

CAN P-ADIC
INTEGRALS
BE
COMPUTED?

Hales

IAS April 6, 2001

I think so....

3 Threads:

Tarski's
decision
procedure

p -adic
integration

motives

First thread

Tarski's
decision
procedure

1930 Alfred Tarski:

There is a decision procedure for sentences in the elementary theory of real closed fields.

Elementary theory:

Language contains

0, 1, +, ×, (,)

\wedge, \vee, \neg (and, or, not)

$\forall x, \exists x$

x_1, x_2, x_3, \dots

$=, >$

Does not contain

$\forall n \in \mathbb{Z}$

$\{x_1, x_2, x_3, \dots\}$

$\pi, e, \ln 2$

$\cos x$

$\int f dx \dots$

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Tarski's method is a mechanical procedure for the elimination of quantifiers (Q.E.)

Example:

$$a \neq 0 \wedge (\exists x)(ax^2 + bx + c = 0)$$

\Downarrow Q.E.

$$b^2 - 4ac \geq 0$$

Example:

positive semi-definite quartic

$$(\forall x)(x^4 + px^2 + qx + r \geq 0)$$

\Downarrow Q.E.

$$[256r^3 - 128p^2r^2 + 144pq^2r + 16p^4r - 27q^4$$

$$- 4p^3q^2 \geq 0 \quad \wedge$$

$$8pr - 9q^2 - 2p^3 \leq 0] \quad \vee$$

$$[27q^2 + 8p^3 \geq 0 \quad \wedge \quad 8pr - 9q^2 - 2p^3 \geq 0]$$

$$\wedge r \geq 0$$

1975 George Collins found a
vastly improved method of
quantifier elimination.

Mathematica 4.0 implements the
algorithm in an experimental
package.

Ax-Kochen, Ershov

Paul J. Cohen 1969

'Decision procedures for
real and p -adic fields'

Later results

Denef, Macintyre, Pas,
on quantifier elimination for
 p -adic fields.

Pas's language

0, 1, +, x, (,)

∧, ∨, ¬

∀x, ∃x (3 sorts)

x₁, x₂, x₃ (3 sorts)

=

ord

\overline{ac} (angular component)

3 sorts:

valued field

value group

residue field

Details omitted

Pas's language does not contain

ω uniformizer
sets, field extensions,
Galois groups,

Pas 1989 (building on earlier results):
the p-adic quantifiers can be
eliminated from this language
for the theory of
henselian fields.

Applications to p-adic integration.

2nd Thread:

p-adic
integration

F p-adic field
 characteristic zero
 \mathfrak{g} reductive Lie algebra
 $X \in \mathfrak{g}$ semisimple

Compute

$\int_{O^{st}(X)} f$ with an
 invariant
 measure

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Prove identities by
computing integrals?

Example $\mathfrak{g} = \mathfrak{so}(5)$

res. char $F \neq 2$ \mathbb{F}_q res. field

$$|\alpha(X)| = q^{-r/2}, \quad r \text{ odd.}$$

P_X char. polynomial

$$P_X = \lambda P_X^0(\lambda); \quad P_X^0(\lambda) \text{ roots } \pm t_1, \pm t_2$$

$$R_X \in \mathbb{F}_q[\lambda] \quad \text{roots } \frac{t_i^2}{w^r}$$

elliptic curve $y^2 = R_X(\lambda^2)$

over \mathbb{F}_q

E_X

$$\int_{\mathcal{O}^{\times}(X)} f = A(q) + B(q) |E_X(q)|$$

$A, B \neq 0$ rational functions

What does it mean to compute the integral?

Answer: Find A, B, E_X

Wrong Answer: For a given p -adic field and parameter X , find the number $A(q) + B(q) |E_X(q)|$.

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What made this calculation possible?

- 1) As X varies, E_X varies in a regular way inside a family of elliptic curves.
- 2) As the local field varies, the "same" family of elliptic curves is obtained.

