The fundamental curve of *p*-adic Hodge theory

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(Joint work with Laurent Fargues)

Introduction

Let C be an algebraically closed field of characteristic 0, complete with respect to a non archimedean non trivial absolute value, whose residue field is of characteristic $p > 0$.
p-adic Hodge theory : $C \mapsto B_{dR}, B_{st}$.
B_{dR} is a field, complete with respect to a discrete valuation, whose residue field is C.
B_{st} is a subring of B_{dR} equipped with two maps $N, \varphi : B_{st} \to B_{st}$, N is a derivation, φ is an endomorphism. $N\varphi = p\varphi N$.

$$B_e = \left\{ b \in B_{st} \mid N(b) = 0 \text{ and } \varphi(b) = b \right\} .$$

Theorem

We have $(B_e)^* = \mathbb{Q}_p^*$. *The ring* B_e *is a principal domain.*

The curve X

We want to compactify the affine scheme $X^e = \text{Spec } B_e$.

$$X = X^e \amalg \{\infty\} \ .$$

$B_e \subset B_{dR}$, v_{dR} = valuation on B_{dR}, $C_e = \mathrm{Frac\ B_e}$,
$\mathcal{O}_{X,\infty} = \{c \in C_e \mid v_{dR}(c) \geq 0\}$.

$$X = X^e \amalg_{\text{Spec } C_e} \text{Spec } \mathcal{O}_{X,\infty} = X^e \amalg_{\text{Spec} B_{dR}} \text{Spec} B_{dR}^+$$
$$\Gamma(U, \mathcal{O}_X) = \begin{cases} \Gamma(U, \mathcal{O}_{X^e}), & \text{if } \infty \notin U \\ \Gamma(U \backslash \{\infty\}, \mathcal{O}_{X^e}) \cap B_{dR}^+, & \text{if } \infty \in U \end{cases}$$

Vector bundles

$$\mathcal{F} = \text{ a vector bundle on } X \ \mapsto \ \text{a pair } (\mathcal{F}^e, \widehat{\mathcal{F}}_\infty) \ \text{ where}$$

– $\mathcal{F}^e = \Gamma(X^e, \mathcal{F})$ is a free B_e-module of finite rank,
– $\widehat{\mathcal{F}}_\infty = B_{dR}^+ \otimes_{\mathcal{O}_{X,\infty}} \mathcal{F}_\infty$ is a B_{dR}^+-lattice in $\mathcal{F}_{dR} = B_{dR} \otimes_{B_e} \mathcal{F}^e$.
This gives an equivalence of categories

Cohomology of vector bundles

$$0 \to H^0(X, \mathcal{F}) \to \mathcal{F}^e \oplus \widehat{\mathcal{F}}_\infty \to \mathcal{F}_{dR} \to H^1(X, \mathcal{F}) \to 0$$

$$0 \to \mathbb{Q}_p \to B_e \oplus B_{dR}^+ \to B_{dR} \to 0$$

$$\implies H^0(X, \mathcal{O}_X) = \mathbb{Q}_p \text{ and } H^1(X, \mathcal{O}_X) = 0.$$

Aims of this talk

1 – Give a more intrinsic definition of the scheme X and study its structure.

2 – Classify vector bundles on X.

3 – All this constructions are functorial. In particular, if C is the p-adic completion of the algebraic closure $\overline{K}$ of a p-adic field K, the group $G_K = \mathrm{Gal}(\overline{K}/K)$ acts on X.

Redo p-adic Hodge theory as a special case of the study of G_K-equivariant vector bundles over X.

(F, E, π)

$F =$ algebraically closed field of characteristic $p > 0$, complete with respect to a non trivial absolute value $|\ |$.
$E =$ non archimedean locally compact field whose residue field $\mathbb{F}_q$ is contained in F.
$\pi =$ uniformizing parameter of E.
To these datas (F, E, π) we'll associate a curve $X = X(F, E, \pi)$.
The curve X defined earlier will be $X(F(C), \mathbb{Q}_p, p)$.

