L-FUNCTIONS, CONVERSE THEOREMS, AND FUNCTORIALITY*

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§1. Background and Functoriality.

$$\begin{aligned} f &= \text{ holomorphic modular cusp form or a Maass form with respect to } \Gamma_0(N) \\ &= \text{ eigenfunction for all the Hecke operator as well as Laplacian} \end{aligned}$$

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) \text{ if } k=0, \text{ where } k \text{ is the weight of } f.$$

$a_n =$ Fourier coefficient of f, $\qquad a_1 = 1$

$a_p = p^{\frac{k-1}{2}}(\alpha_p+\beta_p)$

$\mathbb{Q} =$ rational numbers

$\mathbb{A}_{\mathbb{Q}} =$ ring of adeles of $\mathbb{Q} = \prod'_{p<\infty} \mathbb{Q}_p \cdot \mathbb{R}$, a restricted product with respect to $\prod_{p<\infty} \mathbb{Z}_p$.

There is a natural way of realizing

$$f \mapsto \pi_f \subset L^2_0(\mathbb{A}^*_{\mathbb{Q}} GL_2(\mathbb{Q}) \setminus GL_2(\mathbb{A}_{\mathbb{Q}})),$$

where π_f is an irreducible subrepresentation

$$\pi_f = \bigotimes_{p\le\infty} \pi_p$$

$\pi_p =$ irreducible representation of $GL_2(\mathbb{Q}_p)$.

For all $p \nmid N$, class of $\pi_p \longleftrightarrow \left\{\begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}\right\} \subset GL_2(\mathbb{C})$, a semisimple conjugacy class. Two immediate problems in number theory of great importance are:
$f =$ Maass form.

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Ramanujan-Petersson Conjecture: $|\alpha_p| = |\beta_p| = 1$. (Proved by Deligne for holomorphic forms, 1973).

Selberg Conjecture: The smallest positive eigenvalue for Δ on $\Gamma_0(N) \setminus \mathfrak{h} \geq \frac{1}{4}$, $\mathfrak{h} =$ upper half plane.

More generally:

$$\begin{aligned} F &= \text{ number field} \\ \mathbb{A}_F &= \text{ ring of adeles} \\ \pi &= \text{ irreducible infinite dimensional subrepresentation of} \\ &\quad L^2(\mathbb{A}_F^* GL_2(F) \setminus GL_2(\mathbb{A}_F), \omega), \\ &= \text{ cuspidal representation.} \end{aligned}$$

$$\pi = \bigotimes_v \pi_v$$

$$\forall' v, \qquad \pi_v \leftrightarrow \left\{ \begin{pmatrix} \alpha_v & 0 \\ 0 & \beta_v \end{pmatrix} \right\} \subset GL_2(\mathbb{C}), \text{ semisimple conjugacy class.}$$

Ramanujan-Petersson: $|\alpha_v| = |\beta_v| = 1$.

These problems will be addressed by Duke and Iwanice in their lecture.

Any progress made on these conjectures is very important. In what follows, I will explain how Langlands-Shahidi plus certain recent converse theorems of Cogdell-Piatetski-Shapiro can lead to quantum jumps towards these conjectures by establishing surprising new cases of functoriality with numerous other deep consequences.

Let me now give an important example of functoriality, using GL_2, with important consequences towards these conjectures.

$m =$ positive integer.

$P(x, y) =$ homogeneous polynomial of degree m

$g \in GL_2(\mathbb{C}), Sym^m(g) \in GL_{m+1}(\mathbb{C})$ giving coefficients of $P((x, y)g)$ in terms of those of $P(x, y)$.

