Constructions of Expanders Using Group Theory

Martin Kassabov,
Cornell University and IAS

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Overview

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Expanders

Well connected graphs of small degree are called expanders.

Definition *A graph Γ is called an ε-expander if for any set of vertices A, such that $|A| \leq |\Gamma|/2$ we have $|\partial A| \geq \varepsilon |A|$.*

The maximal ε with this property is called the expanding constant of Γ.

A family of graphs (of bounded degree) is called an expander family if their expanding constants are uniformly bounded.

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Equivalent definitions of Expanders

Theorem *A family of k-regular graphs Γ_i are expanders if any of the following holds:*

- *the expanding constants of Γ_i are bounded from 0,*
- *the Cheeger constants of Γ_i are bounded form 0,*
- *the spectral gaps of the Laplacians (Id- normalized adjacency matrices) are bounded form 0.*

Any of these conditions implies that the random walks on Γ_i mix in $O(\log |\Gamma_i|)$ steps.

If one allows k to increase then the these conditions are NOT equivalent, thus there are several different definitions of unbounded degree expanders.

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Cayley Graphs as Expanders

In the case of Cayley graphs of finite groups, there is an other condition equivalent to “expansion”, which is related to representation theory, more precisely to the “Kazhdan constants”.

Problem *Let be $\{G_i\}$ a family of finite groups. Is it possible to find generating sets S_i, which make the Cayley graphs $\mathcal{C}(G_i, S_i)$ expanders?*

- there are many cases where such generating sets are known to exist, e.g., for quotients of a group with a variant of property T;
- and only a few where it can be proven that such generating sets do not exist, e.g., solvable groups of a fixed class.

Kazhdan Constants

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Unitary Representations

Let $\mathcal{H}$ ba a Hilbert space.
A representation a group of G into $\mathcal{H}$ is called unitary if

$$\|g(v)\| = \|v\|$$

for any $v \in \mathcal{H}$ and $g \in G$.

Definition *A unit vector $v \in \mathcal{H}$ is called ε-almost invariant under the set S if*

$$\|g(v) - v\| \leq \varepsilon \text{ for any } g \in S.$$

One easy way to produce almost invariant vectors is to take perturbations of invariant vectors.

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Kazhdan Property T

Definition *Let G be a group, generated by a finite set S. The maximal ε such that the existence of an ε-almost invariant vector implies the existence of an invariant vector is called Kazhdan constant of G with respect to S and is denoted by $\mathcal{K}(G; S)$.*

The group G has property T if the Kazhdan constant $\mathcal{K}(G; S)$ is positive.

If H is a normal subgroup of G, we can also define about *relative Kazhdan constants*:

Definition *The maximal ε such that the existence of an ε-almost invariant vector implies the existence of an H invariant vector is called relative Kazhdan constant of G with respect to S relative to H and is denoted by $\mathcal{K}(G, H; S)$.*

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Examples of Groups with T

- All finite groups
- Lattices in high rank Lie groups, e.g., $\mathrm{SL}_n(\mathbb{Z})$
- Random group, with sufficiently many relations

Infinite groups with property T can be used to construct expander graphs.

Also, families of finite groups with a generating sets of fixed size lead to expanders, provided that their Kazhdan constants are uniformly bounded.

Moreover, estimates for the Kazhdan constant give to estimates for the mixing time for some random walks.

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Several Kazhdan Constants

Let G be a group, generated by a finite set S.

The Kazhdan constant of G is

$$\mathcal{K}(G; S) = \inf_{\mathcal{H}} \inf_{v \in \mathcal{H}} \max_{s \in S} \frac{||\rho(s)v - v||}{||v||}$$

where $\mathcal{H}$ is a representation of G without invariant vectors

Similarly we can define

$$\mathcal{K}^2_{av}(G; S) = \inf_{\mathcal{H}} \inf_{v \in \mathcal{H}} \frac{1}{|S|} \sum_{s \in S} \frac{||\rho(s)v - v||^2}{||v||^2}$$

$\mathcal{K}^2_{av}(G; S)$ is equal to twice the spectral gap of the Laplacian and the $\inf$ is achieved at some irreducible representation.

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Computation of Kazhdan Constants

Trivial bounds

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It is very easy to see that:

$$2 \geq \mathcal{K}(G;G) > \sqrt{2},$$

and

$$\mathcal{K}^2_{av}(G;G) = 2.$$

These inequalities hold, because if unit vector v is moved by less than $\sqrt{2}$ by any element of the group G then the whole orbit of v lies in some half-space.
Thus, the center of mass of the orbit is a non-zero invariant vector in the representation.

