

CAN P-ADIC  
INTEGRALS  
BE  
COMPUTED?

Hales  
IAS April 6, 2001

I think so....

## 3 Threads:

Tarski's  
decision  
procedure

p-adic  
integration

motives

## First thread

Tarski's  
decision  
procedure

1930 Alfred Tarski :

There is a decision procedure for sentences in the elementary theory of real closed fields.

## Elementary theory:

Language contains

0, 1, +, ×, (, )

^, v,  $\neg$  (and, or, not)

$\forall x$ ,  $\exists x$

$x_1, x_2, x_3, \dots$

=, >

Does not contain

$\forall n \in \mathbb{Z}$

$\{x_1, x_2, x_3, \dots\}$

$\pi, e, \ln 2$

$\cos x$

$\int f dx \dots$

5

Tarski's method is a mechanical procedure for the elimination of quantifiers (Q.E.)

Example:

$$a \neq 0 \wedge (\exists x)(ax^2 + bx + c = 0)$$

↓ Q.E.

$$b^2 - 4ac \geq 0$$

Example:

positive semi-definite quartic

$$(\forall x)(x^4 + px^2 + qx + r \geq 0)$$

↓ Q.E.

$$[256r^3 - 128p^2r^2 + 144pq^2r + 16p^4r - 27q^4$$

$$- 4p^3q^2 \geq 0 \quad \wedge$$

$$8pr - 9q^2 - 2p^3 \leq 0] \quad \vee$$

$$[27q^2 + 8p^3 \geq 0 \quad \wedge \quad 8pr - 9q^2 - 2p^3 \geq 0]$$

$$\wedge \quad r \geq 0$$

1975 George Collins found a vastly improved method of quantifier elimination.

Mathematica 4.0 implements the algorithm in an experimental package.

---

Ax-Kochen, Ershov

Paul J. Cohen 1969

"Decision procedures for  
real and p-adic fields"

Later results

Denef, Macintyre, Pas, . . .  
on quantifier elimination for  
p-adic fields.

# Pas's language

$0, 1, +, \times, (, )$

$\wedge, \vee, \neg$

$\forall x, \exists x$  (3 sorts)

$x_1, x_2, x_3$  (3 sorts)

=

ord

$\bar{ac}$  (angular component)

3 sorts:

valued field

value group

residue field

Details omitted

Pas's language does not contain  
ω uniformizer  
sets, field extensions,  
Galois groups,....

Pas 1989 (building on earlier results):  
the p-adic quantifiers can be  
eliminated from this language  
for the theory of  
henselian fields.

Applications to p-adic integration.

## 2nd Thread:

p-adic  
integration

$F$   $p$ -adic field

characteristic zero

$\mathfrak{g}$  reductive Lie algebra

$X \in \mathfrak{g}$  semisimple

Compute

$$\int f_{G^{\text{st}}(X)}$$

with an  
invariant  
measure

14

Prove identities by  
computing integrals?

Example  $g = \mathrm{so}(5)$

res. char  $F \neq 2$        $\mathbb{F}_q$  res. field

$$|\alpha(x)| = q^{-r/2}, \quad r \text{ odd.}$$

$P_X$  char. polynomial

$$P_X = \lambda P_X^0(\lambda); \quad P_X^0(\lambda) \text{ roots } \pm t_1, \pm t_2$$

$$R_X \in \mathbb{F}_q[\lambda] \quad \text{roots } \frac{t_i^2}{\omega^r}$$

$$\text{elliptic curve} \quad y^2 = R_X(\lambda^2)$$

over  $\mathbb{F}_q$

$E_X$

$$\int_{O^{\text{st}}(X)} f = A(q) + B(q) |E_X(q)|$$

A, B ≠ 0 rational functions

---

What does it mean to compute the integral?

Answer: Find A, B, E<sub>X</sub>

Wrong Answer: For a given p-adic field and parameter X, find the number  $A(q) + B(q) |E_X(q)|$ .

What made this calculation possible?

---

- 1) As  $X$  varies,  $E_X$  varies in a regular way inside a family of elliptic curves.
- 2) As the local field varies, the "same" family of elliptic curves is obtained.

