,y}[\mathbf{1}_U(x)\mathbf{1}_U(y)] \qquad d_H(x,y) = \delta n$$

$$P_2 := \Pr_{x,y}[x \in U, y \in U] = \mathop{\mathbb{E}}_{x,y}[\mathbf{1}_U(x)\mathbf{1}_U(y)] \quad \mathbb{E}[d_H(x,y)] = \delta n$$

Show that: $|P_1 - P_2| = o(1).$

High level of our proof (cont.)

Fix $U \subseteq \{0,1\}^n, |U| \geq \mu 2^n$, define $\mathbf{1}_U(x) = \begin{cases} 1 & x \in U \\ 0 & \text{o.w} \end{cases}$

$P_1 = \mathop{\mathbb{E}}_{x,y}[\mathbf{1}_U(x)\mathbf{1}_U(y)]$ $\quad d_H(x,y) = \delta n$

$P_2 = \mathop{\mathbb{E}}_{x,y}[\mathbf{1}_U(x)\mathbf{1}_U(y)]$ $\quad \mathbb{E}[d_H(x,y)] = \delta n$

Show that:

High level of our proof (cont.)

Fix $U \subseteq \{0,1\}^n, |U| \geq \mu 2^n$, define $\mathbf{1}_U(x) = \begin{cases} 1 & x \in U \\ 0 & \text{o.w} \end{cases}$

$$P_1 = \mathop{\mathbb{E}}_{x,y} [\mathbf{1}_U(x)\mathbf{1}_U(y)] \qquad d_H(x,y) = \delta n$$

$$P_2 = \mathop{\mathbb{E}}_{x,y} [\mathbf{1}_U(x)\mathbf{1}_U(y)] = \mathop{\mathbb{E}}_{x} [\mathbf{1}_U(x)(\mathcal{T}_{1-2\delta}\mathbf{1}_U)(x)] \qquad \mathbb{E}[d_H(x,y)] = \delta n$$

Show that:

$$(\mathcal{T}_{1-2\delta} f)(x) = \mathop{\mathbb{E}}_{y} [f(y)] \qquad \mathbb{E}[d_H(x,y)] = \delta n$$

High level of our proof (cont.)

Fix $U \subseteq \{0,1\}^n, |U| \geq \mu 2^n$, define $\mathbf{1}_U(x) = \begin{cases} 1 & x \in U \\ 0 & \text{o.w} \end{cases}$

$$P_1 = \mathop{\mathbb{E}}_{x,y}[\mathbf{1}_U(x)\mathbf{1}_U(y)] = \mathop{\mathbb{E}}_{x}[\mathbf{1}_U(x)(\mathcal{S}_d\mathbf{1}_U)(x)] \qquad d_H(x,y) = \delta n$$

$$P_2 = \mathop{\mathbb{E}}_{x,y}[\mathbf{1}_U(x)\mathbf{1}_U(y)] = \mathop{\mathbb{E}}_{x}[\mathbf{1}_U(x)(\mathcal{T}_{1-2\delta}\mathbf{1}_U)(x)] \qquad \mathbb{E}[d_H(x,y)] = \delta n$$

Show that:

$$(\mathcal{T}_{1-2\delta}f)(x) = \mathop{\mathbb{E}}_{y}[f(y)] \qquad \mathbb{E}[d_H(x,y)] = \delta n$$

$$(\mathcal{S}_d f)(x) = \mathop{\mathbb{E}}_{y}[f(y)] \qquad d_H(x,y) = \delta n$$

High level of our proof (cont.)

Fix $U \subseteq \{0,1\}^n, |U| \geq \mu 2^n$, define $\mathbf{1}_U(x) = \begin{cases} 1 & x \in U \\ 0 & \text{o.w} \end{cases}$

$$P_1 = \mathop{\mathbb{E}}_{x,y}[\mathbf{1}_U(x)\mathbf{1}_U(y)] = \mathop{\mathbb{E}}_{x}[\mathbf{1}_U(x)(\mathcal{S}_d\mathbf{1}_U)(x)] \qquad d_H(x,y) = \delta n$$

$$P_2 = \mathop{\mathbb{E}}_{x,y}[\mathbf{1}_U(x)\mathbf{1}_U(y)] = \mathop{\mathbb{E}}_{x}[\mathbf{1}_U(x)(\mathcal{T}_{1-2\delta}\mathbf{1}_U)(x)] \qquad \mathbb{E}[d_H(x,y)] = \delta n$$

Show that: $\mathcal{T}_{1-2\delta} \simeq \mathcal{S}_d$

$$(\mathcal{T}_{1-2\delta}f)(x) = \mathop{\mathbb{E}}_{y}[f(y)] \qquad \mathbb{E}[d_H(x,y)] = \delta n$$

$$(\mathcal{S}_d f)(x) = \mathop{\mathbb{E}}_{y}[f(y)] \qquad d_H(x,y) = \delta n$$

High level of our proof ($\mathcal{T}_{1-2\delta} \simeq \mathcal{S}_d$)

- Both are linear operators,

High level of our proof ($\mathcal{T}_{1-2\delta} \simeq \mathcal{S}_d$)

- Both are linear operators,
- Have the same Eigenvectors.

