The Analytic Theory of Automorphic Forms

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Aim: Illustrate and motivate the use of analytic techniques on automorphic forms and associated L-functions

Ramanujan Conjectures

Convexity Breaking for L-functions

Themes: Non-trivial bounds sometimes have qualitative consequences

Existence and distribution of solutions to diophantine equations

Methods based on embedding in families have been developed in a variety of contexts

Quadratic Forms

$Q(x)$ = integral quadratic form in m variables

Representation Problem:

Describe those integers $n = Q(x)$ for integral $x = (x_1, \ldots, x_m)$

(This makes sense over a number field)

Hasse-Minkowski Thm

$Q(x) = n$ has global solutions iff it has local solutions for every place

Over the integers this is much harder. For $m \geq 3$ a local-global principle holds for n suffiently large

$m = 2$: class field theory of quadratic extensions — no local-global result

$m \geq 3$ $\qquad$ $m = 3, 4$ most difficult

$m = 4$ $\quad$ Q positive definite

Theta Series $\qquad\qquad e(z) = e^{2\pi i z}$

$$\Theta(z) = \sum_{\alpha \in \mathbb{Z}^4} e(z\,Q(\alpha)) \qquad z \in H$$

$$= \sum_{n \geq 0} r_Q(n)\, e(nz)$$

$$r_Q(n) = \#\{\alpha \in \mathbb{Z}^4 ;\ Q(\alpha) = n\}$$

$\Theta(z)$ is a holomorphic modular form of weight $\frac{m}{2} = 2$ for some $\Gamma_0(N)$

$$\theta(z) = E(z) + F(z)$$

$$E(z) = \sum_{n \geq 0} \rho(n) e(nz)$$

= linear combination of Eisenstein series

$$F(z) = \sum_{n \geq 1} a(n) e(nz)$$

= linear combination of cusp forms

$$r_Q(n) = \rho(n) + a(n)$$

Fact: $\rho(n) \geq c_\epsilon n^{1-\epsilon} \quad \forall \epsilon > 0$

Provided n is represented primitively over p-adic integers for all p

Need: $a(n) \ll n^{1-\delta}$

for some fixed $\delta > 0$.

Jacobi $\quad Q(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2$

$F = 0 \quad r_4(n) = 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d$

Kloosterman used circle method and innovation: "leveling" to reduce estimation of $a(n)$ to that of

$$S(m,n;p) = \sum_{\substack{x \,(\mathrm{mod}\, p) \\ x\bar{x} \equiv 1}} e\left(\frac{mx + n\bar{x}}{p}\right)$$

$$|S(m,n;p)| \ll p^{3/4}$$

allowing any $\delta < \frac{1}{8}$

$$\sum_{m \,(\mathrm{mod}\, p)} S(m;1;p)^4 = cp^3 + \cdots$$

Weil: $|S(m,n;p)| \leq 2\sqrt{p}$

allowing any $\delta < \frac{1}{4}$

Associated L-function

Assume $F(z)$ is Hecke-eigenform

$$L(s,F) = \prod_p L(s,F_p)$$

$$L(s,F_p) = (1-\alpha_1(p)p^{-s})^{-1}(1-\alpha_2(p)p^{-s})^{-1}$$

$$\alpha_1\alpha_2 = 1, \quad (\alpha_1+\alpha_2)p^{\frac{1}{2}} = a(p)$$

Ramanujan Conjecture

$$|\alpha_i| = 1 \quad \text{or} \quad |a(p)| \leq 2\sqrt{p}$$

allowing any $\delta < \frac{1}{2}$.