Conclusion: These  $p$ -adic integrals could be computed because  $A, B, E_X$  are global objects

$$E_X \quad y^2 = x^{24} - ax^2 + b \quad \text{over } \mathbb{Q}(a, b)$$

An identity needed for the trace formula

$$\int_{\mathrm{O}^{\mathrm{st}}(X)} f = \int_{\mathrm{O}^{\mathrm{st}}(Y)} f' \quad P_X(\lambda) = \lambda P_Y(\lambda)$$

$\mathrm{so}(5) \qquad \qquad \mathrm{sp}(4)$

$$A + B|E_X| = A + B|\tilde{E}_Y|$$

Conclusion This identity is true everywhere locally because of a single global identity of Chow motives:

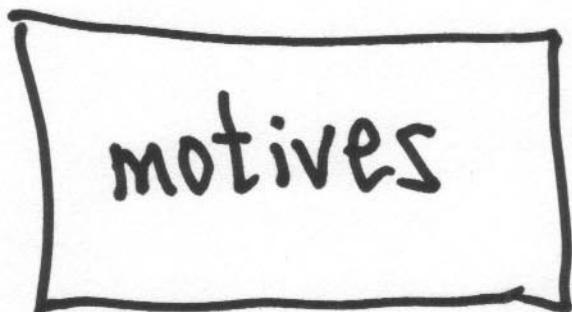
$E_X$  is isogenous to  $\tilde{E}_Y$   
over  $\mathbb{Q}(a, b)$

Thesis The computation of a p-adic integral is an effective algorithm to obtain the underlying virtual Chow motive.

Denef-Loeser principle  
(Strasbourg 2000)

All "natural" p-adic integrals are motivic.

# Third thread



Kontsevich-Denef-Loeser

Introduction to motivic integration

$$\int_{\mathbb{F}_q[[t]]} |x|^k dx = \sum_{l=0}^{\infty} |\mathbb{A}|^{k+l} \int_{|u|=1} \frac{du}{|u|}$$

$$= \left( 1 + \frac{1}{q^{k+1}} + \frac{1}{q^{2(k+1)}} + \dots \right) \left( 1 - \frac{1}{q} \right)$$


---

$$\int_{k[[t]]} |x|^k dx = \left( 1 + \frac{1}{q^{k+1}} + \dots \right) \left( 1 - \frac{1}{q} \right)$$

$$\text{if } k = \mathbb{F}_q \quad q = |\mathbb{A}'(\mathbb{F}_q)|$$

$$\text{if } \text{char } k = 0 \quad q = |\mathbb{L}| = [\mathbb{A}']$$

$k$  char 0.

$K_0(Sch_k)$

Grothendieck ring of algebraic varieties over  $k$

$[S]$   $S$  alg. variety

$$[S \times S'] = [S][S']$$

$$[S] = [S'] \quad S, S' \text{ iso}$$

$$[S] = [S \setminus S'] + [S'] \quad S' \text{ closed in } S.$$

$K_0(Sch_k)_{loc}$

invert  $L$

Ring  $\hat{K}_0^v(\text{Mot}_{k,\bar{\mathbb{Q}}}) \otimes \mathbb{Q}$

$\text{Mot}_{k,\bar{\mathbb{Q}}}$  cat. of Chow motives  
over  $k$   
coefficients in  $\bar{\mathbb{Q}}$   
triples  $(S, p, n)$

$K_0$  Grothendieck group

- image of  $K_0(\text{Sch}_k)_{\text{loc}}$
- ~ complete with respect to  
filtration  $\sum_i S_i \rightarrow 0$  if  
 $\dim S_i - i \rightarrow -\infty$