High level of our proof ($\mathcal{T}_{1-2\delta} \simeq \mathcal{S}_d$)

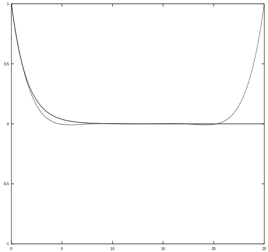
- Both are linear operators,
- Have the same Eigenvectors. $\chi_S(x) = \prod_{i \in S}(-1)^{x_i}$, $S \subseteq [n]$

High level of our proof ($\mathcal{T}_{1-2\delta} \simeq \mathcal{S}_d$)

- Both are linear operators,
- Have the same Eigenvectors. $\chi_S(x) = \prod_{i \in S}(-1)^{x_i}$, $S \subseteq [n]$
- Both have $n+1$ (repeated) eigenvalues.plotted below,

High level of our proof ($\mathcal{T}_{1-2\delta} \simeq \mathcal{S}_d$)

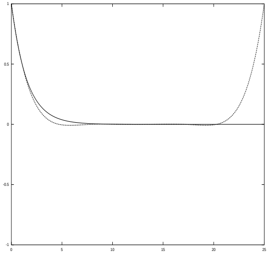
- Both are linear operators,
- Have the same Eigenvectors. $\chi_S(x) = \prod_{i \in S}(-1)^{x_i}$, $S \subseteq [n]$
- Both have $n+1$ (repeated) eigenvalues.plotted below,



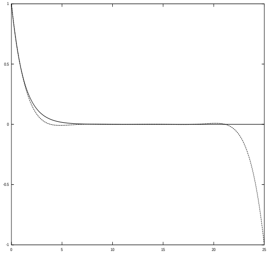
$n = 25, d = 6$

High level of our proof ($\mathcal{T}_{1-2\delta} \simeq \mathcal{S}_d$)

- Both are linear operators,
- Have the same Eigenvectors. $\chi_S(x) = \prod_{i \in S}(-1)^{x_i}$, $S \subseteq [n]$
- Both have $n+1$ (repeated) eigenvalues.plotted below,



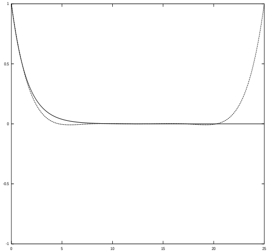
$n = 25, d = 6$



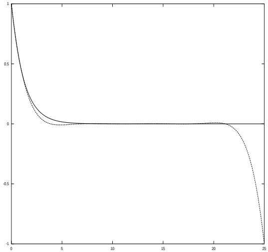
$n = 25, d = 7$

High level of our proof ($\mathcal{T}_{1-2\delta} \simeq (\mathcal{S}_d + \mathcal{S}_{d+1})/2$)

- Both are linear operators,
- Have the same Eigenvectors. $\chi_S(x) = \prod_{i \in S} (-1)^{x_i}$, $S \subseteq [n]$
- Both have $n+1$ (repeated) eigenvalues.plotted below,
- Compare to $(\mathcal{S}_d + \mathcal{S}_{d+1})/2$.



$n = 25, d = 6$



$n = 25, d = 7$

An Isoperimetric Inequality for the Hamming Cube: Open Questions

- Improve the error term.

An Isoperimetric Inequality for the Hamming Cube: Open Questions

- Improve the error term.

Theorem

$\forall\delta \in (0,1)$, and large enough n, if $U \subseteq \{0,1\}^n$, $|U| = \mu 2^n$,

$$\Pr_{x,y}[x \in U, y \in U] > \epsilon \qquad \epsilon = 2(\mu/2)^{\frac{2}{1-|1-2\delta|}} - o(1)$$

x, y chosen randomly so that $d_H(x,y) \simeq \delta n$ is an even integer.

An Isoperimetric Inequality for the Hamming Cube: Open Questions

- Improve the error term.

Theorem

$\forall\delta \in (0,1)$, and large enough n, if $U \subseteq \{0,1\}^n$, $|U| = \mu 2^n$,

$$\Pr_{x,y}[x \in U, y \in U] > \epsilon \qquad \epsilon = 2(\mu/2)^{\frac{2}{1-|1-2\delta|}} - o(1)$$

x, y chosen randomly so that $d_H(x,y) \simeq \delta n$ is an even integer.

Could result in a new proof of [Frankl and Rodl, 1987].

An Isoperimetric Inequality for the Hamming Cube: Open Questions

- Improve the error term.
  Could result in a new proof of [Frankl and Rodl, 1987].
- "Density" version of other Theorems of [Frankl and Rodl, 1987].

An Isoperimetric Inequality for the Hamming Cube: Open Questions

- Improve the error term.
  Could result in a new proof of [Frankl and Rodl, 1987].
- "Density" version of other Theorems of [Frankl and Rodl, 1987].

Theorem ([Frankl and Rodl, 1987])

$$\forall U \subseteq \{0,1\}^n, |U| \geq \mu 2^n : \qquad (\mu = \xi^n)$$

$$\Pr_{x,y}[x \in U, y \in U] > 0 \qquad \sum_i x_i y_i = \delta n$$

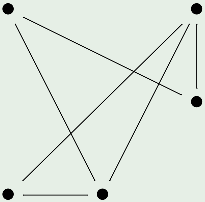
Outline

VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,

Example

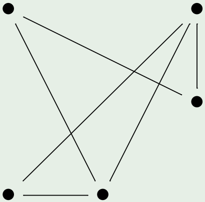


VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:

Example

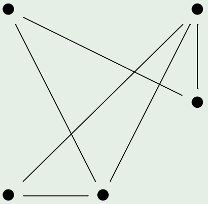


Vertex Cover and Independent Set

Definition (Vertex Cover)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:
- it touches each edge,

Example

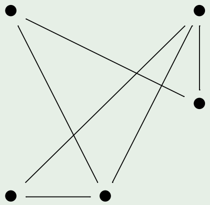


VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:
- it touches each edge,
- $|S|$ is minimized.