RH for curves Eichler (weight 2)
Deligne

Equidistribution Problem

Show that $\frac{x}{\sqrt{n}}$ becomes u.d. on $Q(X) = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ as x runs over solutions to

$$Q(x) = n \quad \text{and} \quad n \to \infty.$$

Harmonic Analysis approach

$\varphi(x)$ = spherical harmonic of degree l

$$\frac{1}{r_4(n)} \sum_{Q(x)=n} \varphi\left(\frac{x}{\sqrt{n}}\right) \to \int_{S^3} \varphi \, d\mu$$

$$\sum_{\alpha \in \mathbb{Z}^4} \varphi(\alpha) e(Q(\alpha) z)$$ cusp form of weight $2+l$

Any nontrivial bound toward Ramanujan solves Equidistribution Problem

Ternary Quadratic Forms

The Representation and Equidistribution Problems are much harder and require nontrivial bounds for L-functions

Thm (D-Iwaniec) Every sufficiently large squarefree integer that is represented by a positive ternary form mod D^2 is integrally represented, where D is the discriminant.

This was originally proved using estimates for sums of Kloosterman sums.

Now it is best seen as convexity breaking of L-functions

Entrance of L-functions already seen with $r_3(n)$

Gauss: $r_3(n) = c_n \pi^{-1} \sqrt{n}\, L(1, \chi_n)$

where
$$c_n = \begin{cases} 0, & n \equiv 0,4,7 \pmod 8 \\ 16, & n \equiv 3 \pmod 8 \\ 24, & n \equiv 1,2,5,6 \pmod 8 \end{cases}$$

$$\chi_n(\cdot) = \left(\frac{-4n}{\cdot}\right)$$

Siegel: $r_3(n) \gg_\epsilon n^{\frac{1}{2}-\epsilon}$

non-effective

A non-trivial bound must be a power savings.

Hecke operators do not touch square-free n for half-integral weights

Using Waldspurger's Theorem we are led to bounding in n

$$L(\tfrac{1}{2}, f \otimes \chi_n)$$

where f has integral weight.

The method of families, to be explained, works to do this, and allows extensions to number fields K.

Hilbert's 11th Problem:

Kneser: Integral Local-Global holds over number fields in 4 or more variables

Analytic methods for estimating L-functions typically yield weaker results over number fields: Circle method breaks down, Kloosterman sums hard to handle.

Most difficult case:

K totally real and Q positive

Cogdell-P-S-Sarnak: Integral Local-Global holds in this case

Remark: As over $\mathbb{Q}$, ternary results are non-effective, in that no explicit constant may be given in "sufficiently large"

Indefinite $Q(x)$

Most interesting problems here are about equi-distribution and lead to non-holomorphic automorphic forms.

$\underline{m=4}$ $\quad x_1 x_2 - x_3 x_4 = n$

Selberg eigenvalue conjecture arises in such equations where $x_1 \equiv 0 \pmod{N}$ large N

Higher Degree : $\det x = n \quad GL_m$

$\underline{m=3}$ $\quad x_1^2 - 4x_2 x_3 = n$

Hyperbolic equidistribution of CM points and closed geodesics "arithmetically" ordered $- SL_2(\mathbb{Z})$.

Over Totally real K

Zhang : Cases of Andre-Oort conjecture reduce to convexity breaking for certain Rankin convolutions

L-functions on GL_m

K = number field degree d
A = adele ring of K
π = cuspidal automorphic representation of $GL_m(A)$ with unitary central character

Standard L-function

$$\Lambda(s,\pi) = \prod_v L(s,\pi_v)$$

$\mathrm{Re}\, s > 1$

For unramified v

$$L(s,\pi_v) = \begin{cases} \prod_{j=1}^{m} \left(1 - \alpha_{j,\pi}(v) N(v)^{-s}\right)^{-1}, & v \text{ finite} \\ \prod_{j=1}^{m} \Gamma_v(s - \mu_{j,\pi}(v)), & v \text{ archim} \end{cases}$$

$$\Gamma_v(s) = \begin{cases} \pi^{-s/2}\Gamma\left(\frac{s}{2}\right), & v \text{ real} \\ (2\pi)^{-s}\Gamma(s), & v \text{ complex} \end{cases}$$

