Modularity of Galois Representations

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The beginning...

$f(z) = q + a_2 q^2 + \cdots$ cuspidal eigenform level N weight $k \geq 1$ char. χ

$T_\ell f = c_\ell(f) \cdot f$

(Shimura, Eichler, Deligne, Serre, ...)

p = prime

$\exists\ \rho_{f,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\overline{\mathbb{Q}}_p)$

- continuous, semisimple
- irreducible
- $\det \rho_{f,p} = \chi \varepsilon^{k-1}$
- unramified at all $\ell \nmid Np$
- trace $\rho_{f,p}(\mathrm{Frob}_\ell) = c_\ell(f)$ if $\ell \nmid Np$

(... Langlands, Carayol, Wiles, Taylor...)

$$f \to \pi_f = \bigotimes_v \pi_{f,v}$$

$$\rho_{f,p} = \rho_{\pi_f,p}$$

$$\rho_{\pi_f,p}\Big|_{W_v} \cong \rho(\pi_{f,v}), \quad v \neq p$$

W_v = Weil group

$\rho(\pi_{f,v})$ = rep. from local Langlands correspondence

Can replace $\mathbb{Q}$ with F = totally real

f with Hilbert modular eigenform

$$L(\rho_{f,p}, s) = L(\pi_f, s)$$

- analytic continuation
- functional equation

everything you could possibly want!

HOPE:

All 'reasonable' Galois representations come from modular forms.

MODULARITY CONJECTURE (Fontaine - Mazur)

Suppose $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\bar{\mathbb{Q}}_p)$

is

- continuous, semisimple, irreducible
- unramified at all but finitely many ℓ
- potentially semistable at p
- odd

Then

$$\rho \simeq \rho_{f,p} \otimes \varepsilon^m$$

Tate twist

for some f and m.

residual representations:

Suppose $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_p)$ is continuous.

$\exists\ \mathcal{O}$ = ring of integers of a finite extension of $\mathbb{Q}_p$ s.t. (poss. after change of basis)

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathcal{O}).$$

$\bar{\rho}$:= semi-simplification of

$\rho \bmod \lambda : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathcal{O}/\lambda})$

residual rep.

(λ a uniformizer of $\mathcal{O}$)

MOD P MODULARITY CONJECTURE (SERRE)

$\mathbb{F}$ = finite field of char. p

Suppose $\rho_0 : Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F})$

is

- continuous
- irreducible
- odd

Then

(A) $\rho_0 \simeq \bar{\rho}_{f,p}$ for some f

(B) $\rho_0 \simeq \bar{\rho}_{f,p}$ for some f of prescribed weight, level, char.

THM. (Ribet, ...) Suppose p odd.

$$(A) \implies (B)$$

What's known about MOD. CONJ.?

Early results:

- CM elliptic curves
  CM representations

- Faltings - Serre method

The breakthrough...

THM. (Wiles, Taylor-W.) p odd

Suppose $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_p)$ is

- continuous, irreducible
- unramified away from finitely many primes
- odd
- flat or ordinary at p with $\det\rho \simeq \chi\varepsilon^{k-1}$, $k \geq 2$.

Suppose further that $\overline{\rho}$ is

- irreducible when restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{\pm p}))$
- 'nice' (local conditions)
- modular

Then $\rho \simeq \rho_{f,p}$ for some f.

The ingredients ...

IDEA: identify deformation rings and Hecke rings

deformations

Fix $\mathcal{O}$ s.t. $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O})$.

DEFORMATION

- A = complete local Noeth. $\mathcal{O}$-algebra with residue field $= \mathcal{O}/\lambda$
- $\phi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(A)$

s.t. $\phi \bmod \mathfrak{m}_A = \bar{\rho}$.

deformation problem

$$\mathcal{D} = (\vartheta, \Sigma, *)$$

places at which ramification is permitted

restrictions at places in Σ

$$\rho_{\mathcal{D}} : G_\Sigma \longrightarrow GL_2(R_{\mathcal{D}})$$

universal deformation for $\mathcal{D}$

$\phi : G_\Sigma \to GL_2(A)$ type-$\mathcal{D}$

$\exists !\ R_{\mathcal{D}} \xrightarrow{\psi} A$ s.t.

$$\phi \cong \psi \circ \rho_{\mathcal{D}}$$

Hecke rings

$\mathbb{T}_{\mathcal{D}}$ = Hecke ring generated by U and T_ℓ's acting on space of cuspforms spanned by those eigenforms f s.t. $\rho_{f,p} \rightsquigarrow$ deformation of type-$\mathcal{D}$.