Example



VERTEX COVER and INDEPENDENT SET

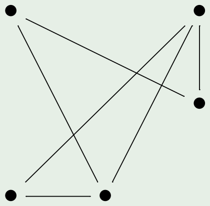
Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:
- it touches each edge,
- $|S|$ is minimized.

Definition (INDEPENDENT SET)

Input: Graph $G = (V, E)$,
Goal: Finding subset $\overline{S} \subseteq V$:,
- no edge has both ends in $\overline{S}$,
- $|\overline{S}|$ is maximized.

Example



VERTEX COVER and INDEPENDENT SET

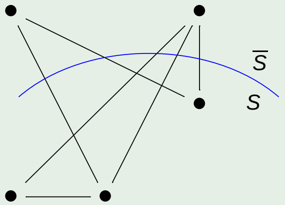
Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:
- it touches each edge,
- $|S|$ is minimized.

Definition (INDEPENDENT SET)

Input: Graph $G = (V, E)$,
Goal: Finding subset $\overline{S} \subseteq V$:,
- no edge has both ends in $\overline{S}$,
- $|\overline{S}|$ is maximized.

Example



VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:
- it touches each edge
-

Definition (INDEPENDENT SET)

Input: Graph $G = (V, E)$,
Goal: Finding subset $\overline{S} \subseteq V$:,
- no edge has both ends in $\overline{S}$

What is known: General Graphs

	VERTEX COVER	INDEPENDENT SET

VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:

- it touches each edge
-

Definition (INDEPENDENT SET)

Input: Graph $G = (V, E)$,
Goal: Finding subset $\overline{S} \subseteq V$:,

- no edge has both ends in $\overline{S}$

What is known: General Graphs

	VERTEX COVER	INDEPENDENT SET
Best algorithm	$2 - o(1)$	$O(n/\text{polylog}(n))$

[Karakostas, 2005] [Feige, 2004]

VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:
- it touches each edge
-

Definition (INDEPENDENT SET)

Input: Graph $G = (V, E)$,
Goal: Finding subset $\overline{S} \subseteq V$:,
- no edge has both ends in $\overline{S}$

What is known: General Graphs

	VERTEX COVER	INDEPENDENT SET
Best algorithm	$2 - o(1)$	$O(n/\text{polylog}(n))$
NP-hardness	1.36	$\Omega(n^{1-\epsilon})$

[Karakostas, 2005] [Feige, 2004] [Dinur and Safra, 2005] [Håstad, 1996]

VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:
- it touches each edge
-

Definition (INDEPENDENT SET)

Input: Graph $G = (V, \underline{E})$,
Goal: Finding subset $\overline{S} \subseteq V$:,
- no edge has both ends in $\overline{S}$

What is known: General Graphs

	VERTEX COVER	INDEPENDENT SET
Best algorithm	$2 - o(1)$	$O(n/\text{polylog}(n))$
NP-hardness	1.36	$\Omega(n^{1-\epsilon})$
UGC-hardness	$2 - \epsilon$	

[Karakostas, 2005] [Feige, 2004] [Dinur and Safra, 2005] [Håstad, 1996] [Khot and Regev, 2008]

VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:
- it touches each edge
-

Definition (INDEPENDENT SET)

Input: Graph $G = (V, E)$,
Goal: Finding subset $\overline{S} \subseteq V$:,
- no edge has both ends in $\overline{S}$

What is known: General Graphs

	VERTEX COVER	INDEPENDENT SET
Best algorithm	$2 - o(1)$	$O(n/\operatorname{polylog}(n))$
NP-hardness	1.36	$\Omega(n^{1-\epsilon})$
UGC-hardness	$2 - \epsilon$	
Hierarchy IGs	$2 - \epsilon$ (LS$^+$, SA) 1.36 (Lasserre)	$n/2^{O(\sqrt{\log n \log\log n})}$ (Lasserre)

[Karakostas, 2005] [Feige, 2004] [Dinur and Safra, 2005] [Håstad, 1996] [Khot and Regev, 2008] [Tulsiani, 2009] [Charikar et al., 2009], ...

VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:
- it touches each edge
-

Definition (INDEPENDENT SET)

Input: Graph $G = (V, E)$,
Goal: Finding subset $\overline{S} \subseteq V$:,
- no edge has both ends in $\overline{S}$

What is known: Graphs of bounded degree (d)

	VERTEX COVER	INDEPENDENT SET

VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:
- it touches each edge
-

Definition (INDEPENDENT SET)

Input: Graph $G = (V, E)$,
Goal: Finding subset $\overline{S} \subseteq V$:,
- no edge has both ends in $\overline{S}$

What is known: Graphs of bounded degree (d)

	VERTEX COVER	INDEPENDENT SET
Best algorithm	$2 - (2 - o_d(1))\frac{\log\log d}{\log d}$	$O\left(\frac{d \log\log d}{\log d}\right)$

[Halperin, 2002]

VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:
- it touches each edge
-

Definition (INDEPENDENT SET)

Input: Graph $G = (V, E)$,
Goal: Finding subset $\bar{S} \subseteq V$:,
- no edge has both ends in $\bar{S}$

What is known: Graphs of bounded degree (d)