Equivalently, we use that the normalized adjacency matrix of a complete graph has a spectral gap equal to 1.

Dependance on the generating set I

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Let G be a group and let S be a set (containing the identity) then

$$\mathcal{K}(G;S) \geq \frac{1}{k}\mathcal{K}(G;S^k),$$

where S^k denotes all group elements which can be written as a product of k elements from S.

These inequality holds, because any ϵ-almost invariant vector w.r.t. S is also $k\epsilon$-almost invariant w.r.t. to S^k.

Dependance on the generating set II

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If H is a normal subgroup of G then

$$\mathcal{K}(G;S) \geq \frac{1}{2}\mathcal{K}(G;H \cup S)\mathcal{K}(G,H;S),$$

where $\mathcal{K}(G,H;S)$ is the relative Kazhdan constant.

These inequality holds, because any ϵ-almost invariant vector w.r.t. S is also $2\mathcal{K}(G,H;S)^{-1}\epsilon$-almost invariant w.r.t. to H.

Lemma *If $H_i \lhd N_i$ are subgroups of G such that*

$$\mathcal{K}(N_i,H_i;S_i) \geq L.$$

Then

$$\mathcal{K}(G;\cup S_i) \geq \frac{1}{2}L \times \mathcal{K}(G;\cup H_i).$$

Some Relative Kazhdan Constants

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Theorem (Burger, Shalom, K.)
Let R be an associative ring generated by $\alpha_i, i = 1,\dots,k$. Consider the subgroup $\mathrm{GL}_2(R) \ltimes R^2 \subset \mathrm{GL}_3(R)$.

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix}$$

Then

$$\mathcal{K}(\mathrm{GL}_2(R) \ltimes R^2, R^2; F) \geq \frac{1}{5(\sqrt{k}+3)},$$

where F is the set consisting of $4(k+1)$ elementary matrices with ± 1 or $\pm\alpha_i$ off the diagonal and the two standard generators of R^2.

The proof uses almost invariant measures on the unitary dual of R^2 considered as additive group.

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Definition *A group G is said to be boundedly generated by the ordered multi-set $S = \{s_1, s_2, \dots, s_k\}$ if any element $g \in G$ can be written a a product*

$$g = s_1^{a_1} s_2^{a_2} \dots s_k^{a_k}$$

for some integers a_i.

This is a very strong condition and implies strong restrictions on the group G.

Example: $\mathrm{SL}_n(\mathbb{Z})$

Theorem (Carter-Keller) *Any matrix in $\mathrm{SL}_n(\mathbb{Z})$ can be written as a product of less than $2n^2 + 100$ elementary matrices.*

The proof uses reduction from $\mathrm{SL}_n(\mathbb{Z})$ to $\mathrm{SL}_{n-1}(\mathbb{Z})$ for $n \geq 3$, which only uses existence of primes in arithmetic progression.

The main step is to show that any element in $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_3(\mathbb{Z})$ can be written as a product of 100 elementary matrices.
This step uses K-theory and playing with Mennicke symbols.

If we are only interested in $\mathrm{SL}_n(\mathbb{F}_p)$ then the proof becomes much easier and uses only Gauss elimination.

Uniform bounded elementary generation

For groups like $\mathrm{SL}_n(R)$ it is better to consider the following property

Definition *A group G called is boundedly generated by the subgroups H_i, if there exists N such that any $g \in G$ can be written as*

$$g = g_1 g_2 \dots g_N, \quad \text{where } g_i \in \bigcup H_i.$$

Theorem *Let R be a "nice" finite associative ring with 1, then any element in $\mathrm{EL}_3(R)$ can be written as a product of 50 elementary matrices.*

A "nice" ring is any product of matrix rings over fields, or commutative rings.

It is unknown if similar result holds for $\mathrm{SL}_n(\mathbb{Z}[x])$ for a big n.
It if false for $\mathrm{SL}_n(\mathbb{C}[x])$.