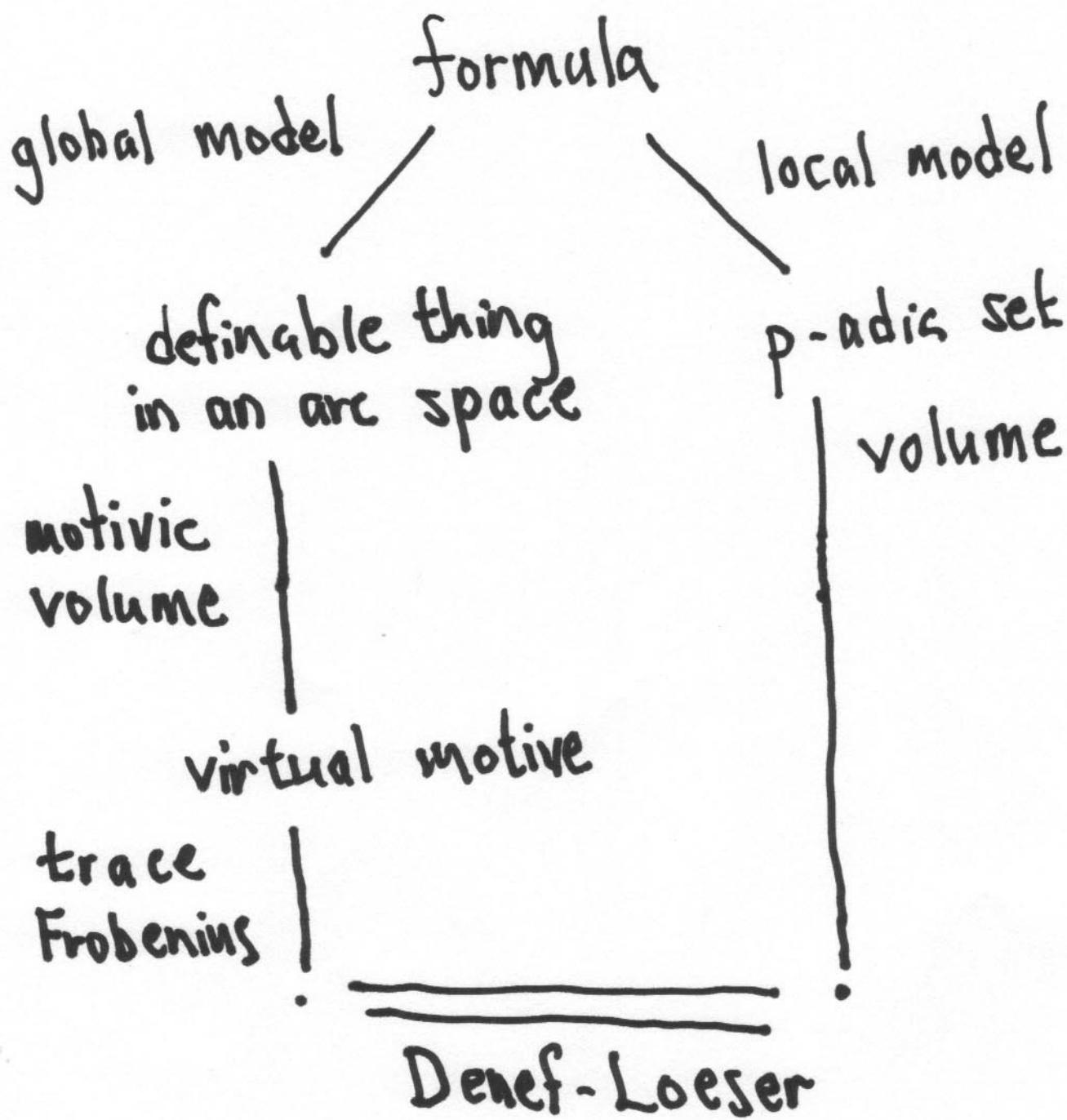
Denef-Loeser 1999:

"Definable sets, motives and  
p-adic integrals"

[xxx.lanl.gov](http://xxx.lanl.gov)

Roughly,

- Motives can be attached to formulas in Pas's language
- The trace of Frobenius on the motive equals the p-adic integral over the set defined by the formula.



My results put orbital  
integrals into this  
framework.

---

$\mathbb{F}$  p-adic char 0

Parameters  $n, k, r$

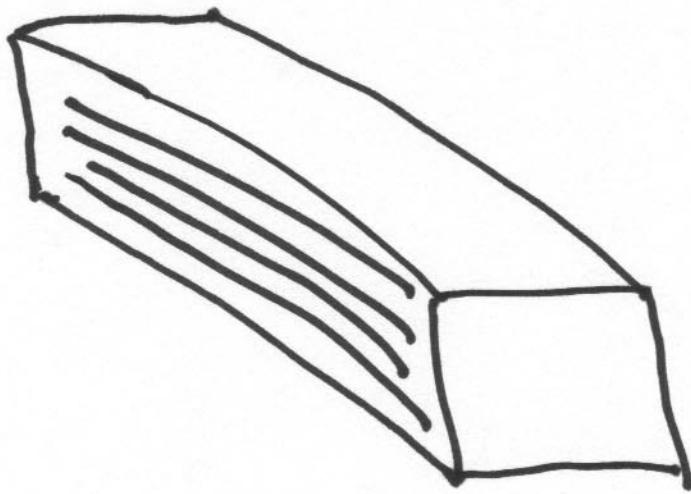
$$n \geq 1 \quad k \leq n$$

$$r \in \mathbb{Q} \quad r = \frac{l}{h} \quad (l, h) = 1.$$

$$g = SO(2n+1)$$

$$h = SO(2k+1) \times SO(2n-2k+1)$$

endoscopic



strip( $r$ ) :