	VERTEX COVER	INDEPENDENT SET
Best algorithm	$2 - (2 - o_d(1))\frac{\log\log d}{\log d}$	$O\left(\frac{d \log\log d}{\log d}\right)$
NP-hardness		$\frac{d}{2^{O(\sqrt{\log d})}}$

[Halperin, 2002] [Samorodnitsky and Trevisan, 2000]

VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:
- it touches each edge
-

Definition (INDEPENDENT SET)

Input: Graph $G = (V, \underline{E})$,
Goal: Finding subset $\overline{S} \subseteq V$:,
- no edge has both ends in $\overline{S}$

What is known: Graphs of bounded degree (d)

	VERTEX COVER	INDEPENDENT SET
Best algorithm	$2 - (2 - o_d(1))\frac{\log\log d}{\log d}$	$O\left(\frac{d \log\log d}{\log d}\right)$
NP-hardness		$\frac{d}{2^{O(\sqrt{\log d})}}$
UGC-hardness	$2 - (2 + o_d(1))\frac{\log\log d}{\log d}$	$\Omega\left(\frac{d}{\log^2 d}\right)$

[Halperin, 2002] [Samorodnitsky and Trevisan, 2000]
[Austrin et al., 2009]

VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:
- it touches each edge
-

Definition (INDEPENDENT SET)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:,
- no edge has both ends in S

What is known: Graphs of bounded degree (d)

	VERTEX COVER	INDEPENDENT SET
Best algorithm	$2 - (2 - o_d(1))\frac{\log\log d}{\log d}$	$O\left(\frac{d \log\log d}{\log d}\right)$
NP-hardness		$\frac{d}{2^{O(\sqrt{\log d})}}$
UGC-hardness	$2 - (2 + o_d(1))\frac{\log\log d}{\log d}$	$\Omega\left(\frac{d}{\log^2 d}\right)$
Hierarchy IGs		

[Halperin, 2002] [Samorodnitsky and Trevisan, 2000]
[Austrin et al., 2009]

VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:

- it touches each edge
-

Definition (INDEPENDENT SET)

Input: Graph $G = (V, E)$,
Goal: Finding subset $\overline{S} \subseteq V$:,

- no edge has both ends in $\overline{S}$

What is known: Graphs of bounded degree (d)

	VERTEX COVER	INDEPENDENT SET
Best algorithm	$2-(2-o_d(1))\frac{\log\log d}{\log d}$	$O\left(\frac{d\log\log d}{\log d}\right)$
NP-hardness		$\frac{d}{2^{O(\sqrt{\log d})}}$
UGC-hardness	$2-(2+o_d(1))\frac{\log\log d}{\log d}$	$\Omega\left(\frac{d}{\log^2 d}\right)$
Hierarchy IGs	$2-O(\frac{\log\log d}{\log d})$	$\Omega\left(\frac{d}{\log d}\right)$

[Halperin, 2002] [Samorodnitsky and Trevisan, 2000]
[Austrin et al., 2009]

VERTEX COVER and INDEPENDENT SET

Definition (VERTEX COVER)

Input: Graph $G = (V, E)$,
Goal: Finding subset $S \subseteq V$:
- it touches each edge
-

Definition (INDEPENDENT SET)

Input: Graph $G = (V, E)$,
Goal: Finding subset $\overline{S} \subseteq V$:,
- no edge has both ends in $\overline{S}$

What is known: Graphs of bounded degree (d)

	VERTEX COVER	INDEPENDENT SET
Best algorithm	$2 - (2 - o_d(1))\frac{\log\log d}{\log d}$	$O\left(\frac{d \log\log d}{\log d}\right)$
NP-hardness		$\frac{d}{2^{O(\sqrt{\log d})}}$
UGC-hardness	$2 - (2 + o_d(1))\frac{\log\log d}{\log d}$	$\Omega\left(\frac{d}{\log^2 d}\right)$
Hierarchy IGs	$2 - O(\frac{\log\log d}{\log d})$ (LS$^+$)	$\Omega\left(\frac{d}{\log d}\right)$ (SA)

[Halperin, 2002] [Samorodnitsky and Trevisan, 2000]
[Austrin et al., 2009]

LP relaxation for VERTEX COVER

IP Formulation

$$\text{Minimize} \sum_{i \in V(G)} x_i \tag{1}$$

Variables: $x_1, \ldots, x_n \in \{0, 1\}$

Subject to:

$\forall ij \in E(G)\; x_i + x_j \geq 1$

LP relaxation for Vertex Cover

IP Formulation

$$\text{Minimize} \sum_{i \in V(G)} x_i \tag{1}$$

Variables: $x_1, \ldots, x_n \in \{0, 1\}$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

LP relaxation

$$\text{Minimize} \sum_{i \in V(G)} x_i \tag{2}$$

Variables: $x_1, \ldots, x_n \in [0, 1]$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

LP relaxation for VERTEX COVER

IP Formulation

$$\text{Minimize} \sum_{i \in V(G)} x_i \quad (1)$$

Variables: $x_1, \ldots, x_n \in \{0, 1\}$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

Exact, NP-hard to solve.

LP relaxation

$$\text{Minimize} \sum_{i \in V(G)} x_i \quad (2)$$

Variables: $x_1, \ldots, x_n \in [0, 1]$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

Not Exact, easy to solve.

LP relaxation for VERTEX COVER

IP Formulation

$$\text{Minimize} \sum_{i \in V(G)} x_i \qquad (1)$$

Variables: $x_1, \ldots, x_n \in \{0, 1\}$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

Exact, NP-hard to solve.