Kazhdan Constants for $\mathrm{SL}_n(\mathbb{F}_p)$

$\mathrm{SL}_n(\mathbb{F}_p)$ for fixed $n \geq 3$

The homomorphisms from $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ to $\mathrm{SL}_n(\mathbb{F}_p)$, and estimates for the relative Kazhdan constant of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ imply that

$$\mathcal{K}(\mathrm{SL}_n(\mathbb{F}_p); E_n(\pm 1)) \geq \frac{1}{20} \mathcal{K}(\mathrm{SL}_n(\mathbb{F}_p); E_n),$$

where $E_n(\pm 1)$ is the set of elementary matrices with ± 1 off the diagonal and E_n is the set of all el. matrices.

By bounded generation ${E_n}^{2n^2} = \mathrm{SL}_n$, therefore

$$\mathcal{K}(\mathrm{SL}_n(\mathbb{F}_p); E_n) \geq \frac{1}{2n^2} \mathcal{K}(\mathrm{SL}_n(\mathbb{F}_p); \mathrm{SL}_n(\mathbb{F}_p)) \geq \frac{1}{2n^2}$$

Theorem (Shalom)

$$\mathcal{K}(\mathrm{SL}_n(\mathbb{F}_p); E_n(\pm 1)) > \frac{1}{40n^2}.$$

$\mathrm{SL}_{3n}(\mathbb{F}_p)$ for a all n and p

Block decomposition implies $\mathrm{SL}_{3n}(\mathbb{F}_p) \simeq \mathrm{EL}_3(\mathrm{Mat}_n(\mathbb{F}_p))$.

Since the ring $\mathrm{Mat}_n(\mathbb{F}_p)$ is 2-generated, and $\mathrm{EL}_3(\mathrm{Mat}_n(\mathbb{F}_p))$ has uniform bounded generation property, we have:

$$\mathcal{K}(\mathrm{SL}_{3n}(\mathbb{F}_p); E_3(\pm 1, A, B)) \geq \frac{1}{20} \mathcal{K}(\mathrm{SL}_{3n}(\mathbb{F}_p); E_3),$$

$$\mathcal{K}(\mathrm{SL}_{3n}(\mathbb{F}_p); E_3)) \geq \frac{1}{40} \mathcal{K}(\mathrm{SL}_{3n}(\mathbb{F}_p); \mathrm{SL}_{3n}(\mathbb{F}_p)) \geq \frac{1}{40}$$

where $E_3(\pm 1, A, B)$ is the set of block elementary matrices with generators of $\mathrm{Mat}_n(\mathbb{F}_p)$ off the diagonal and E_3 is the set of all block el. matrices.

Theorem (K) *There resists a generating set S of $\mathrm{SL}_{3n}(\mathbb{F}_p)$ with $|S| < 20$ and*

$$\mathcal{K}(\mathrm{SL}_{3n}(\mathbb{F}_p); S) \geq \frac{1}{1000}.$$

Using Geometry to (dis-)Prove Property T

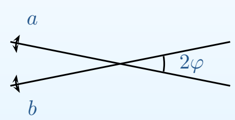
Example D_∞

The infinite dihedral group

$$D_\infty = \langle a, b \,|\, a^2 = b^2 = 1 \rangle$$

does not have property T.

For any φ there is a representation of D_∞ on $\mathbb{R}^2$ where a, b act as reflections across two lines at angle 2φ.
This representation does not have invariant vectors, but it has a unit vector which is $2\sin\varphi$ invariant.



This example shows that we need some bound for the angle between the subspaces $\mathcal{H}^{G_1}$ and $\mathcal{H}^{G_2}$ in order to prove property T for $G = \langle G_1, G_2 \rangle$.

Groups generated by involutions

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Assume that the generating set S consist of involutions.
Let $\mathcal{H}_s$ denotes the fixed subspace of $s \in S$.
Notice that:

- $\mathcal{H}$ has no G invariant vectors $\Leftrightarrow \bigcap \mathcal{H}_s = \{0\}$
- $\rho(s)v - v = 2d(v, \mathcal{H}_s)$

Thus, the group G has property T if for any representation of G on $\mathcal{H}$, any vector which is close to all subspaces $\mathcal{H}_s$ is not too far from the intersection $\bigcap \mathcal{H}_s$.

Angle between two subspaces

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Definition *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two subspaces of a Hilbert space $\mathcal{H}$. The angle $\sphericalangle(\mathcal{H}_1, \mathcal{H}_2)$ is the smallest angle between vectors v_1 and v_2, such that $v_i \in \mathcal{H}_i$ and $v_i \perp \mathcal{H}_1 \cap \mathcal{H}_2$.*

Notice that the operator norm of the addition

$$sum : \mathcal{H}_1/(\mathcal{H}_1 \cap \mathcal{H}_2) \oplus \mathcal{H}_2/(\mathcal{H}_1 \cap \mathcal{H}_2) \to \mathcal{H}/(\mathcal{H}_1 \cap \mathcal{H}_2)$$

is less than $\sqrt{1 + \cos \sphericalangle(\mathcal{H}_1, \mathcal{H}_2)}$.