The map

$$g \mapsto Sym^m(g)$$

gives a $(m+1)$-dimensional irreducible representation of $GL_2(\mathbb{C})$

$$Sym^m : GL_2(\mathbb{C}) \to GL_{m+1}(\mathbb{C}),$$

m–th **symmetric power representation** of $GL_2(\mathbb{C})$

$$\pi = \underset{v}{\otimes}\pi_v$$

$$\forall' v, \qquad \pi_v \leftrightarrow \{t_v\} = \left\{ \begin{pmatrix} \alpha_v & 0 \\ 0 & \beta_v \end{pmatrix} \right\} \subset GL_2(\mathbb{C})$$

$$Sym^m(t_v) = \text{ diag}(\alpha_v^m, \alpha_v^{m-1}\beta_v, \ldots, \beta_v^m) \in GL_{m+1}(\mathbb{C}).$$

$$Sym^m(t_v) \leftrightarrow Sym^m(\pi_v),$$

$$Sym^m(\pi_v) = \text{ spherical representation of } GL_{m+1}(F_v).$$

Even if π_v is not spherical, Harris-Taylor, and Henniart can be used to define $Sym^m(\pi_v)$. In fact, if $\varphi_v : W'_{F_v} \longrightarrow GL_2(\mathbb{C})$ parametrizes π_v by Kutzko and Langlands, $Sym^m(\pi_v)$ is attached to $Sym^m(\varphi_v) = Sym^m \cdot \varphi_v$.

Conjecture (Langlands). $\underset{v}{\otimes} Sym^m(\pi_v)$ *is an automorphic representation of* $GL_{m+1}(\mathbb{A}_F)$, *i.e., it appears in* $L^2(\mathbb{A}_F^* GL_{m+1}(F) \setminus GL_{m+1}(\mathbb{A}_F))$. *Thus* $Sym^m : GL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$ *has a dual* $Sym^m : Aut(GL_2) \rightarrow Aut(GL_{m+1})$.

If true for all m, we get Ramanujan-Petersson and Selberg at once.

$2. Results on Functoriality with Applications.

Theorem 1. a) (Kim-Shahidi) $Sym^3(\pi) = \underset{v}{\otimes} Sym^3(\pi_v)$ *is automorphic.*

b) (Kim) $Sym^4(\pi) = \underset{v}{\otimes} Sym^4(\pi_v)$ *is automorphic.*

Corollary. *a) (Kim-Shahidi)* $q_v^{-1/9} < |\alpha_v|$ *and* $|\beta_v| < q_v^{1/9}$.

b) (Kim-Sarnak) $F = \mathbb{Q}$, $p^{-7/64} \leq |\alpha_p| \& |\beta_p| \leq p^{7/64}$; *smallest positive eigenvalue for* $\Delta \geq 0.2376\ldots$

c) (Kim-Shahidi) On Sato–Tate: For m *up to* $m = 9$, $L(s, \pi, Sym^m)$ *is meromorphic and satisfies a functional equation. If* π *has a trivial central character then* $L_S(s, \pi, Sym^m)$ *is invertible at* $s = 1$ *for* m *up to* $m = 8$. *Moreover,* $ord_{s=1} L_S(s, \pi, Sym^9) \in \{0, 1, -1\}$. *Consequently (Serre), if* π *satisfies RP, then given* $\epsilon > 0$, *there are sets of positive lower densities for which* $a_v = \alpha_v + \beta_v > 1.68\ldots - \epsilon$ *and* $a_v < -1.68\ldots + \epsilon, 1.68\ldots = 2\cos(2\pi/11)$.

d) (Kim-Shahidi) Modulo Arthur's multiplicity conjecture for $GSp_4(\mathbb{A}_F)$, *Siegel modular cusp forms of weight* 3 *exist. They are in the same* L*-packet as* $Sym^3(\pi)$, *when* π *is a non-CM form of weight* 2, *and where* $Sym^3(\pi)$ *is considered as a form on* $GSp_4(\mathbb{A}_F)$.

Corollary (Kim-Shahidi). *Suppose* σ *is a two dimensional (irreducible) representation of* $Gal(\overline{F}/F)$ *of icosahedral type. Assume* $\pi(\sigma)$ *exists (Buzzard–Baron-Shepard–Dickinson–Taylor). Then* $L(s, Sym^3(\sigma))$, *which is a primitive four dimensional Artin* L*-functions, is entire.*

Corollary (Sarnak). *Maass forms with integral coefficients are all Galois and therefore Selberg conjecture is valid for them.*

Remark 1. Other applications are proved and significant applications to arithmetic geometry are expected.

Remark 2. Theorem 1a) follows from the following theorem by letting $\pi_1 = \pi$ and $\pi_2 = Ad(\pi) =$ Gelbart-Jacquet lift of π.

