Lecture 4

## TODAY'S TASK

**To understand the relation between the Euclidean geometrical constructions and algebra, especially addition, subtraction, multiplication, division, and the extraction of square roots.**

The historical development of Cartesian geometry took place over about a century and a half, from the middle of the sixteenth to the end of the seventeenth, and was neither begun by Descartes nor ended by him. His concerns were none the less to some extent ours, the discovery and analysis of geometric constructions by algebraic means. On the other hand, the use of a rectilinear coordinate system, so familiar to us from various cities, especially New York, is not to be found in Descartes, where indeed one does not see a coordinate system at all. Descartes published his views early in the seventeenth century. Coordinate systems in a sense approximating ours did not appear until late in the century, in particular, in the works of Newton and Leibniz.

Nevertheless Descartes concerns are closer to ours than are those of other authors, whom I am in any case not yet in a position to discuss. Descartes not only published in the vernacular but also has been widely translated, so that he is much more accessible than many of the others.

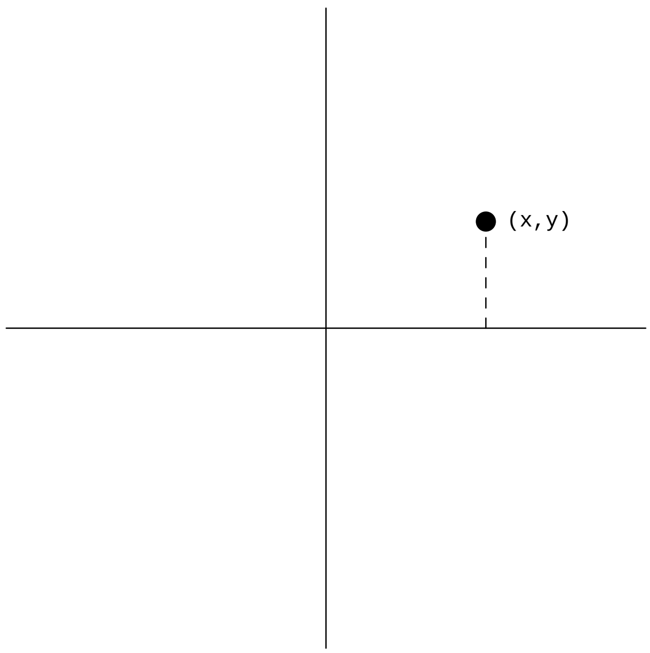
Since some familiarity with coordinate systems and their manipulation is almost universal in the modern world, it will be most efficient to be quite ahistorical and to run through a standard, textbook treatment, so that we can get on to Gauss within a reasonable time Even so a few scattered references to Descartes will be useful, just for fun.

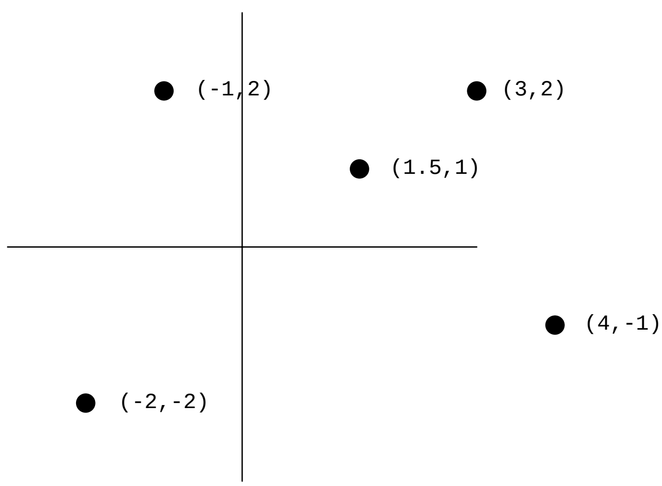
We shall have to analyze algebraically the constructions of Euclidean geometry, thus analyze these constructions in terms of Cartesian geometry. We examine first translation and rotation. Once the coordinates are chosen, we can consider translation by a *given* $(a, b)$, thus translation by $a$ parallel to one axis and then by $b$ parallel to another, where it is understood that $a$ or $b$ could be negative, thus comprise both a magnitude and a direction and that then we are to translate to the left or downward. Consider the first step. To translate the point $(x, y)$ parallel to the horizontal axis, thus parallel to the axis of abscissas, we draw a line through $p = (x, y)$ parallel to this axis, a construction that, according to Euclid, can be carried with a ruler and compass. Then with the help of a compass, we mark on it a point to the right or left of $p$, according to the sign of $a$, that is at a distance from it equal to the magnitude of $a$. The second translation is effected in a similar fashion. In particular, therefore, the cosntructions of Euclidean geometry allow us to add or subtract two numbers, which could be, for example, $a$ and $x$.