Conclusion: These p -adic integrals could q be computed because

A, B, E_X are global objects

$$E_X \quad y^2 = x^2 + ax + b$$

over $\mathbb{Q}(a, b)$

An identity needed for the trace formula

$$\int_{\text{so}(5)} f = \int_{\text{sp}(4)} f'$$

$$P_X(\lambda) = \lambda P_Y(\lambda)$$

$$A + B |E_X| = A + B |\tilde{E}_Y|$$

Conclusion This identity is true everywhere locally because of a single global identity of Chow motives:

E_X is isogenous to \tilde{E}_Y
over $\mathbb{Q}(a, b)$

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Thesis The computation of a p -adic integral is an effective algorithm to obtain the underlying virtual Chow motive.

Denef-Loeser principle
(Strasbourg 2000)

All "natural" p -adic integrals are motivic.

Third thread

motives

Kontsevich - Denef - Loeser

Introduction to motivic integration

$$\int_{\mathbb{F}_q[[t]]} |x|^k dx = \sum_{l=0}^{\infty} |t^l|^{k+1} \int_{|u|=1} \frac{du}{|u|}$$

$$= \left(1 + \frac{1}{q^{k+1}} + \frac{1}{q^{2(k+1)}} + \dots \right) \left(1 - \frac{1}{q} \right)$$

$$\int_{k[[t]]} |x|^k dx = \left(1 + \frac{1}{q^{k+1}} + \dots \right) \left(1 - \frac{1}{q} \right)$$

$$\text{if } k = \mathbb{F}_q \quad q = |A'(\mathbb{F}_q)|$$

$$\text{if } \text{char } k = 0 \quad q = \mathbb{L} = [A']$$

k char 0.

$K_0(\text{Sch}_k)$

Grothendieck ring of algebraic varieties over k

$[S]$ S alg. variety

$$[S \times S'] = [S][S']$$

$$[S] = [S'] \quad S, S' \text{ iso}$$

$$[S] = [S \setminus S'] + [S'] \quad S' \text{ closed in } S.$$

$K_0(\text{Sch}_k)_{\text{loc}}$ invert \mathbb{L}

Ring $\hat{K}_0^v(\text{Mot}_{k, \bar{\mathbb{Q}}}) \otimes \mathbb{Q}$

$\text{Mot}_{k, \bar{\mathbb{Q}}}$ cat. of Chow motives
over k

coefficients in $\bar{\mathbb{Q}}$

triples (S, p, n)