Theorem 2 (Kim-Shahidi). *Let $\pi_1 = \underset{v}{\otimes}\pi_{1v}$, $\pi_2 = \underset{v}{\otimes}\pi_{2v}$ be cusp forms on $GL_2(\mathbb{A}_F)$ and $GL_3(\mathbb{A}_F)$, respectively. For each v, let*

$$\varphi_{iv} : W'_{F_v} \longrightarrow GL_{i+1}(\mathbb{C}) \qquad\qquad i = 1, 2$$

be attached to π_{iv}. Let $\pi_{1v} \boxtimes \pi_{2v}$, representation of $GL_6(F_v)$, be attached to $\varphi_{1v} \otimes \varphi_{2v}$. Set $\pi_1 \boxtimes \pi_2 = \underset{v}{\otimes}(\pi_{1v} \boxtimes \pi_{2v})$. Then $\pi_1 \boxtimes \pi_2$ is an automorphic representation of $GL_6(\mathbb{A}_F)$. Thus the functorial dual of $GL_2(\mathbb{C}) \times GL_3(\mathbb{C}) \to GL_6(\mathbb{C})$ exists.

§3. The method.

Theorem 2 is proved by applying a recent converse theorem of Cogdell-Piatetski-Shapiro, in fact a fairly recent one, to analytic properties of L-functions all obtained from the method of Langlands-Shahidi.

Converse theorems are an important tool in proving functoriality when one compares groups with $GL(n)$. They have been used by several mathematicians including in Lafforgue's proof of global Langlands correspondence for $GL(n)$ over function fields and non-normal cubic base changes for $GL(2)$ and $GL(3)$; Langlands-Tunnel's proof of Artin's conjecture, as well as existence of Sym^2 by Gelbart–Jacquet.

In the method of integral representations for $GL(m) \times GL(n)$, Rankin-Selberg product L-functions $L(s, \pi \times \pi')$ are expressed as Mellin transforms which allow us to prove the invariance on the left by the corresponding $GL(\quad, F)$, $F =$ global field. This implies a converse theorem. One then needs $L(s, \pi \times \pi')$ to have the appropriate analytic properties. $\pi = \underset{v}{\otimes}\pi_v$, $\pi' = \underset{v}{\otimes}\pi'_v$ cuspidal representation of $GL_m(\mathbb{A}_F)$ and $GL_n(\mathbb{A}_F)$, respectively.

$$\forall' v, \qquad\qquad \pi_v \longleftrightarrow t_v \in GL_m(\mathbb{C}) \text{ and } \pi'_v \longleftrightarrow t'_v \in GL_n(\mathbb{C}).$$

Define:

$$L(s, \pi_v \times \pi'_v) = \det(I - (t_v \otimes t'_v)q_v^{-s})^{-1}$$

and at every other place use Harris-Taylor-Henniart, $\varphi_v : W'_{F_v} \longrightarrow GL_m(\mathbb{C})$, $\varphi'_v : W'_{F_v} \longrightarrow GL_n(\mathbb{C})$.

$$\begin{aligned} L(s, \pi_v \times \pi'_v) &= L(s, \varphi_v \otimes \varphi'_v) \\ &= \text{ Artin } L\text{-function} \end{aligned}$$

set

$$L(s,\pi\times\pi')=\prod_v L(s,\pi_v\times\pi'_v).$$

Then, Jacquet-Piatetski-Shapiro-Shalika, Moeglin-Waldspurger, Shahidi, proved all the basic analytic properties of $L(s,\pi\times\pi')$ that:

1. It is entire, unless $\pi'\cong\tilde{\pi}\otimes|\det|^{s_0}$, $s_0\in\mathbb{C}$.
2. It satisfies a functional equation $s\mapsto 1-s$.
3. It is bounded in vertical strips of finite width (Gelbart-Shahidi, quite generally for all the L–functions in our method, Luo-Rudnick-Sarnak for $GL(n)$).
4. non-zero for $Re(s)\geq 1$.

We can now state a converse theorem (1998) which we used in proving Theorem 2.