Let $L(s,\pi) = \prod_{v \text{ finite}} L(s,\pi_v) = \sum_n b(n) N(n)^{-s}$

a Dirichlet Series

Godement - Jacquet :

$\Lambda(s,\pi)$ is entire (except when $m=1$, π trivial)

and

$$\Lambda(1-s,\tilde{\pi}) = \bar{\epsilon}_\pi N_\pi^{s-\frac{1}{2}} \Lambda(s,\pi)$$

$N_\pi \in \mathbb{Z}^+$ (conductor of π)

$|\epsilon_\pi| = 1$ $\tilde{\pi}$ = contragredient of π

Expect after Langlands that all L-functions are products of these (even over $\mathbb{Q}$)

Generalized Ramanujan Conjecture

For ν unramified :

$$|\alpha_{j,\pi}(\nu)| = 1$$

$$Re(\mu_{j,\pi}(\nu)) = 0$$

Known for holomorphic forms $m=2$, $K=\mathbb{Q}$ Deligne and some other cases.

For $K = \mathbb{Q}$ at infinite place Ramanujan is equivalent to Selberg eigenvalue conjecture. when $m = 2$.

Jacquet - Shalika : local bounds

$$|\alpha(v)| < N(v)^{\frac{1}{2}}, \quad v \text{ finite}$$

$$|\mathrm{Re}(\nu)| < \frac{1}{2}, \quad v \text{ archim.}$$

<u>Problem</u> Find $\delta > 0$ s.t. for π unramified at v

$$|\alpha(v)| \leq N(v)^{\frac{1}{2}-\delta}$$

$$|\mathrm{Re}(\nu)| \leq \frac{1}{2} - \delta$$

Luo - Rudnick - Sarnak

$$\delta = \frac{1}{m^2+1}$$

Kim - Shahidi $\quad m = 2$

$$\delta = \cancel{\frac{1}{9}}\ \frac{7}{18}$$

These are uniform in degree of K.

Analytic Approach

$$L(s, \pi \otimes \tilde{\pi}) = \sum b(n) N(n)^{-s}$$

J-PS-S Known analytic properties

"Landau lemma"

$$\sum_{N(n) \leq x} b(n) = cx + O\left(x^{\frac{dm^2-1}{dm^2+1}+\epsilon}\right)$$

Since $b(n) \geq 0 \Rightarrow \delta = \frac{1}{dm^2+1}$

<u>Method</u> of <u>Families</u>:

twist by characters and use root number instead of Gamma factors. (D-I) over $\mathbb{Q}$

$$\Delta(p, \ell) = \sum_{\substack{n \equiv \ell \,(p) \\ n \sim \ell}} b(n) - \frac{1}{p-1} \sum_{n \sim \ell} b(n)$$

$$\Delta(p, \ell) \ll p^{\frac{m^2}{2}+\epsilon} \sum_{x_1 \cdots x_{m^2} \equiv a \,(\mathrm{mod}\, p)} e_p(x_1 + \cdots + x_{m^2})$$

Deligne $\Rightarrow$ $\Delta(p,l) \ll p^{m^2-\frac{1}{2}+\epsilon}$

Summing over many p's and using that for $n \equiv l$

$$n \equiv l \pmod p$$

holds for all p we derive

$$\delta = \frac{1}{m^2+1}$$

This does not require $b(n) \geq 0$.

Luo-Rudnick-Sarnak treat all places at the same time and also were able to treat arbitrary K using a construction of enough characters due to Rohrlich.

Convexity Breaking

GRH The zeros of $\Lambda(s,\pi)$ lie on $\mathrm{Re}(s) = \frac{1}{2}$.