The field $F(C)$

Let C be as earlier (alg. closed of char. 0, complete with $|\ |$, $|p| = p^{-1}$).

$$F(C) = \left\{x = (x^{(n)})_{n\in\mathbb{N}} \mid x^{(n)} \in C \text{ and } (x^{(n+1)})^p = x^{(n)}\right\} .$$

Set $(x+y)^{(n)} = \lim_{m\mapsto+\infty}(x^{(n+m)} + y^{(n+m)})^{p^m}$, $(xy)^{(n)} = x^{(n)}y^{(n)}$ and $|x|_C = |x^{(0)}|_p$.
Then $F(C)$ is an algebraically closed field of char. p complete for $|\ |_C$.

Construction of $X(F, E, \pi)$

$\mathcal{E}$ = the unique field containing E which is complete with respect to a discrete valuation extending the valuation on E, such that π is a uniformizing parameter of $\mathcal{E}$ and that the residue field of $\mathcal{E}$ is F.
Therefore,
– Either (EC) $E = \mathbb{F}_q((\pi))$ and $\mathcal{E} = F((\pi))$,
– Or (MC) $[E : \mathbb{Q}_p] < +\infty$ and $\mathcal{E} = E \otimes_{W(\mathbb{F}_q)} W(F)$.

The projection $\mathcal{O}_{\mathcal{E}} \to F$ has an unique multiplicative section $a \mapsto [a]$ (get $[a] = a$ in (EC)).

$$\mathcal{E} = \Big\{ \sum_{n>>-\infty} [a_n]\pi^n \mid a_n \in F \Big\} .$$

$$B^b = \Big\{ \sum_{n>>-\infty} [a_n]\pi^n \in \mathcal{E} \mid \exists C \text{ such that } |a_n| \leq C,\ \forall n \Big\} .$$

For $f = \sum_{n>>-\infty} [a_n]\pi^n \in B^b$ and $\rho \in [0, 1]$, set

$$|f|_\rho = \begin{cases} q^{-r} \text{where } r \text{ is the smallest integer such that } a_r \neq 0 & \text{if } \rho = 0 \\ \sup_{n\in\mathbb{Z}} |a_n|\rho^n & \text{if } \rho > 0 \end{cases}$$

$|\ |_\rho$ is a multiplicative norm (=absolute value) (obvious if (EC)).

Rigid analytic « functions » in mixed characteristic

For $I \subset [0,1]$ an interval, set

$$B_I = \text{ completion of } B^b \text{ w.r.t. the } |\ |_\rho\text{'s with } \rho \in I \quad \text{and} \quad B = B_{]0,1[}\ .$$

Assume $I \subset]0,1[$. Then,

– If (EC)

1. $B_I =$ ring of rigid analytic functions in $\{z \in F \mid |z| \in I\}$.
2. Any $f \in B_I$ may be written uniquely

$$f = \sum_{n \in \mathbb{Z}} [a_n]\pi^n \text{ with } a_n \in F \text{ s.t. } \forall \rho \in I, |a_n|\rho^n \mapsto 0 \text{ for } n \mapsto +\infty \text{ and } n \mapsto -\infty\ .$$

– If (MC), any such series defines an element of B_I. It's unlikely that any $f \in B_I$ may be written uniquely like that.

– If $I \subset J$, the map $B_J \to B_I$ is injective.