Equivalently, we can define the angle using the spectral gap near $\sqrt{2}$ of the addition

$$sum : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}$$

Notice that the angle between two subspaces is 'not defined' if one is subspace of the other. In this case we say that the angle is $\pi/2$ and the subspaces are perpendicular.

Angle between many subspaces

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Definition *Let $\mathcal{H}_i$, $i = 1, \dots, n$ be subspaces of a Hilbert space $\mathcal{H}$. We say that the angle $\sphericalangle(\mathcal{H}_i)$ between them is ϕ if*

$$\cos\phi = \frac{1}{n-1}\left(\sup \frac{||\sum v_i||^2}{\sum ||v_i||^2} - 1\right)$$

where $\sup$ is taken over $v_i \in \mathcal{H}_i$ and $v_j \perp \bigcap_i \mathcal{H}_i$.

Equivalently the norm of the addition

$$sum : \mathcal{H}_1/\cap\mathcal{H}_i \oplus \dots \mathcal{H}_n/\cap\mathcal{H}_i \to \mathcal{H}/\cap\mathcal{H}_i$$

is $\sqrt{1 + (n-1)\cos\phi}$.

Notice that the angle between any collection of subspaces in a finite dimensional spaces is positive, but this is not true in general.

Also any lower bound for $\sphericalangle(\mathcal{H}_i)$ gives a lower bound of maximal distance from a unit vector v to the subspaces $\mathcal{H}_i$.

Bounding the angle between subspaces

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Lemma *If the the involutions s and t generate a dihedral group of order $2k$ then the angle $\sphericalangle(\mathcal{H}_s, \mathcal{H}_t) \geq \pi/k$, where $\mathcal{H}_s$ and $\mathcal{H}_t$ are the fixed points of s and t in any unitary representation of G.*

The proof uses the representation theory of the dihedral group D_k.

Similar result hold if s and t are elements of order p and $\langle s, t \rangle$ is the Heisenberg group mod p. In this case we have

$$\cos \sphericalangle(\mathcal{H}_s, \mathcal{H}_t) \leq \frac{1}{\sqrt{p}}.$$

Again it suffices to verify this bound for any irreducible representation of the Heisenberg group.

Groups generated by 3 involutions

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Let G be a group generated by 3 involutions, i.e., G is a Coxeter group

$$G = \langle s_1, s_2, s_3 \;|\; s_i^2 = 1, \; (s_i s_j)^{m_{ij}} = 1 \rangle$$

Question *When does G has property T?*

Using the classification of 3 generated Coxeter groups one can see that the following are equivalent:

- G has property T
- G is finite
- $\frac{1}{m_{12}} + \frac{1}{m_{23}} + \frac{1}{m_{31}} > 1$

3 Subspaces

Let $\mathcal{H}_i$ be 3 subspaces in a Hilbert space $\mathcal{H}$, with trivial intersection.

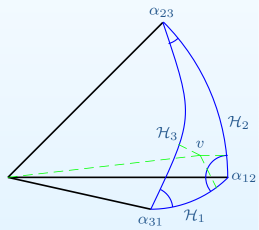
Lemma: Let α_{ij} denote the angle between $\mathcal{H}_i$ and $\mathcal{H}_j$. If $\sum \alpha_{ij} > \pi$ than any vector v which is closed to all $\mathcal{H}_i$, is short, i.e.,

$$||v||^2 \leq C \sum d(v, \mathcal{H}_i)^2,$$

where the constant C depends only in α_{ij}.
Notice that the condition

$$\sum \alpha_{ij} > \pi$$

is equivalent to the existence of spherical triangle with angles α_{ij}.