LP relaxation

$$\text{Minimize} \sum_{i \in V(G)} x_i \qquad (2)$$

Variables: $x_1, \ldots, x_n \in [0, 1]$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

Not Exact, easy to solve.

- *Integrality gap:* The ratio (1)/(2).

LP relaxation for VERTEX COVER

IP Formulation

$$\text{Minimize} \sum_{i \in V(G)} x_i \qquad (1)$$

Variables: $x_1, \ldots, x_n \in \{0, 1\}$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

Exact, NP-hard to solve.

LP relaxation

$$\text{Minimize} \sum_{i \in V(G)} x_i \qquad (2)$$

Variables: $x_1, \ldots, x_n \in [0, 1]$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

Not Exact, easy to solve.

- *Integrality gap:* The ratio (1)/(2).
  Standard for how good the relaxation is.

LP relaxation for VERTEX COVER

IP Formulation

$$\text{Minimize} \sum_{i \in V(G)} x_i \tag{1}$$

Variables: $x_1, \ldots, x_n \in \{0, 1\}$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

Exact, NP-hard to solve.

LP relaxation

$$\text{Minimize} \sum_{i \in V(G)} x_i \tag{2}$$

Variables: $x_1, \ldots, x_n \in [0, 1]$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

Not Exact, easy to solve.

- *Integrality gap:* The ratio (1)/(2).
  Standard for how good the relaxation is.
- $\text{IG} \leq 2.$

LP relaxation for VERTEX COVER

IP Formulation

$$\text{Minimize} \sum_{i \in V(G)} x_i \qquad (1)$$

Variables: $x_1, \ldots, x_n \in \{0, 1\}$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

Exact, NP-hard to solve.

LP relaxation

$$\text{Minimize} \sum_{i \in V(G)} x_i \qquad (2)$$

Variables: $x_1, \ldots, x_n \in [0, 1]$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

Not Exact, easy to solve.

- *Integrality gap:* The ratio (1)/(2).
  Standard for how good the relaxation is.
- $2 - o(1) \leq \text{IG} \leq 2$.

LP relaxation for VERTEX COVER

IP Formulation

$$\text{Minimize} \sum_{i \in V(G)} x_i \qquad (1)$$

Variables: $x_1, \ldots, x_n \in \{0, 1\}$

Subject to:

$\forall ij \in E(G)\; x_i + x_j \geq 1$

LP relaxation

$$\text{Minimize} \sum_{i \in V(G)} x_i \qquad (2)$$

Variables: $x_1, \ldots, x_n \in [0, 1]$

Subject to:

$\forall ij \in E(G)\; x_i + x_j \geq 1$

Exact, NP-hard to solve.

Not Exact, easy to solve.

- *Integrality gap:* The ratio (1)/(2).
  Standard for how good the relaxation is.
- $2 - o(1) \leq \text{IG} \leq 2$.
  factor 2 is inherent in (simple) LP based approaches.

"Strengthening" the LP relaxation

LP relaxation

Minimize $\sum_{i \in V(G)} x_i$

Variables: $x_1, \ldots, x_n \in [0, 1]$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

"Strengthening" the LP relaxation

- A distribution μ of VERTEX COVERS,

LP relaxation

Minimize $\sum_{i \in V(G)} x_i$

Variables: $x_1, \ldots, x_n \in [0, 1]$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

"Strengthening" the LP relaxation

- A distribution μ of VERTEX COVERS,
- $x_i = \Pr_{S \sim \mu}[i \in S]$,

LP relaxation

Minimize $\sum_{i \in V(G)} x_i$

Variables: $x_1, \ldots, x_n \in [0, 1]$

Subject to:

$\forall ij \in E(G)\; x_i + x_j \geq 1$

"Strengthening" the LP relaxation

- A distribution μ of Vertex Covers,
- $x_i = \Pr_{S\sim\mu}[i \in S]$,
- Add variable/constraints to encode more information about μ:

LP relaxation

Minimize $\sum_{i\in V(G)} x_i$

Variables: $x_1, \ldots, x_n \in [0,1]$

Subject to:

$\forall ij \in E(G)\; x_i + x_j \geq 1$

"Strengthening" the LP relaxation

- A distribution μ of VERTEX COVERS,
- $x_i = \Pr_{S \sim \mu}[i \in S]$,
- Add variable/constraints to encode more information about μ: $x_{ij} = \Pr_{S \sim \mu}[i \in S; j \in S]$

"Strong" LP relaxation

Minimize $\sum_{i \in V(G)} x_i$

Variables: $x_1, \ldots, x_n \in [0, 1]$

$x_{ij} \in [0, 1]$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

"Strengthening" the LP relaxation

- A distribution μ of VERTEX COVERS,
- $x_i = \Pr_{S\sim\mu}[i \in S]$,
- Add variable/constraints to encode more information about μ: $x_{ij} = \Pr_{S\sim\mu}[i \in S; j \in S]$

"Strong" LP relaxation

Minimize $\sum_{i\in V(G)} x_i$

Variables: $x_1, \ldots, x_n \in [0,1]$
$x_{ij} \in [0,1]$

Subject to:
$\forall ij \in E(G)\ x_i + x_j \geq 1$
$\forall ij\ x_i + x_j - x_{ij} \geq 0$

"Strengthening" the LP relaxation

- A distribution μ of VERTEX COVERS,
- $x_i = \Pr_{S\sim\mu}[i \in S]$,
- Add variable/constraints to encode more information about μ:
  $x_{ij} = \Pr_{S\sim\mu}[i \in S; j \in S]$