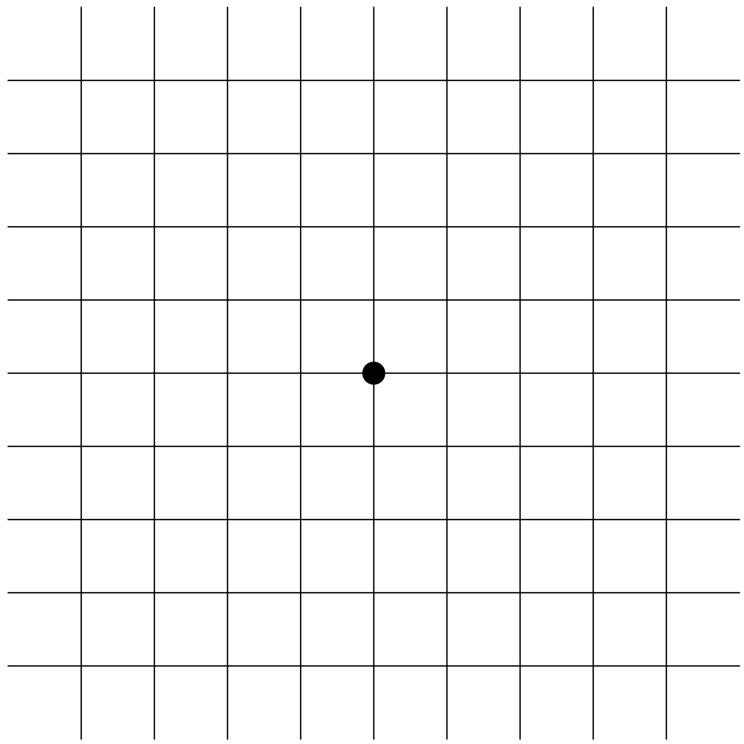
Multiplication is a different matter, because there is a philosophical point to discuss first. In Cartesian geometry we have a fundamental length, so that it is appropriate to identify lengths and numbers. We can therefore add two lengths or two numbers, and scarcely notice the difference. Multiplying two lengths yields however an area, which is not yet identified with a number, so that we have to be careful. We should therefore be explicit about the fundamental length. We call it $\lambda$. Then another length is $\mu$ and it is only $\mu/\lambda$ that is a number, a proportion or a ratio in the language of Euclid. How then do we multiply two proportions $\mu/\lambda$ and $\nu/\lambda$. We use similar triangles. We can divide in a similar way. All we need is exchange the roles of $\nu$ and $\eta$.

Since rotation is an operation that can, as we have seen, be carried out by multiplying coordinates, rotation of a given point, and thus of a given line, determined by any two points on it, can be carried out explicitly provided that the angle is given, either by its sine and cosine or by the two lines that form it, for from them the sine and cosine can be determined.