Subspace Arrangements

Subspace Arrangements

Definition *A* subspace arrangement *is the collection of all possible n-tuples of subspaces $\{\mathcal{H}_i\}_{i\in I}$ in some Hilbert space $\mathcal{H}$, satisfying a fixed set of conditions of the following type.*

- *for some subsets $J \subset I$ there exists α_J such that*

$$\sphericalangle(\mathcal{H}_j)_{j\in J} \geq \alpha_J$$

- *for some subsets $J \subset I$ and some $i \in I$ we have $\mathcal{H}_i \supset \bigcap_{j\in J} \mathcal{H}_j$*

A collection is called "good", if there is a strictly positive lower bound for

$$\sphericalangle(\mathcal{H}_i)_{i\in I},$$

which is valid for all n-tuples of subspaces which satisfy the conditions listed above.

3 Subspaces

The previous lemma is equivalent to saying that any 3 subspaces such that

- $\sphericalangle(\mathcal{H}_1, \mathcal{H}_2) \geq \alpha_{12}$
- $\sphericalangle(\mathcal{H}_2, \mathcal{H}_3) \geq \alpha_{23}$
- $\sphericalangle(\mathcal{H}_3, \mathcal{H}_1) \geq \alpha_{31}$

satisfy $\sphericalangle(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) \geq \alpha_0 > 0$ if

$$\alpha_{12} + \alpha_{23} + \alpha_{31} > \pi.$$

In other words the arrangement of 3 subspaces $\mathcal{H}_i$, satisfying the above conditions is "good".

Relationship with groups

Our motivation for studying such arrangements of subspaces comes from representation theory.

Let $\mathcal{H}$ be a representation of a group G and let G_i be finite subgroups of G. Denote $\mathcal{H}_i = \mathcal{H}^{G_i}$ the set of vectors fixed by each subgroup.

The two type of restrictions on the subspaces are related to properties of the group G (and the collections of subgroups G_i):

- representation theory of the group $\langle G_{i_1}, \ldots, G_{i_k}\rangle$ yields lower bounds for the angle $\sphericalangle(\mathcal{H}_{i_1}, \ldots, \mathcal{H}_{i_k})$,
- inclusions $G_j \subset \langle G_{i_1}, \ldots, G_{i_k}\rangle$ correspond $\mathcal{H}_j \supset \cap_k \mathcal{H}_{i_k}$.

Easy Observation

Theorem *Let G be a group generated by n finite subgroups G_i. Suppose that the inclusions between G_J for $J \subset \{1, \ldots, n\}$ and the representation theory of these groups define a "good" arrangement of subspaces.*
Then G has property T.

Easy Example

An Example

Suppose that a collection of subspaces is given only by conditions of the first type, in the simplest case we have bounds for the angles α_{ij} between any pair of $\mathcal{H}_i$-es.
If we find sufficient condition for having a good collection we can deduce property T for some groups.
Previous results: ($\varepsilon_{i,j} = \cos \alpha_{i,j}$)

- Dymara-Januszkiewicz proved that the collection is "good" if $\varepsilon_{ij} < 2^{-12n}$ for all i, j
- Ershov-Jaikin improved the bound to $\varepsilon_{ij} < \frac{1}{n-1}$ for all i, j
- They also showed that the collection is good if $n = 3$ and some inequality is satisfied.

Main result

Theorem (K.) *Let $\mathcal{H}_i$ are subspaces of a Hilbert space $\mathcal{H}$. Denote $\varepsilon_{i,j} = \cos(\sphericalangle(\mathcal{H}_i, \mathcal{H}_j))$. Suppose that the matrix*

$$A = \begin{pmatrix} 1 & -\varepsilon_{1,2} & -\varepsilon_{1,3} & \dots & -\varepsilon_{1,n} \\ -\varepsilon_{2,1} & 1 & -\varepsilon_{2,2} & \dots & -\varepsilon_{2,n} \\ -\varepsilon_{3,1} & -\varepsilon_{3,2} & 1 & \dots & -\varepsilon_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\varepsilon_{n,1} & -\varepsilon_{n,2} & -\varepsilon_{n,3} & \dots & 1 \end{pmatrix}$$

is positive definite, then there is lower bound for $\sphericalangle(\mathcal{H}_i)$, which depends only the matrix A.

Equivalent Statement

Theorem *The arrangement of n subspaces $\mathcal{H}_i$ satisfying the conditions*

$$\sphericalangle(\mathcal{H}_i, \mathcal{H}_j) \geq \alpha_{ij}$$

for all i, j, is "good" if there exists an n-dimensional spherical simplex with angles between the faces equal to α_{ij}.

Actually, "if" can be replaced with "if and only if", but the other direction is much easier.