  (Equivalent to Sherali-Adams Hierarchy)

"Strong" LP relaxation

Minimize $\sum_{i \in V(G)} x_i$

Variables: $x_1, \ldots, x_n \in [0, 1]$

$x_{ij} \in [0, 1]$

Subject to:

$\forall ij \in E(G)\ x_i + x_j \geq 1$

$\forall ij\ x_i + x_j - x_{ij} \geq 0$

"Strengthening" the LP relaxation

- A distribution μ of VERTEX COVERS,
- $x_i = \Pr_{S \sim \mu} [i \in S]$,
- Add variable/constraints to encode more information about μ:
  $x_{ij} = \Pr_{S \sim \mu} [i \in S; j \in S]$
  $M = \left[x_{ij}\right]_{1 \leq i,j \leq n} \succeq 0$

"Strong" LP relaxation

$$\text{Minimize} \sum_{i \in V(G)} x_i$$

$$\text{Variables: } x_1, \ldots, x_n \in [0,1]$$

$$x_{ij} \in [0,1]$$

Subject to:

$$\forall ij \in E(G)\ x_i + x_j \geq 1$$

$$\forall ij\ x_i + x_j - x_{ij} \geq 0$$

$$M \succeq 0$$

"Strengthening" the LP relaxation

- A distribution μ of VERTEX COVERS,
- $x_i = \Pr_{S\sim\mu}[i \in S]$,
- Add variable/constraints to encode more information about μ:
  $x_{ij} = \Pr_{S\sim\mu}[i \in S; j \in S]$
  $M = \left[x_{ij}\right]_{1\le i,j\le n} \succeq 0$
  (Equivalent to SDP relaxation of VERTEX COVER)

"Strong" LP relaxation

Minimize $\sum_{i\in V(G)} x_i$

Variables: $x_1, \ldots, x_n \in [0,1]$
$x_{ij} \in [0,1]$

Subject to:
$\forall ij \in E(G)\ x_i + x_j \ge 1$
$\forall ij\ x_i + x_j - x_{ij} \ge 0$
$M \succeq 0$

Lift and Project methods

Lift-and-Project methods

- Axiomatic methods to strengthen a relaxation.

Lift and Project methods

Lift-and-Project methods

- Axiomatic methods to strengthen a relaxation.
- Often have a parameter “level” ℓ adjusting the strength.

Lift and Project methods

Lift-and-Project methods

- Axiomatic methods to strengthen a relaxation.
- Often have a parameter “level” ℓ adjusting the strength.
- $\ell = 0$: original formulation, $\ell = O(1)$: tractable, $\ell = n$: exact formulation.

Lift and Project methods

Lift-and-Project methods

- Axiomatic methods to strengthen a relaxation.
- Often have a parameter “level” ℓ adjusting the strength.
- $\ell = 0$: original formulation, $\ell = O(1)$: tractable, $\ell = n$: exact formulation.
- Relaxation used in many algorithms is weaker than $\ell = 4$.

Lift and Project methods

Lift-and-Project methods

- Axiomatic methods to strengthen a relaxation.
- Often have a parameter "level" ℓ adjusting the strength.
- $\ell = 0$: original formulation, $\ell = O(1)$: tractable, $\ell = n$: exact formulation.
- Relaxation used in many algorithms is weaker than $\ell = 4$.
- Used in algorithms [Chlamtac, 2007], [Bateni et al., 2009], [Barak et al., 2011],...

Lift and Project methods

Lift-and-Project methods

- Axiomatic methods to strengthen a relaxation.
- Often have a parameter "level" ℓ adjusting the strength.
- $\ell = 0$: original formulation, $\ell = O(1)$: tractable, $\ell = n$: exact formulation.
- Relaxation used in many algorithms is weaker than $\ell = 4$.
- Used in algorithms [Chlamtac, 2007], [Bateni et al., 2009], [Barak et al., 2011],...
- Integrality Gap studied extensively [Arora et al., 2006], [Charikar, 2002], [de la Vega and Kenyon-Mathieu, 2007], [Georgiou et al., 2007], [Schoenebeck, 2008], [Raghavendra and Steurer, 2009], ...

General Strategy

- Start with IG for VERTEX COVER (unbounded degree)

General Strategy

- Start with IG for VERTEX COVER (unbounded degree): $G = (V, E)$ such that,
  - If $S \subset V(G)$ is "small", there is an edge with both ends outside S.

General Strategy

- Start with IG for VERTEX COVER (unbounded degree): $G = (V, E)$ such that,
  - If $S \subset V(G)$ is "small", there is an edge with both ends outside S.
  - The value of the VERTEX COVER relaxation for G is small.

General Strategy

- Start with IG for Vertex Cover (unbounded degree): $G = (V, E)$ such that,
  - If $S \subset V(G)$ is "small", there is an edge with both ends outside S.
  - The value of the Vertex Cover relaxation for G is small.
- Sample $|V(G)|d/4$ edges of G at random, call the result $\tilde{G}$

General Strategy

- Start with IG for Vertex Cover (unbounded degree): $G = (V, E)$ such that,
  - If $S \subset V(G)$ is "small", there is an edge with both ends outside S.
  - The value of the Vertex Cover relaxation for G is small.
- Sample $|V(G)|d/4$ edges of G at random, call the result $\tilde{G}$
- $\tilde{G}$ has average degree $d/2$, maximum degree $\leq d$.