$\bar{\rho}$ irreducible

$$\Rightarrow \exists\ \rho_{\mathcal{D}}^{\mathrm{mod}} : G_{\Sigma} \longrightarrow GL_2(\mathbb{T}_{\mathcal{D}})$$

def. of type-$\mathcal{D}$

s.t.

$$\mathrm{Trace}\, \rho_{\mathcal{D}}^{\mathrm{mod}}(\mathrm{Frob}_\ell) = T_\ell$$

The connection

By universality

$\exists \ \phi_{\mathfrak{p}} : R_{\mathfrak{p}} \twoheadrightarrow \mathbb{T}_{\mathfrak{p}}$

$\mathrm{Trace}\,\rho_{\mathfrak{p}}(\mathrm{Frob}_\ell) \mapsto T_\ell \quad (\ell \notin \Sigma)$

CONJECTURE:

$\phi_{\mathfrak{p}}$ is an isomorphism

Consequence

If $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathcal{O})$ is type-$\mathfrak{p}$ then ρ is modular.

Pf: $\exists \ \mathbb{T}_{\mathfrak{p}} \xrightarrow{\psi} \mathcal{O}$, $T_\ell \mapsto \mathrm{trace}\,\rho(\mathrm{Frob}_\ell)$

$\exists \ f$ s.t. $c_\ell(f) = \psi(T_\ell)$ $\Rightarrow$ $\rho \simeq \rho_{f,\mathfrak{p}}$

The strategy ...

Step 1

Formulate a minimal deformation problem $\mathcal{D}_0$ — ess. minimal poss. ramification

Prove that $\mathbb{T}_{\mathcal{D}_0}$ exists.

- uses theorem of Ribet, ...

Step 2 "Then a miracle occurs ..."

Prove that $\phi_{\mathcal{D}_0}$ is an isomorphism.

(of complete intersections)

Step 3

Show that

$$\phi_{\wp_0} = \text{isom.} \implies \phi_{\wp} = \text{isom.}$$

$\Phi_{\wp}, \eta_{\wp}$ – numerical invariants attached to $R_{\wp}, \mathbb{T}_{\wp}$

$$\phi_{\wp} = \text{isom. of c.i.'s} \iff \Phi_{\wp} = \eta_{\wp}$$

change in $\Phi_{\wp}$ controlled by Galois cohomology

change in $\eta_{\wp}$ controlled by congruences

Generalizations, simplifications, applications ...

/$\mathbb{Q}$:

- Diamond removed many of the conditions in '$\bar{\rho}$ = nice'

THM Suppose $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathcal{O})$. ...

Suppose further that $\bar{\rho}$ is

- irred. when restricted to $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\sqrt{\pm p}))$
- modular
- flat or ordinary & D_p-distinguished

Then $\rho \simeq \rho_{f,p}$ for some f.

- Breuil, Conrad, Diamond, Taylor

  weaken restrictions at p

  (enough to prove modularity of all elliptic curves)

- Diamond, Fujiwara

  technical simplifications in Step 2

  (eliminate necessity of proving Gorenstein-ness of $\mathbb{T}_{\wp}$)

Ramakrishna, Khare

Avoid Step 2 !

idea: Formulate a deformation problem forcing ramification at certain primes $- \mathcal{D}_Q$.

Choose Q so that $R_{\mathcal{D}_Q} \simeq \mathcal{O}$

(prove that def. to $GL_2(\mathcal{O})$ exists)

This forces $R_{\mathcal{D}_Q} \simeq \mathbb{T}_{\mathcal{D}_Q}$.

Deduce $R_{\mathcal{D}} \simeq \mathbb{T}_{\mathcal{D}}$ when

$\mathcal{D} = (\mathcal{O}, \Sigma, *)$, Σ large enough.

/ F = totally real

- Fujiwara

  $R_{\mathfrak{p}} = \mathbb{T}_{\mathfrak{p}}$ in many cases

  - technical hypotheses incl. existence of a minimal lift

- S.-Wiles

  results for residually reducible representations.

<u>THM</u> p odd

Suppose $\rho : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p) \ldots$

- $\rho|_{D_v} \cong \begin{pmatrix} \psi_{1,v} & * \\ & \psi_{2,v} \end{pmatrix} \quad \forall\, v | p$
- $\bar{\rho} \cong \chi_1 \oplus \chi_2$, $\chi_1 = \bar{\psi}_{1,v} \neq \bar{\psi}_{2,v} = \chi_2$ on D_v
- $F(\chi_1/\chi_2)$ abelian $\Rightarrow \rho \cong \rho_{f,p}$

S.-Wiles

results for $\rho : \mathrm{Gal}(\bar{F}/F) \to GL_2(\bar{\mathbb{Q}}_p)$

ordinary at all $v \mid p$.

Both results make great use

of base change.

$p=2$/

M. Dickinson

Artin's Conjecture/

many new cases

Buzzard
Dickinson
Sh.-B.
Taylor

Other groups / Harris - Taylor

some results for GL_n

What is needed?

- Galois representations!

  may only need $\rho_\pi|_{W_v} \cong \rho(\pi_v)$

  for π_v a principal series rep.

- $R_\wp$ - conditions at p?
- minimal lifts — might get around these using base change + congruences
- congruences

  some progress (Russ Mann, Clozel)