## Cartesian or analytic geometry



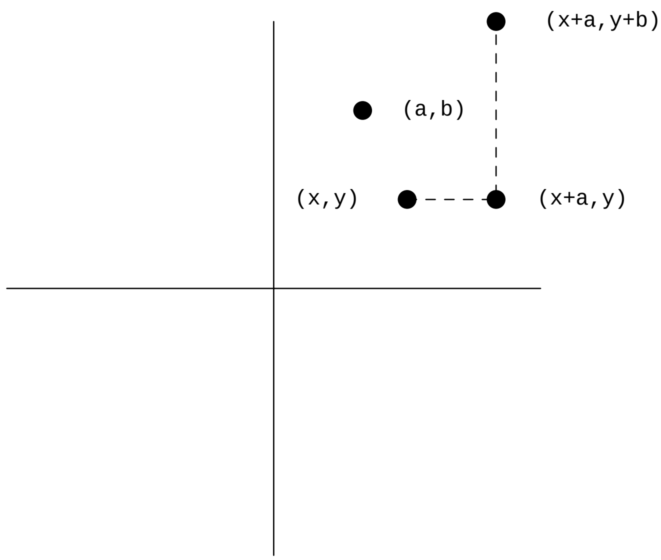





Observe that in Cartesian geometry a length is only implicit, or if one prefers there is no length, it has been replaced by a pure number. Thus in Cartesian geometry the notion of number is primary and independent of length, whereas in euclidean geometry the notion of number is secondary and is derived from that of length.

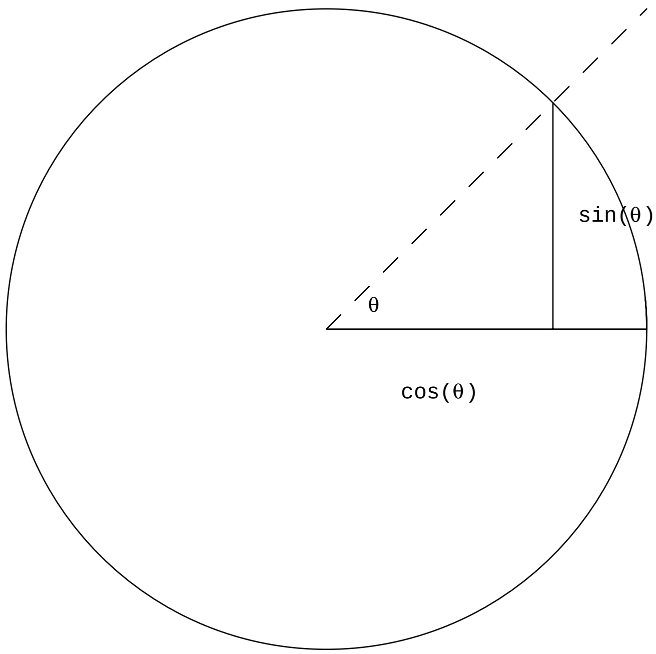
The notion of *conguence* that is so essential for Euclidean geometry has now to be made explicit as a combination of *translations* and *rotations*. I recall the formulas.

**Translation**

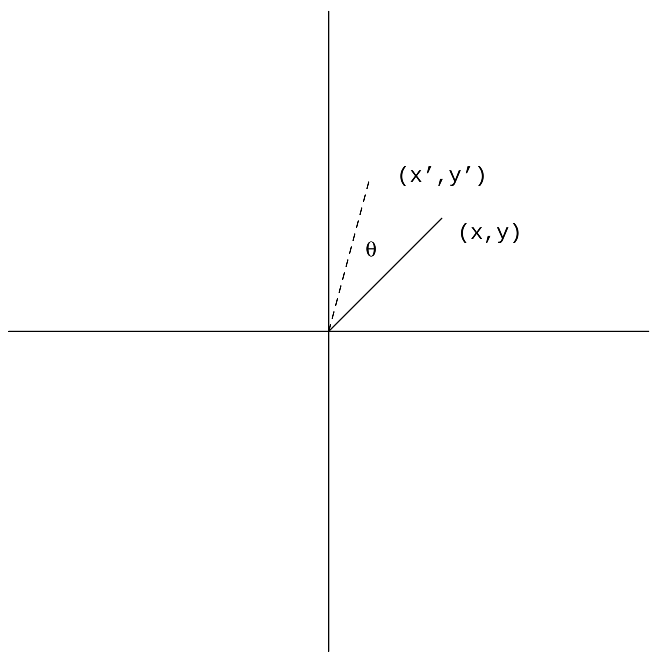


## Rotation

Rotation is more difficult. Recall that rotation of a figure and especially of a point is a turning about some given point that we can take at first to be the origin. A rotation is through an *angle* and we have first to be able to specify an angle. This is done – as you will no doubt be delighted to discover – through its sine and cosine.