The curve X

Let $\mathcal{I}$ the set of closed intervals of $]0,1[$, ordered by inclusion. For $I \in \mathcal{I}$, set $Y_I = \operatorname{Spec} B_I$. $Y = (Y_I)_{I\in\mathcal{I}}$ is an ind-scheme, i.e. it is a directed system of schemes (but the transition maps are not immersions).
The automorphism φ of B^b defined by

$$\varphi\Big(\sum_{n>>-\infty}[a_n]\pi^n\Big) = \sum_{n>>-\infty}[a_n^q]\pi^n$$

extends to
– an automorphism of the ind-scheme Y (φ induces an isomorphism of $Y_{[a^q,b^q]}$ onto $Y_{[a,b]}$),
– an automorphism of the ring $B = B_{]0,1[} = \varprojlim_{I\in\mathcal{I}} B_I$. The cyclic group $\varphi^{\mathbb{Z}}$ acts on Y and on B. We may define the scheme X
i) either as the quotient (in the category of ind-schemes) of Y by the action of this group,
ii) or as

$$X = \operatorname{Proj} P \ \text{ with } \ P = \bigoplus\nolimits_{d\in\mathbb{N}} P_d,$$

where

$$P_d = \big\{ b \in B \mid \varphi(b) = \pi^d b \big\} \ .$$

The ring B_I with $I \subset [0,1[$ and closed

Proposition

Let I an interval contained in $]0,1[$.
i) If $\lambda \in F$ is such that $|\lambda| \in I$, then the ideal of B_I generated by $\pi - [\lambda]$ is maximal.
ii) If the interval I is closed, B_I is a principal domain. If $\mathfrak{p}$ is a maximal ideal, there exists $\lambda \in F$ with $|\lambda| \in I$ such that $\mathfrak{p} = \pi - [\lambda]$.

If (EC), the proof is easy (Newton polygon),λ is unique and $B_I/\mathfrak{p} = F$.
If (MC), λ is not unique ! And $B_I/\mathfrak{p} = C_\mathfrak{p}$ is an algebraically closed field of characteristic 0, containing E and complete with an absolute value $|\ |_\mathfrak{p}$ satisfying $|\pi|_\mathfrak{p} = |\lambda|$.
Moreover $F = F(C_\mathfrak{p})$ (canonically) and the projection $\theta : B_I \to C_\mathfrak{p}$ is the extension by continuity to B_I from the map $B^b \to C_\mathfrak{p}$ defined by

$$\sum_{n>>-\infty} [a_n]\pi^n = \sum_{n>>-\infty} a_n^{(0)}\pi^n \ .$$

Therefore the possible choices for λ are the $\lambda \in F$ such that $\lambda^{(0)} = \pi$.

The graded ring P

We assume $E = \mathbb{Q}_p$ and $\pi = p$ (similar description in the general case using the Lubin-Tate formal group for E relative to π)).
We have $P_0 = \mathbb{Q}_p$.
Set $U = 1 + \mathfrak{m}_F$ (it is a $\mathbb{Q}_p$-vector space). The map

$$u \mapsto \log[u]$$

is an isomorphism of U onto P_1.
For each closed max. ideal $\mathfrak{m}$ of B, there exists $\lambda \in \mathfrak{m}_F$ (non unique) such that $\mathfrak{m} = (p - [\lambda])$. But $\mathfrak{m} \cap P_1$ is the log of a unique $\mathbb{Z}_p$-line $\varepsilon^{\mathbb{Z}_p}$ in U.

If $d > 0$, and if $x \in P_d$ is non zero, there exists $t_1, t_2, \ldots, t_d \in P_1$ such that

$$x = t_1 t_2 \ldots t_d$$

(uniqueness up to obvious changes).

The curve

The following proposition is a formal consequence of the previous results :

Proposition

X is a separated integral noetherian scheme of dimension 1. We have bijections

$$\textit{closed points of } X \ \leftrightarrow \mathbb{Q}_p\textit{-lines in } P_1 \ \leftrightarrow |Y|/\varphi^{\mathbb{Z}}.$$

If x is a closed point, $\mathcal{O}_{X,x}$ is a d.v.r. whose residue field is an algebraically closed field complete with respect to a real valued valuation extending the given valuation on E.

$$X\backslash\{x\} = Spec B_{e,x} \textit{ with } B_{e,x} \textit{ a principal domain.}$$