"Analytic conductor" of π

$$C(\pi,t) = N_\pi \prod_{j=1}^{m} \prod_{v \text{ arch.}} \left(1 + |\mu_{j,\pi}(v) + it|^{d(v)} \right)$$

$$d(v) = \begin{cases} 1, & v \text{ real} \\ 2, & v \text{ complex} \end{cases}$$

Measures "size" of π

Lindelöf Hypothesis : $\forall \epsilon > 0$

$$L(\tfrac{1}{2}+it, \pi) \ll_\epsilon C(\pi,t)^\epsilon$$

GRH $\Rightarrow$ Lindelöf

Convexity Bound (Phragmen-Lindelöf)

$$L(\tfrac{1}{2}+it, \pi) \ll_\epsilon C(\pi, t)^{\frac{1}{4}+\epsilon}$$

Subconvexity Problem:

Replace exponent by $\frac{1}{4}-\delta$ for $\delta > 0$.

In practice for applications it is usually enough to obtain such a bound in only one parameter.

For applications of type discussed we concentrate on N_π aspect since we must estimate

$$L(\tfrac{1}{2}, \pi \otimes \chi_n)$$

Other aspects are also interesting e.g. infinite places and the equidistribution of Mass conjecture of Rudnick - Sarnak

Watson - Triple product special value

Over $\mathbb{Q}$ $m = 2$ level aspect

Series of works by

D, Friedlander, Iwaniec

Over K totally real

Cogdell, P-S, Sarnak

Recently over $\mathbb{Q}$ when $m = 1$

Conrey and Iwaniec have improved the long standing convexity beating estimate of Burgess

Some higher degree cases have been successfully treated.

Kowalski - Michel - Vanderkam

break convexity for $L(s, \pi_1 \otimes \pi_2)$ where π_1, π_2 are holomorphic on GL_2 and one is fixed.

Amplification / Families

$$L(\tfrac{1}{2},\pi) = 2\sum_{n} \frac{b_\pi(n)}{\sqrt{N(n)}} W\left(\frac{N(n)}{X}\right)$$

W smooth rapid decay

$X = \sqrt{C(\pi,0)}$

One may recover the convexity bound from this and Ramanujan on average:

$$\sum_{N(n)\le Y} |b_\pi(n)|^2 \ll Y\, C(\pi,0)^\epsilon$$

To go beyond, consider

$$S(\mathcal{F}) = \sum_{\pi\in\mathcal{F}} |L(\tfrac{1}{2},\pi)|^2$$

where conductors of $\pi \in \mathcal{F}$ are in some range.

Lindelöf would give

$$S(\mathcal{F}) \ll_{\epsilon} |\mathcal{F}|\, C(\pi_0)^{\epsilon}$$

for particular π_0 and this is typically provable for suitable $\mathcal{F}$ with $|\mathcal{F}| = \sqrt{C}$ using orthogonality, yielding convexity bound again, using positivity.

Amplification Bound

$$\sum_{\pi \in \mathcal{F}} \Big| \sum_{N(\ell) \in Y} \alpha(\ell)\, b_{\pi}(\ell) \Big|^2 \, |L(\tfrac{1}{2}, \pi)|^2$$

$$\ll |\mathcal{F}|\, Y\, X^{\epsilon}$$

Make choice $\alpha(\ell) = \overline{b_{\pi_0}(\ell)}$.

Key Feature:

Off diagonal terms must be handled:

$$\sum_{\nu\alpha - \mu\beta = h} b_\pi(\alpha)\, \bar{b}_\pi(\beta)$$

For $L(s, \pi \otimes \chi)$ over $\mathbb{Q}$ one uses for $\mathcal{F}$ all χ mod q and handles off diagonal terms using Kloosterman sums.

Over K, one cannot use ray class characters, which must be trivial on units.

Cogdell - P-S - Sarnak have found that Selberg's original approach to handling off-diagonal terms via Poincaré series may be employed here with a larger family arising from twists by characters of $(\mathcal{O}_K / q)^*$.

Feature: The whole spectrum of $GL_2(A_K)$ is used to obtain a result for holomorphic forms.

Question: For Ramanujan over GL_2 the correct objects to study are symmetric powers, not powers.

For Lindelöf, is there a "correct" form of tensor power which will properly embed GL_1 or GL_2 L-functions into GL_m ?