General Strategy

- Start with IG for Vertex Cover (unbounded degree): $G = (V, E)$ such that,
  - If $S \subset V(G)$ is "small", there is an edge with both ends outside S.
  - The value of the Vertex Cover relaxation for G is small.
- Sample $|V(G)|d/4$ edges of G at random, call the result $\tilde{G}$
- $\tilde{G}$ has average degree $d/2$, maximum degree $\leq d$.
- Is $\tilde{G}$ an IG instance?

General Strategy

- Start with IG for VERTEX COVER (unbounded degree): $G = (V, E)$ such that,
  - If $S \subset V(G)$ is "small", there is an edge with both ends outside S.
  - The value of the VERTEX COVER relaxation for G is small.
- Sample $|V(G)|d/4$ edges of G at random, call the result $\tilde{G}$
- $\tilde{G}$ has average degree $d/2$, maximum degree $\leq d$.
- Is $\tilde{G}$ an IG instance?

  - The value of the VERTEX COVER relaxation for $\tilde{G}$ is small.

General Strategy

- Start with IG for VERTEX COVER (unbounded degree): $G = (V, E)$ such that,
  - If $S \subset V(G)$ is "small", there is an edge with both ends outside S.
  - The value of the VERTEX COVER relaxation for G is small.
- Sample $|V(G)|d/4$ edges of G at random, call the result $\tilde{G}$
- $\tilde{G}$ has average degree $d/2$, maximum degree $\leq d$.
- Is $\tilde{G}$ an IG instance?
  - If $S \subset V(\tilde{G})$ is "small", is there an edge with both ends outside S?
  - The value of the VERTEX COVER relaxation for $\tilde{G}$ is small.

General Strategy

- Start with IG for VERTEX COVER (unbounded degree): $G = (V, E)$ such that,
  - If $S \subset V(G)$ is "small", there is an edge with both ends outside S.
  - The value of the VERTEX COVER relaxation for G is small.
- Sample $|V(G)|d/4$ edges of G at random, call the result $\tilde{G}$
- $\tilde{G}$ has average degree $d/2$, maximum degree $\leq d$.
- Is $\tilde{G}$ an IG instance?
  - If $S \subset V(\tilde{G})$ is "small", is there an edge with both ends outside S?
  - The value of the VERTEX COVER relaxation for $\tilde{G}$ is small.
- We have to show $\bar{S}$ is dense in G!

Frankl-Rödl Graphs[Frankl and Rodl, 1987]

- $G_\lambda^{(n)} = (\{0,1\}^n, E)$.

Frankl-Rödl Graphs[Frankl and Rodl, 1987]

- $G_\lambda^{(n)} = (\{0,1\}^n, E)$.
- $(x, y) \in E(G) \iff d_H(x, y) = (1 - \lambda)n$.

Frankl-Rödl Graphs[Frankl and Rodl, 1987]

- $G_\lambda^{(n)} = (\{0,1\}^n, E)$.
- $(x,y) \in E(G) \iff d_H(x,y) = (1-\lambda)n$.

Theorem ([Frankl and Rodl, 1987])

If $U \subseteq \{0,1\}^n$ and $|U| > \xi^n 2^n$ implies U is not independent.

Frankl-Rödl Graphs[Frankl and Rodl, 1987]

- $G_\lambda^{(n)} = (\{0,1\}^n, E)$.
- $(x,y) \in E(G) \iff d_H(x,y) = (1-\lambda)n$.

Theorem ([Frankl and Rodl, 1987])

If $U \subseteq \{0,1\}^n$ and $|U| > \xi^n 2^n$ implies U is not independent.

We need to show such U is also dense!

Frankl-Rödl Graphs[Frankl and Rodl, 1987]

- $G_\lambda^{(n)} = (\{0,1\}^n, E)$.
- $(x,y) \in E(G) \iff d_H(x,y) = (1-\lambda)n$.

Theorem ([Frankl and Rodl, 1987])

If $U \subseteq \{0,1\}^n$ and $|U| > \xi^n 2^n$ implies U is not independent.

We need to show such U is also dense!

Theorem

$$\forall U \subseteq \{0,1\}^n, |U| \geq \mu 2^n :$$
$$\Pr_{x,y : d_H(x,y)=(1-\lambda)n} [x,y \in U] > \epsilon = \epsilon(\mu,\lambda).$$

Integrality Gap results

Theorem

For any constant ℓ, the Integrality Gap for level-ℓ Lovasz-Schrijver SDP relaxation (LS^+) for VERTEX COVER *in graphs of maximum degree d is* $2 - O\left(\frac{\log\log d}{\log d}\right)$.

Integrality Gap results

Theorem

For any constant ℓ, the Integrality Gap for level-ℓ Lovasz-Schrijver SDP relaxation (LS^+) for VERTEX COVER *in graphs of maximum degree d is $2 - O\left(\frac{\log \log d}{\log d}\right)$.*

Proof: Apply the above construction to [Georgiou et al., 2007].

Integrality Gap results

Theorem

For any constant ℓ, the Integrality Gap for level-ℓ Lovasz-Schrijver SDP relaxation (LS^+) for VERTEX COVER *in graphs of maximum degree d is $2 - O\left(\frac{\log\log d}{\log d}\right)$.*

Proof: Apply the above construction to [Georgiou et al., 2007].