Vector bundles over X

$$Pic(X) \simeq \mathbb{Z}$$

(for each $d \in \mathbb{Z}$, any line bundle on X of degree d is isomorphic to $\mathcal{O}_X(d) = \text{Proj } P[d]$).

$$H^1(X, \mathcal{O}_X(d)) = 0 \text{ if } \ d \geq 0 \text{ but } H^1(X, \mathcal{O}_X(d)) \neq 0 \ \text{ if } \ d < 0 \ .$$

Any non zero vector bundle $\mathcal{F}$ over X has a rank $r (\in \mathbb{N}^*)$ and a degree ($\mathcal{F}$ is of degree d if $\wedge^r \mathcal{F} \simeq \mathcal{O}_X(d)$), hence a slope $\mu(\mathcal{F}) = d/r \in \mathbb{Q}$.
A vector bundle $\mathcal{E}$ is semi-stable if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ for any sub vector bundle $\mathcal{F}$.
Harder-Narasimhan th. holds : If $\mathcal{E}$ is a v. bundle, there is a unique filtr.

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_{i-1} \subset \mathcal{E}_i \subset \ldots \subset \mathcal{E}_{m-1} \subset \mathcal{E}_m = \mathcal{E}$$

by strict sub v. bundles with $\mathcal{E}_i/\mathcal{E}_{i-1} \neq 0$, semi-stable and

$$\mu(\mathcal{E}_1/\mathcal{E}_0) > \mu(\mathcal{E}_2/\mathcal{E}_1) > \ldots > \mu(\mathcal{E}_m/\mathcal{E}_{m-1}) \ .$$

Moreover, *the Harder-Narasiman filtration splits* (non canonically).

For $h \geq 1$, $X_h = Y/\{\varphi^{h\mathbb{Z}}\} = X(F, E_h, \pi)$ (with E_h/E unramified of degree h).

$$\nu_h : X_h \to X \text{ (cyclic covering of degree h).}$$
$$X_h = X \times_{\mathrm{Spec}\ E} \mathrm{Spec}\ E_h\ .$$

For $\lambda \in \mathbb{Q}$, set $\lambda = d/h$ with $h \geq 1$ and $(d, h) = 1$. Set

$$\mathcal{O}_X(\lambda) = \nu_{h*}\mathcal{O}_{X_h}(d)\ .$$

Then any stable vector bundle of slope λ is isomorphic to $\mathcal{O}_X(\lambda)$.

The action of G_K

$K =$ a field of characteristic 0 complete with respect to a discrete valuation, with perfect residue field of char. $p > 0$, $\overline{K} =$ an algebraic closure of K, $C =$ the completion of $\overline{K}$.
$F = F(C)$, $X = X(F, \mathbb{Q}_p, p)$, $\widehat{\infty} =$ the ideal of B which is the kernel of the canonical map $\theta : B \to C$, ∞ the associated closed point of X. Then

$$B_{dR}^+ = (\ker\ \theta)\text{-adic completion of } B\ .$$

Set $\varepsilon, \lambda \in \mathcal{F}$ such that $\varepsilon^{(0)} = 1$, $\varepsilon^{(1)} \neq 1$, $\lambda^{(0)} = p$.Then, $\widehat{\infty} = (p - [\lambda])$ (resp. ∞) corresponds to the $\mathbb{Z}_p$-line (resp. $\mathbb{Q}_p$-line) generated by ε. If

$$t = \log[\varepsilon] \in P_{1,1} \cap \widehat{\infty} \text{ and } u = \log([\lambda]/p) \in B_{dR}^+\ ,$$

we may use $B_{cr} = B[1/t]$ and $B_{lcr} = B_{cr}[u]$ instead of B_{cris} $(= (B_{st})_{N=0})$ and B_{st}.