"Proof"

It seems "obvious" that the most difficult case is when $\mathcal{H}_i$ are hyper-planes (they have co-dimension 1 in $\mathcal{H}$).

In this case WLOG we can assume that $\dim \mathcal{H} = n$ and we can find unit vectors u_i perpendicular to the hyper-planes $\mathcal{H}_i$. By construction we have $|\langle u_i, u_j \rangle| \leq \varepsilon_{i,j}$.
The volume of the simplex spanned by u_i is

$$V^2 = \det(\langle u_i, u_j \rangle)_{i,j} \geq \det A$$

is bounded from below, thus any interior point is "far" form at least one face, which translates to a lower bound for the angle between the subspaces $\mathcal{H}_i$.

In the general case one need to do a clever induction to justify the "obvious".

Application to property T

Theorem (Dymara-Januszkiewicz) *Let G be a group generated by a finite subgroup G_i. If for any i, j the group $G_{i,j} = \langle G_i, G_j \rangle$ has property T and the Kazhdan constant $\mathcal{K}(G_{ij}, G_i \cup G_j)$ is sufficiently big, than G has property T.*

Applications

Coxeter Groups

Theorem (K.) *A Coxeter group G has property T if and only if it is finite. Moreover, the Kazhdan constant $\mathcal{K}(G,S)$ can be computed using the defining representation of the group G.*

Corollary *The mixing time of the random walk on a Coxeter group G with respect to the standard generating set S is bounded by $n^3 \log n$, where n is the rank of G.*

$\mathrm{SL}_n(\mathbb{F}_p[t_1,\dots,t_k])$

Theorem *The groups $\mathrm{SL}_n(\mathbb{F}_p[t_1,\dots,t_k])$ have property T for any $n \geq 3$ and $p > 4$.*

Proof: The subgroups

$$G_i = \mathrm{Id} + \mathbb{F}_p E_{i,i+1}$$

and

$$G_n = \mathrm{Id} + (\mathbb{F}_p + t_1\mathbb{F}_p + \dots + t_k\mathbb{F}_p) E_{n,1}$$

$$\begin{pmatrix} 1 & G_1 & 0 & 0 \\ 0 & 1 & G_2 & 0 \\ 0 & 0 & 1 & G_3 \\ G_4 & 0 & 0 & 1 \end{pmatrix}$$

generate the group $\mathrm{SL}_n(\mathbb{F}_p[t_1,\dots,t_k])$.

Any two of these group either commute of generate a Heisenberg group mod p.

Thus the "matrix" of cosines of the angles between the subspaces of fixed points is:

$$\begin{pmatrix} 1 & -\varepsilon & 0 & \dots & -\varepsilon \\ -\varepsilon & 1 & -\varepsilon & \dots & 0 \\ 0 & -\varepsilon & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\varepsilon & 0 & 0 & \dots & 1 \end{pmatrix}$$

where $\varepsilon = p^{-1/2}$.
This matrix is positive definite if $\varepsilon < 1/2$.

Same result holds for Steinberg groups over any finitely generated ring of characteristic $p \geq 10$.

Groups graded by root systems

Definition *Let Φ be a finite root system. A group G is called graded by Φ if there exists root subgroup X_α for $\alpha \in \Phi$ such that*

- *G is generated by X_α*
- *if α and β are linearly independent then $[X_\alpha, X_\beta] \subset \langle X_\gamma \rangle$, where $\gamma = a\alpha + b\beta$ for $a, b \geq 1$*
- *if $\beta = a\alpha$ for $a \geq 1$ then $[X_\alpha, X_\beta] \subset \langle X_\gamma \rangle$, where $\gamma = b\beta$ for $b \geq 1$*
- *for each $\alpha \in \Phi$ there exists a set of positive roots Φ^+ such that $\alpha \in \Phi^+$ and*
  $$X_\alpha \subset \langle X_\gamma \,|\, \gamma \in \Phi^+,\, \gamma \neq \alpha \rangle.$$

Theorem (Ershov-Jaikin-K.) *Let G be a group graded by the root system Φ. Then the Kazhdan constant $\mathcal{K}(G, \cup X_\alpha) > 0$, i.e., G almost has property T.*

The proof is a bit messy...

Corollary *Let $G = St_\Phi(R)$ be a (twisted) Steinberg group of rank ≥ 2 over a finitely generated ring R.
Then G has property T and there is an lower bound for the Kazhdan constant $\mathcal{K}(G,S) > 0$, where S is the "natural" generating set of G.*