Theorem (Cogdell-Piatetski-Shapiro). $\Pi=\bigotimes_v \Pi_v$ *irreducible admissible representation of* $GL_m(\mathbb{A}_F)$ *whose central character is a grossëncharacter. Let* S *be a finite set of finite places of* F, $\tau^S(n)=$ *cuspidal representations of* $GL_n(\mathbb{A}_F)$, *unramified for all* $v\in S$. *For each* $\sigma\in\tau^S(n)$, $n\leq m-2$, *let* $L(s,\Pi\times\sigma)=\prod_v L(s,\Pi_v\times\sigma_v)$. *Assume:*

a) $L(s,\Pi\times\sigma)$ *converges absolutely for* $Re(s)\gg 0$ *and is entire,*

b) $L(s,\Pi\times\sigma)$ *is bounded in vertical strips of finite width,*

c) and $L(s,\Pi\times\sigma)$ *satisfies a standard functional equation, i.e.,* $L(s,\Pi\times\sigma)$ *is "nice". Then there exists an automorphic representation* Π' *of* $GL_m(\mathbb{A}_F)$ *such that* $\Pi_v\cong\Pi'_v$, $\forall v\notin S$ *and in particular for all* $v=\infty$.

To apply this to Theorem 2, let

$$S=\{v<\infty|\pi_{1v}\text{ or }\pi_{2v}\text{ is ramified}\}.$$
$$\sigma=\bigotimes_v\sigma_v\in\tau^S(n),\qquad n=1,2,3,4$$

$\chi=\bigotimes_v\chi_v=$ highly ramified at some $v_0\in S$, grössencharacter. Then

$$L(s,\pi_{1v}\times\pi_{2v}\times(\sigma_v\otimes\chi_v))\text{ and }\epsilon(s,\pi_{1v}\times\pi_{2v}\times(\sigma_v\otimes\chi_v),\psi_v)$$

are defined by Langlands-Shahidi method at every v, extending the factors for triple products at unramified places. In particular, they are Artin factors for $v=\infty$. Moreover:

$$L(s,\pi_1\times\pi_2\times(\sigma\otimes\chi))=\prod_v L(s,\pi_{1v}\times\pi_{2v}\times(\sigma_v\otimes\chi_v))$$

is "nice" by techniques of the same method. More precisely

a) Follows from an observation of Kim (Langlands Lemma on holomorphy of constant terms of Eisenstein series if the inducing representation is not fixed by any non-trivial Weyl group element.) — Canadian Journal.

b) New result of Gelbart-Shahidi (JAMS), subtle; uses non-constant term of Eisenstein series as one needs to deal with inverses of L-functions up to and including $Re(s) = 1$.

c) Proved in full generality by Shahidi (Annals 1990), where local factors were defined and a number of their properties were proved (and some were conjectured). They are fundamental in all the cases of functoriality proved by us. Moreover for $v \notin S$:

$$L(s, \pi_{1v} \times \pi_{2v} \times (\sigma_v \otimes \chi_v)) = L(s, (\pi_{1v} \boxtimes \pi_{2v}) \times (\sigma_v \otimes \chi_v))$$

as well as for root numbers. One needs new local results on normalized intertwining operators (Casselman–Shahidi, Kim, Muic, Zhang, Asgari).

By the converse theorem, there exists an automorphic representation $\Pi = \underset{v}{\otimes}\Pi_v$ such that $\Pi_v = \pi_{1v} \boxtimes \pi_{2v}$, $\forall v \notin S$. Using weak Ramanujan type arguments (Ramakrishnan, Cogdell-Piatetski-Shapiro), one gets

$$\Pi = \sigma_1 \boxplus \cdots \boxplus \sigma_k$$

σ_i's = unitary cuspidal representations of $GL_{r_i}(\mathbb{A}_F), r_i = 2, 3, 4$.

We then embark on a path to prove the strong lift, i.e., that our factors are the same as $L(s, (\pi_{1v} \boxtimes \pi_{2v}) \times \sigma_v)$ for all v. Similarly for root numbers. We use base change, both normal (Arthur-Clozel,Langlands) or non-normal (JPSS) and our machinery. (We even need a K-type result at the end, provided to us by Bushnell-Henniart.)