G_K-equivariant vector bundles

$$\mathcal{F} = \text{ a } G_K\text{-equivariant vector bundle on } X \mapsto (\mathcal{F}^e, \widehat{\mathcal{F}}_\infty) \text{ where now}$$

$\mathcal{F}^e$ is a ≪ B_e-representation ≫ and the B_{dR}^+-lattice $\widehat{\mathcal{F}}_\infty$ is stable under G_K.
This gives an equivalence of categories (Berger's B_e-pairs).

B_e-representations of G_K

Proposition

The category of B_e-representations of G_K is abelian. This is a $\mathbb{Q}_p$-linear tannakian category.

Proof : B_e is principal and has no proper ideal which is G_K-stable.

Remark : One dim'l B_e-representation = One dimensional p-adic representation.

B_e-representations and (φ, N, G)-modules

If $B_?$ is any sub-ring of B_{lcr} stable under G_K (e.g. $B_? = \mathbb{Q}_p$ or B_e) we say :

a $B_?$-representation $\mathcal{V}$ of G_K is *log-crystalline* (old term. : semi-stable) (resp. *potentially log-crystalline*) if $B_{lcr} \otimes_{B_?} \mathcal{V}$ is generated, as a B_{lcr}-module by the element fixed under G_K (resp. under a sufficiently small open subgroup).

Similarly a G_K-equivariant vector bundle $\mathcal{F} = (\mathcal{F}^e, \widehat{\mathcal{F}}_\infty)$ is .... if $\mathcal{F}^e$ is.

A *(φ, N)-module* is a finite dim'l K_0-vector space D equipped with an endomorphism N and a semi-linear bijective map $\varphi : D \to D$ such that

$$N\varphi = p\varphi N.$$

There is an equivalence of tannakian categories

$$\text{log-crystalline } B_e\text{-representations} \iff (\varphi, N)\text{-modules} .$$

$$D_{lcr}(\mathcal{V}) = (B_{st} \otimes_{B_e} \mathcal{V})^{G_K} \ , \ \mathcal{V}_{lcr}(D) = (B_{st} \otimes_{K_0} D)_{N=0, \varphi=1} \ .$$

Extends to an equivalence of tannakian categories

$$\text{pot. log-crystalline } B_e\text{-representations} \iff (\varphi, N, G)\text{-modules} .$$

Remark : If $[K : \mathbb{Q}_p] < +\infty$

iso. classes of (φ, N, G)-mod. $\iff$ iso. classes of repr's of WD_K over $\mathbb{Q}_p^{nr}$.

Pot. log-crystalline G_K-equivariant vector bundles

Any B_e-representation $\mathcal{V}$ of G_K which is de Rham can be ≪ compactified ≫ canonically into a G_K-equivariant vector bundle

$$\mathcal{V} \mapsto \mathcal{F}(\mathcal{V}) = \big(\mathcal{V}, B_{dR}^{+} \otimes_K D_{dR}(\mathcal{V})\big) \text{ with } D_{dR}(V) = (B_{dR} \otimes_{B_e} \mathcal{V})^{G_K} \ .$$

$K \otimes_{K_0} B_{lcr} \subset B_{dR} \implies$ any $B_?$-repre'n of G_K which is pot. log-crys. is de Rham .

Let $\mathcal{F} = (\mathcal{F}^e, \widehat{\mathcal{F}}_\infty)$ a G_K-equivariant vector bundle which is pot. log-crys.

Then $\mathcal{F}$ is a G_K-equivariant modification of $\mathcal{F}(\mathcal{F}^e)$.
May be modified only at ∞. The modifications corresponds bijectively to filtration on $D_K = D_{dR}(\mathcal{F}^e)$ and D_K can be computed from the (φ, N, G)-module associated to $\mathcal{F}^e$.
Get an equivalence of categories

pot. log-cryst. G_K-equiv. vect bundles $\iff$ Filtered (ϕ, N, G)-modules over K.