Theorem

For any constant ℓ, the Integrality Gap for level-ℓ Sherali-Adams LP relaxation for INDEPENDENT SET *in graphs of maximum degree d is $\Omega\left(\frac{d}{\log d}\right)$.*

Integrality Gap results

Theorem

For any constant ℓ, the Integrality Gap for level-ℓ Lovasz-Schrijver SDP relaxation (LS^+) for VERTEX COVER *in graphs of maximum degree d is $2 - O\left(\frac{\log\log d}{\log d}\right)$.*

Proof: Apply the above construction to [Georgiou et al., 2007].

Theorem

For any constant ℓ, the Integrality Gap for level-ℓ Sherali-Adams LP relaxation for INDEPENDENT SET *in graphs of maximum degree d is $\Omega\left(\frac{d}{\log d}\right)$.*

Proof: Apply the above construction to [Benabbas et al., 2011].

Integrality Gaps in Bounded-degree Graphs: Open Questions

- Improve the UGC-hardness of INDEPENDENT SET in degree-bounded graphs,

Integrality Gaps in Bounded-degree Graphs: Open Questions

- Improve the UGC-hardness of INDEPENDENT SET in degree-bounded graphs,
- Extend our results to other hierarchies.

Integrality Gaps in Bounded-degree Graphs: Open Questions

- Improve the UGC-hardness of INDEPENDENT SET in degree-bounded graphs,
- Extend our results to other hierarchies.

Thank you!

Arora, S., Bollobás, B., Lovász, L., and Tourlakis, I. (2006).
Proving integrality gaps without knowing the linear program.
Theory of Computing, 2(2):19–51.

Austrin, P., Khot, S., and Safra, M. (2009).
Inapproximability of Vertex Cover and Independent Set in bounded degree graphs.
Annual IEEE Conference on Computational Complexity, pages 74–80.

Barak, B., Raghavendra, P., and Steurer, D. (2011).
Rounding semidefinite programming hierarchies via global correlation.
In *FOCS'11*.
To appear.

Bateni, M. H., Charikar, M., and Guruswami, V. (2009).
MaxMin allocation via degree lower-bounded arborescences.
In *STOC'09*, pages 543–552. ACM Press.

Benabbas, S., Chan, S. O., Georgiou, K., and Magen, A. (2011).
Tight integrality gap for Sherali-Adams SDPs for Vertex Cover.
to appear in FSTTCS.

Charikar, M. (2002).
On semidefinite programming relaxations for graph coloring and Vertex Cover.
In *SODA'02: Proceedings of the 13th annual ACM-SIAM symposium on Discrete algorithms*, pages 616–620, Philadelphia, PA, USA. ACM Press.

Charikar, M., Makarychev, K., and Makarychev, Y. (2009).
Integrality gaps for Sherali-Adams relaxations.
In *STOC'09: Proceedings of the 41st annual ACM symposium on Theory of computing*, pages 283–292, New York, NY, USA. ACM Press.

Chlamtac, E. (2007).
Approximation Algorithms Using Hierarchies of Semidefinite Programming Relaxations.
In *IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 691–701.

de la Vega, W. F. and Kenyon-Mathieu, C. (2007).
Linear programming relaxations of Max Cut.
In *SODA'07*, pages 53–61. ACM Press.

Dinur, I. and Safra, S. (2005).

On the Hardness of Approximating Minimum Vertex Cover.
Annals of Mathematics, 162(1):439–485.

Feige, U. (2004).
Approximating Maximum Clique by Removing Subgraphs.
SIAM J. Discrete Math., 18(2):219–225.

Frankl, P. and Rodl, V. (1987).
Forbidden intersections.
Transactions of the American Mathematical Society, 300(1):259–286.

Georgiou, K., Magen, A., Pitassi, T., and Tourlakis, I. (2007).
Integrality Gaps of $2 - o(1)$ for Vertex Cover SDPs in the Lovász-Schrijver Hierarchy.
In *IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 702–712.

Halperin, E. (2002).
Improved Approximation Algorithms for the Vertex Cover Problem in Graphs and Hypergraphs.
SIAM J. Comput., 31(5):1608–1623.

Håstad, J. (1996).
Clique is hard to approximate within $n^{1-\epsilon}$.
In *IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 627–636.

Karakostas, G. (2005).
A Better Approximation Ratio for the Vertex Cover Problem.
In *International Colloquium on Automata, Languages and Programming (ICALP)*, pages 1043–1050.

Khot, S. and Regev, O. (2008).
Vertex Cover Might be Hard to Approximate to Within $2-\epsilon$.

Journal of Computer and System Sciences, 74(3):335–349.

Mossel, E., O'Donnell, R., Regev, O., Steif, J. E., and Sudakov, B. (2006).
Non-interactive correlation distillation, inhomogeneous Markov chains, and the reverse Bonami-Beckner inequality.
Israel Journal of Mathematics, 154:299–336.

Raghavendra, P. and Steurer, D. (2009).
Integrality gaps for strong SDP relaxations of Unique Games.
In *FOCS'09*, pages 575–585. IEEE Computer Society.

Samorodnitsky, A. and Trevisan, L. (2000).
A PCP characterization of NP with optimal amortized query complexity.

In *ACM Symposium on Theory of Computing (STOC)*, pages 191–199.

Schoenebeck, G. (2008).
Linear Level Lasserre Lower Bounds for Certain k-CSPs.
In *IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 593–602.

Tulsiani, M. (2009).
CSP gaps and reductions in the Lasserre hierarchy.
In *STOC'09: Proceedings of the 41th annual ACM symposium on Theory of computing*, pages 303–312, New York, NY, USA. ACM Press.