Langlands-Shahidi method exploits the analytic properties of Eisenstein series. One has a triple $(\mathbf{G}, \mathbf{M}, \pi)$, where $\mathbf{G}$ is a reductive group, $\mathbf{M}$ is a maximal Levi, and π is a cuspidal representation of $M = \mathbf{M}(\mathbb{A}_F)$. In the constant term of the corresponding Eisenstein series, several L-functions show up. So do their inverses in a non-constant term, if we assume π is globally generic. We can then deduce the properties we need for L-functions by inductive use of such triples.

We get $L(s, \pi_1 \times \pi_2 \times (\sigma \otimes \chi))$, $\sigma \in \tau^S(n)$, for $n = 1, 2, 3, 4$ from triples:

$n = 1$ $\quad \mathbf{G} = GL_5 \quad \mathbf{M} = GL_2 \times GL_3$

$n = 2$ $\quad \mathbf{G} = Spin(10) \quad \mathbf{M}_D =$ derived group of $\mathbf{M} = SL_3 \times SL_2 \times SL_2$

$n = 3$ $\quad \mathbf{G} =$ simply connected E_6, $\quad \mathbf{M}_D = SL_3 \times SL_2 \times SL_3$

$$n = 4 \qquad \mathbf{G} = \text{ simply connected } E_7, \quad \mathbf{M}_D = SL_3 \times SL_2 \times SL_4$$

The number of L-functions appearing in the constant term in these cases are $1, 2, 3$ and 4, respectively. We observe that this functorial product is not endoscopic of any kind.

$Sym^4(\pi)$ is proved by applying

$$\Lambda^2\colon Aut(GL_4(\mathbb{A}_F)) \longrightarrow Aut(GL_6(\mathbb{A}_F)),$$

dual to the exterior square map

$$\Lambda^2\colon GL_4(\mathbb{C}) \longrightarrow GL_6(\mathbb{C}),$$

to $Sym^3(\pi)$. Functoriality of Λ^2 is proved by Kim by applying our machinery to $\mathbf{G} = Spin(2n)$, $n = 4, 5, 6, 7$, and $\mathbf{M}_D = SL(n-3) \times SL(4)$, $n = 4, 5, 6, 7$, respectively. This is a twisted endoscopic case of highest subtlety.

Classical Groups: Last, but not least, we like to mention that we have also proved the functoriality of

$$i : Sp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C}),$$

i.e., the existence of

$$i : Aut(SO_{2n+1}(\mathbb{A}_F)) \longrightarrow Aut(GL_{2n}(\mathbb{A}_F)),$$

in a joint work: Cogdell-Kim-Piatetski-Shapiro-Shahidi, using the very same machinery as others, for the generic spectrum. The fact that our lift is strong is proved by Ginzburg–Rallis–Soudry using their backward lift. Modulo a result of theirs, this is also proved by Kim. Beside converse theorems, we needed to borrow one more result from Rankin-Selberg: The stability of local root numbers under highly ramified twists. (The general case is being studied by Arthur modulo fundamental lemmas.)

To generalize this to arbitrary quasisplit classical groups we need this lemma. In fact, one hopes to prove the stability in full generality of our method for any quasisplit reductive group. We can then prove the following new cases of functoriality:

$$i : GSO_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$$

and

$$i : GSp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$$

leading to

$$i : Aut(GSpin_{2n}(\mathbb{A}_F)) \longrightarrow Aut(GL_{2n}(\mathbb{A}_F))$$

and

$$i : Aut(GSpin_{2n+1}(\mathbb{A}_F)) \longrightarrow Aut(GL_{2n}(\mathbb{A}_F)).$$

(Functoriality of Λ^2 and $GSp_4(\mathbb{C}) \longrightarrow GL_4(\mathbb{C})$ are among these.) These are examples of the most general kind of twisted endoscopic transfer. There is also the work of Friedberg-Goldberg to generalize our method to non-generic ones. But that is still in very early stages. One can also approach this from the point of view of a conjecture on genericity of tempered L–packets.

Beyond these we need new ideas. Two possible paths to follow are the theory of infinite dimensional groups and Langlands new ideas: "Beyond Endoscopy". These remain to be seen.