K_0 Grothendieck group

\vee image of $K_0(\text{Sch}_k)_{\text{loc}}$

\wedge complete with respect to

filtration $\frac{S_i}{L^i} \rightarrow 0$ if

$\dim S_{i-i} \rightarrow -\infty$

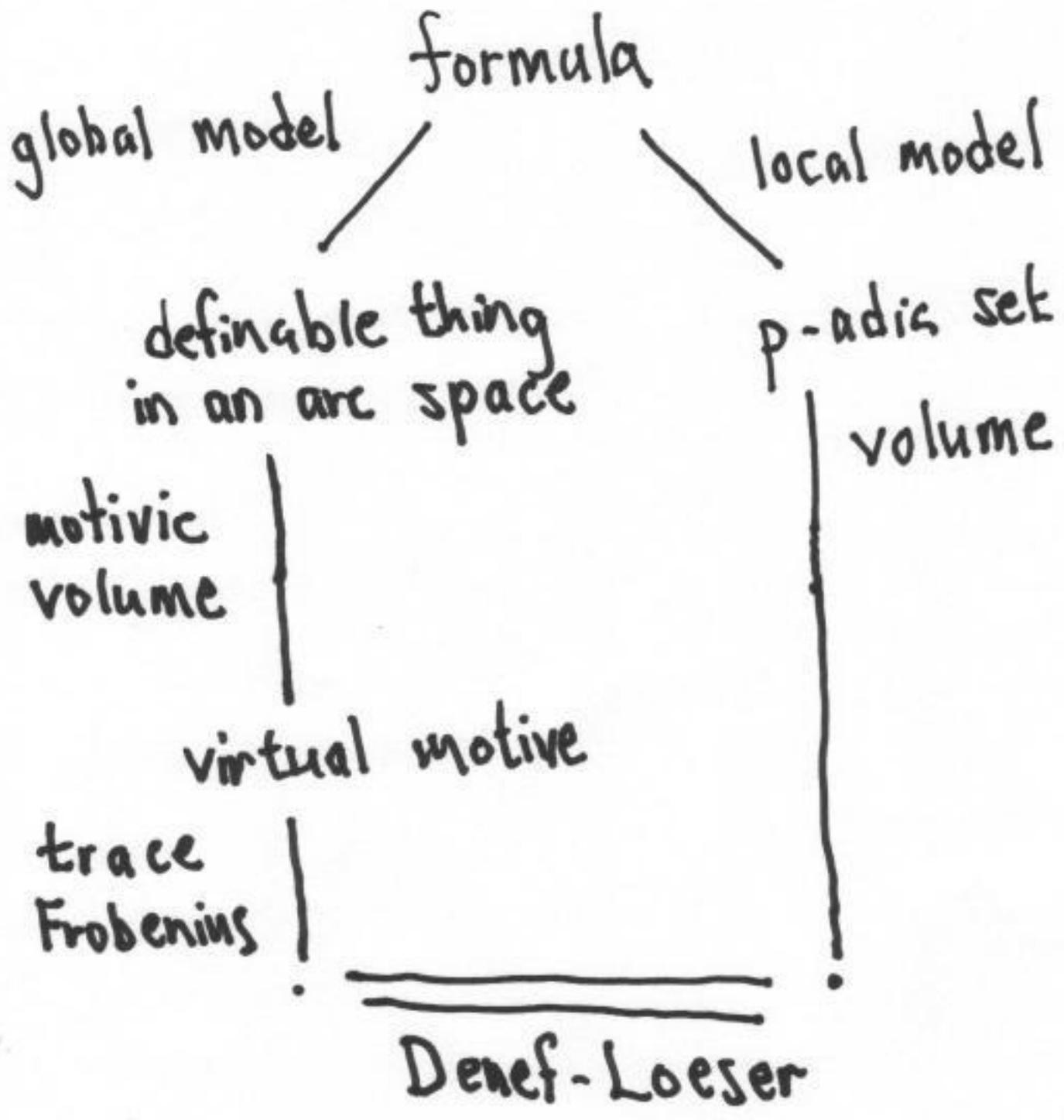
Denef-Loeser 1999:

"Definable sets, motives and p-adic integrals"

xxx.lanl.gov

Roughly,

- ▶ Motives can be attached to formulas in Pas's language
- ▶ The trace of Frobenius on the motive equals the p-adic integral over the set defined by the formula.



My results put orbital integrals into this framework.

\mathbb{F} p -adic char 0

Parameters n, k, r

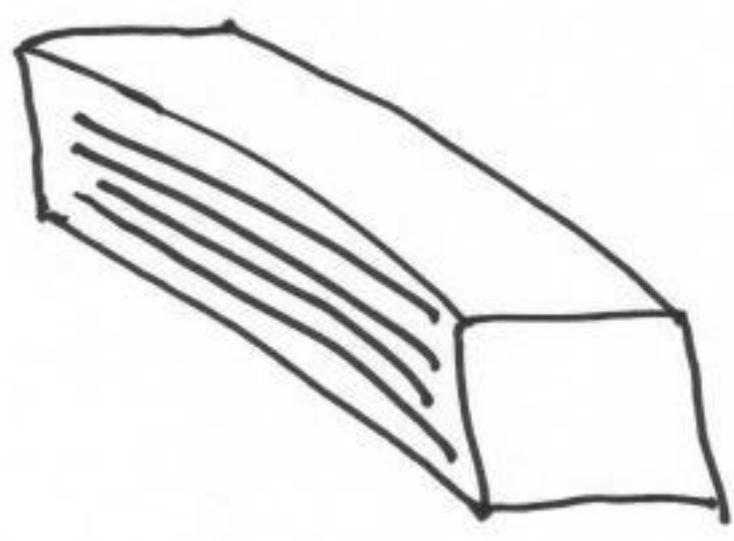
$$n \geq 1 \quad k \leq n$$

$$r \in \mathbb{Q} \quad r = \frac{\ell}{h} \quad (\ell, h) = 1.$$

$$\mathfrak{so} = \mathfrak{so}(2n+1)$$

$$\mathfrak{h} = \mathfrak{so}(2k+1) \times \mathfrak{so}(2n-2k+1)$$

endoscopic



strip(r) :