Corollary

≪ *weakly admissible* = *admissible* ≫ .

de Rham representations

Theorem

Let $\mathcal{V}$ be either a p-adic representation of G_K, a B_e-representation of G_K or a G_K-equivariant vector bundle over X. If $\mathcal{V}$ is de Rham, then $\mathcal{V}$ is potentially log-crystalline.

Sketch of the proof :
– We are easily reduced to the case where k is algebraically closed.
– Setting $\mathcal{F}^e = B_e \otimes_{\mathbb{Q}_p} \mathcal{V}$ (resp. $\mathcal{F}^e = \mathcal{V}$, resp. $\mathcal{F}^e = \Gamma(X^e, \mathcal{V})$), we are reduced to prove the theorem for a B_e-representation $\mathcal{F}^e$.

– Consider the canonical completion $\mathcal{F} = \mathcal{F}(\mathcal{F}^e) = (\mathcal{F}^e, \widehat{\mathcal{F}}_\infty)$ of $\mathcal{F}_e$.
i) If $\mathcal{F}$ is pure of slope 0, then $H^0(X, \mathcal{F})$ is a p-adic representation which is de Rham, hence Hodge-Tate and the Hodge-Tate weights are all 0. By a theorem of Sen, this implies the action of G_K on V is finite. Therefore V and $\mathcal{F}^e$ are potentially crystalline.
ii) If $\mathcal{F}$ is pure of slope λ, then $\mathcal{F} \otimes \mathcal{O}_X(-\lambda)$ is pure of slope 0. Easy to see that $\mathcal{O}_X(-\lambda)$ is equipped with a natural action of G_K and is crystalline, therefore $\mathcal{F} \otimes \mathcal{O}_X(-\lambda)$ is de Rham, therefore potentially crystalline. Therefore $\mathcal{F} \otimes \mathcal{O}_X(-\lambda) \otimes \mathcal{O}_X(\lambda)$ also and $\mathcal{F}$ who is a direct summand as well.

(If $\lambda = d/h$ with $(d,h) = 1, h \geq 1$, then $\mathcal{O}_X(\lambda)$ is a modification of $\mathcal{O}_X \otimes V_\lambda$ with $V_\lambda = \mathrm{Symm}^d_{\mathbb{Z}_{p^h}} T_p(\Gamma_h)[1/p]$ where Γ_h is a Lubin-Tate formal group for $\mathbb{Q}_{p^h}$.)
iii) In general, by functoriality of Harder-Narasimhan filtration, G_K acts on the associated graded and we know that each piece is potentially crystalline. Using the usual tannakian trick, we are reduced to show, that, if $\mathcal{F}_0^e$ is a log-crystalline B_e-representation such that all its Harder-Narasimhan slopes are > 0, and if we have a short exact sequence of G_K-equivariant vector bundles

$$0 \to \mathcal{F}_0 \to \mathcal{F} \to \mathcal{O}_X \to 0$$

then $\mathcal{F}^e$ is log-crystalline.

– There is a modification $\mathcal{F}'$ of $\mathcal{F}$ which is semi-stable of slope 0 and such that, if $V = H^0(X, \mathcal{F}')$, we have an exact sequence of p-adic representations

$$0 \to V_0 \to V \to \mathbb{Q}_p \to 0$$

where V_0 is a successive extension of V_{λ_i}'s with $\lambda_i > 0$. As k is algebraically closed, there is no H^2. We are reduced to show that, for all $\lambda > 0$, if we have a short exact sequence

$$0 \to V_\lambda \to W \to \mathbb{Q}_p \to 0$$

of Galois representations, then W is log-crystalline.
– If $\lambda = 1$, this is an easy and well known computation using Kummer theory.
– If $\lambda \neq 1$, we use (φ, Γ)-modules to show that W is crystalline (this is a very special case of a computation of Laurent Berger).

Reference : http ://www.math.u-psud.fr/∼fargues/Prepublications.html