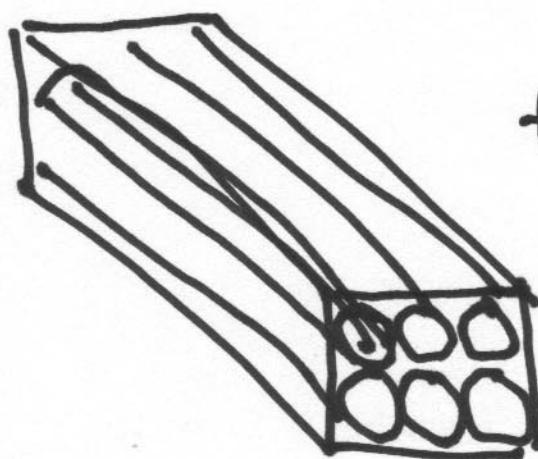
$$X \in g \quad |\alpha(X)| = q^{-r} \quad \forall \alpha$$

$$P_X(\lambda) = \lambda P_X^0(\lambda) \quad \text{char. poly.}$$

$$R_X \in \mathbb{F}_q[\lambda] \quad \frac{}{\text{roots}} \quad t_i^h / \omega^l \in \overline{\mathbb{F}_q}$$

The orbital integrals are expected to degenerate outside these strips.

Hope: If  $R_x = R_{x'}$ , then  
their orbital integrals are equal.



tube  $\subset$  strip( $r$ )  $\subset \mathcal{G}$

$X \rightarrow R_x$  partitions the strip( $r$ )  
into tubes.

Fundamental lemma

Conjecture (Langlands-Shelstad)

$$\left[ q^{\sigma(r)} \sum_X \begin{cases} \operatorname{sgn}(X, Y, Z) &= 1 \\ G(X) \cap g(O_F) & G(Y) \times G(Z) \cap h(O_F) \end{cases} \right]$$

$$P_X^0 = P_Y^0 P_Z^0$$

$$\sigma(r) = r \cdot \sigma_0$$

$$\operatorname{sgn}(X, Y, Z) \in \{0, 1, -1\}.$$

$X, Y, Z$  in  $\operatorname{strip}(r)$

Proposition    res.char  $F \gg 0$  :

$\text{sgn}^{-1}(x)$ ,  $x \in \{0, 1, -1\}$  is given by a formula in the language of rings.

(ord,  $\bar{a}c$ ,  $\forall n \in \mathbb{Z}$ , ... not used)

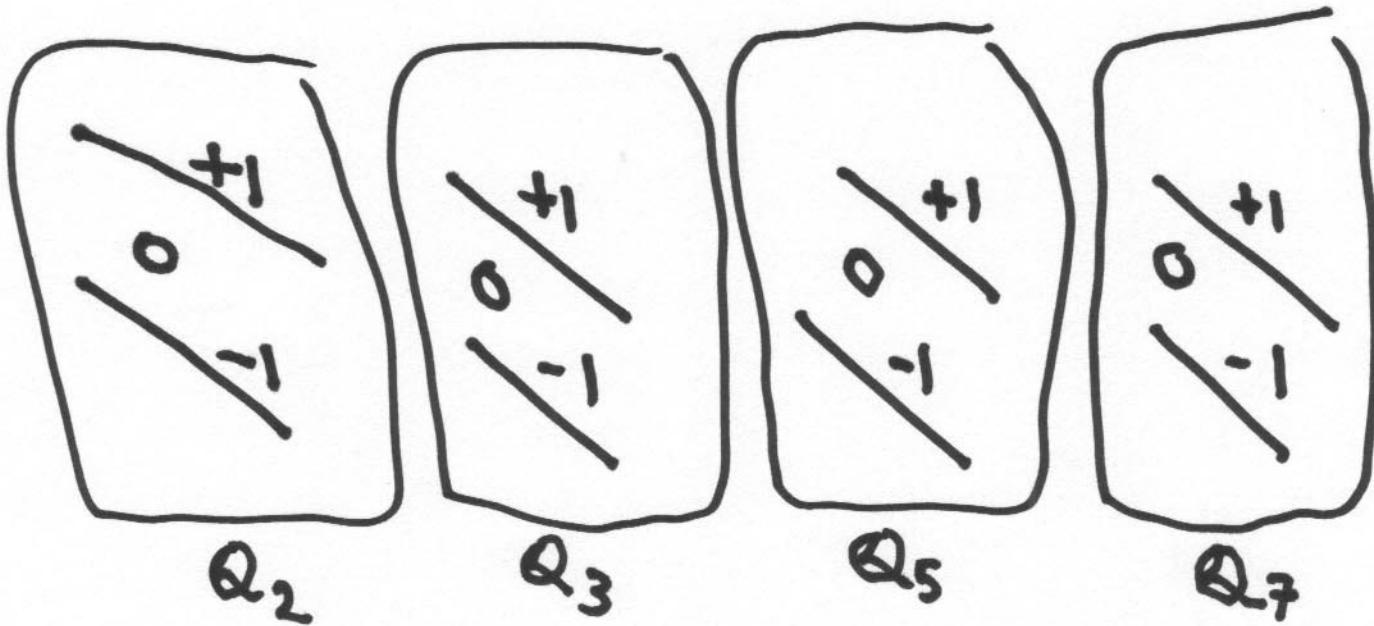
Formula for  $\text{sgn}^{-1}\{1, -1\}$ :