$$X \in \mathfrak{g} \quad |\alpha(X)| = q^{-r} \quad \forall \alpha$$

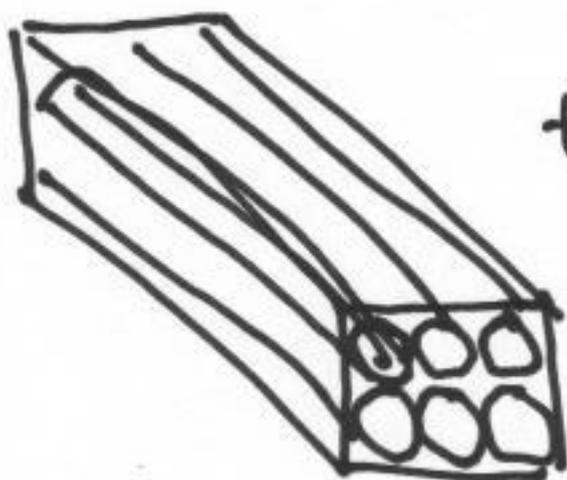
$$P_X(\lambda) = \lambda P_X^0(\lambda) \quad \text{char. poly.}$$

$$R_X \in \mathbb{F}_q[\lambda] \quad \overline{x_i^h / \omega^2} \in \overline{\mathbb{F}_q}$$

roots

The orbital integrals are expected to degenerate outside these strips.

Hope: If $R_X = R_{X'}$, then
their orbital integrals are equal.



tube \subset strip $(r) \subset \mathfrak{g}$

$X \rightarrow R_X$ partitions the strip (r)
into tubes.

Fundamental Lemma

Conjecture (Langlands-Selstad)

$$\left[\int_{\mathcal{X}} \sum_x \text{sgn}(X, Y, Z) \right]_{\substack{O(X) \sim \mathfrak{g}(O_F) \\ P_x^0 = P_y^0 = P_z^0}} = \int_{O(Y) \times O(Z) \sim \mathfrak{h}(O_F)} 1$$

$$\sigma(r) = r \cdot \sigma_0$$

$$\text{sgn}(X, Y, Z) \in \{0, 1, -1\}$$

X, Y, Z in strip(r)

Proposition $\text{res.char } F \gg 0$:

$\text{sgn}^{-1}(x)$, $x \in \{0, 1, -1\}$ is

given by a formula in the language of rings.

(ord, $\bar{a}c$, $\forall n \in \mathbb{Z}$, ... not used)

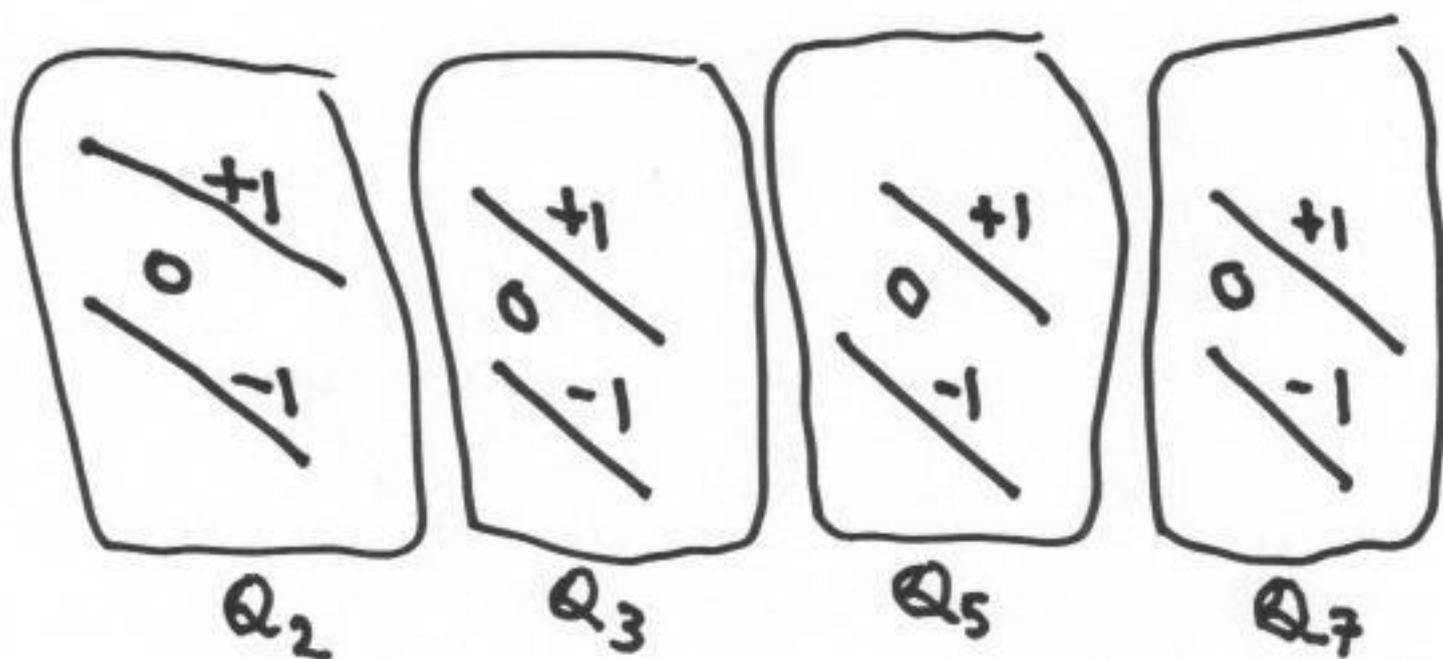
Formula for $\text{sgn}^{-1}\{1, -1\}$:

$$\lambda P_x = P_y P_z$$

$$\text{resultant}(P_x, P_x') \neq 0$$

Formula for $\text{sgn}^{-1}(1)$:

Start with Waldspurger's formula (2001)



$$so(2n+1) \times so(2k+1) \times so(2n-2k+1)$$

Denef-Loeser apparatus gives a
 $+1$ -motive $\in K_0^v(M_{\mathbb{A}, \bar{\mathbb{Q}}})_{\mathbb{Q}}$
 -1 -motive

The Langlands-Shelstad transfer
 factor "is" a motive.

Individual orbits are not given
by a formula in Pas's language:

$$P_x \in F[\lambda]$$

↑ can't express individual
p-adic numbers

But tubes are OK

$$R_x \in \mathbb{F}_q[\lambda]$$

$$\check{R}_x \in \mathcal{O}_k[\mathbb{Z}][\lambda] = S[\lambda]$$

$$\mathbb{Z} = \mathbb{Z}_r = \sigma\tau'/A\Delta A$$

for some Lie algebra σ

$$[k:\mathbb{Q}] < \infty$$

in $\mathcal{L}_{\text{Pas}}(\mathcal{O}_k[\mathbb{Z}])$

Conjecture Given n, k, r

$$\mathbb{L}^{r \cdot \sigma_0} \left(\mathbb{H}_{n, k, r}^{G, +} - \mathbb{H}_{n, k, r}^{G, -} \right) = \mathbb{H}_{n, k, r}^{H, st}$$

in $\bar{K}_0^v(M_{\mathbb{Q}(Z_r), \bar{\mathbb{Q}}})_{loc, \mathbb{Q}}$

(Motivic fundamental lemma)

This single identity governs
the fundamental lemma over
the entire strip (r) at almost
all places.

Remark The Denef-Loeser comparison theorem relates the trace of Frobenius on these motives to the traditional fundamental lemma.

Remark There are two local-global pathways

- 1) standard one used in applications of the trace formula
- 2) new one given by the Denef-Loeser apparatus

Problem 1 Give effective algorithms

to find

$$\textcircled{4}^{*,*}_{n,k,N}$$

(compute the p-adic integrals)

Problem 2 Prove the "hope": $R_X = R_{X'}$

\Rightarrow orbital integrals of $O^{\text{st}}(X)$, $O^{\text{st}}(X')$ equal

Problem 3 Extends to degenerate $X \notin \text{strip}(r)$:

Find finitely many motives that govern the fundamental lemma for all $X \in \mathfrak{g}$

* Problem 4 Prove the motivic

fundamental lemma.

(Develop a conceptual understanding of the motives that arise).

Conclusion

The Denef-Loeser apparatus seems to mesh well with certain p -adic integrals that arise in representation theory.

We should investigate how far motives permeate representation theory of p -adic groups.

This should allow us to "calculate" p -adic integrals that we have found hard to calculate.