$$\lambda P_X = P_Y P_Z$$

$$\text{resultant}(P_X, P_X') \neq 0$$

Formula for  $\text{sgn}^{-1}(1)$ :

Start with Waldspurger's formula (2001)



$$\mathrm{SO}(2n+1) \times \mathrm{SO}(2k+1) \times \mathrm{SO}(2n-2k+1)$$

Denef-Loeser apparatus gives a  
 $\pm 1$ -motive  $\in K^v_{\mathbb{Q}, \bar{\mathbb{Q}}/\mathbb{Q}}$

$-1$ -motive

The Langlands-Shelstad transfer  
 factor "is" a motive.

Individual orbits are not given by a formula in Pas's language:

$$P_x \in F[\lambda]$$

↑ can't express individual  
p-adic numbers

But tubes are OK

$$R_x \in \mathbb{F}_q[\lambda]$$

$$\check{R}_x \in \mathcal{O}_k[z][\lambda] = S[\lambda]$$

$$Z = Z_r = \alpha'/\text{Ad}A$$

for some Lie algebra  $\alpha$

$$[k:\mathbb{Q}] < \infty$$

in  $L_{\text{Pas}}(\mathcal{O}_k[z])$

Conjecture Given  $n, k, r$

$$\mathbb{L}^{r, \sigma_0} \left( \mathbb{H}_{n, k, r}^{G, +} - \mathbb{H}_{n, k, r}^{G, -} \right) = \mathbb{H}_{n, k, r}^{H, \text{st}}$$

in  $\bar{K}_0^v(M_{\mathbb{Q}(Z_r), \bar{\mathbb{Q}}})_{\text{loc}, \mathbb{Q}}$

(Motivic fundamental lemma)

This single identity governs  
the fundamental lemma over  
the entire strip( $r$ ) at almost  
all places.

Remark The Denef-Loeser comparison theorem relates the trace of Frobenius on these motives to the traditional fundamental lemma.

Remark There are two local-global pathways

- 1) standard one used in applications of the trace formula
- 2) new one given by the Denef-Loeser apparatus

Problem 1 Give effective algorithms  
to find  $\textcircled{H}_{n,k,r}^{*,*}$   
(compute the p-adic integrals)

Problem 2 Prove the "hope":  $R_X = R_{X'}$   
 $\Rightarrow$  orbital integrals of  $O^{\text{st}}(X)$ ,  $O^{\text{st}}(X')$  equal

Problem 3 Extends to degenerate  $X \notin \text{stripl}(r)$ :  
Find finitely many motives that govern  
the fundamental lemma for all  $X \in \mathfrak{g}$

\* Problem 4 Prove the motivic  
fundamental lemma.

(Develop a conceptual understanding  
of the motives that arise).

## Conclusion

The Denef-Loeser apparatus seems to mesh well with certain  $p$ -adic integrals that arise in representation theory.

We should investigate how far motives permeate representation theory of  $p$ -adic groups.

This should allow us to "calculate"  $p$ -adic integrals that we have found hard to calculate.