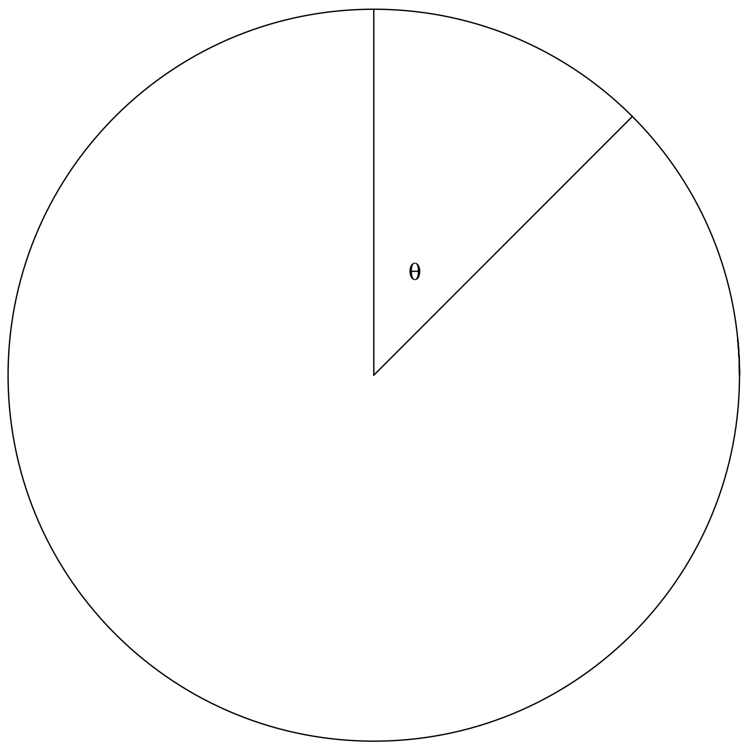



We want to rotate.

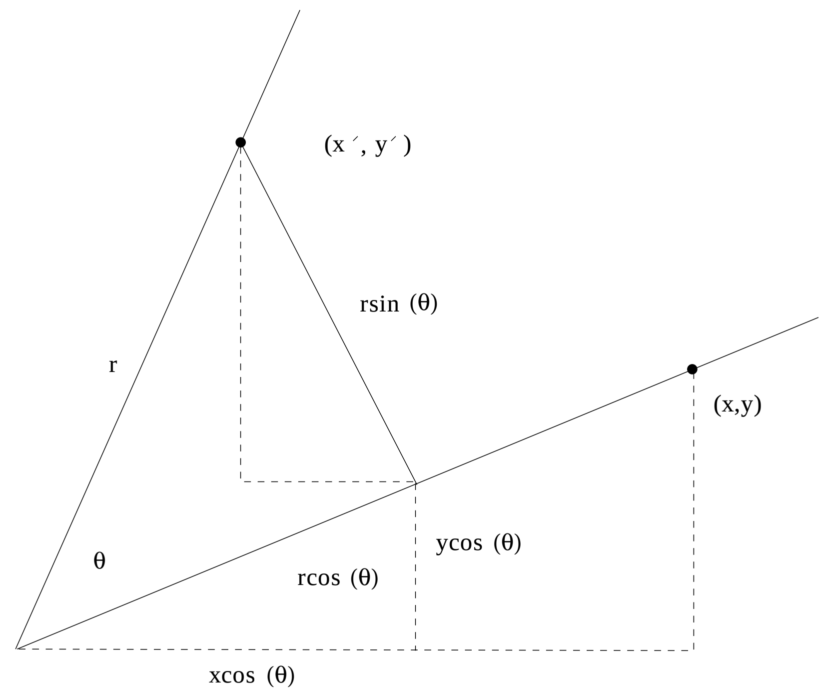



How do we find the coordinates $(x', y')$ in terms of $(x, y)$?

## Measurement of angles.



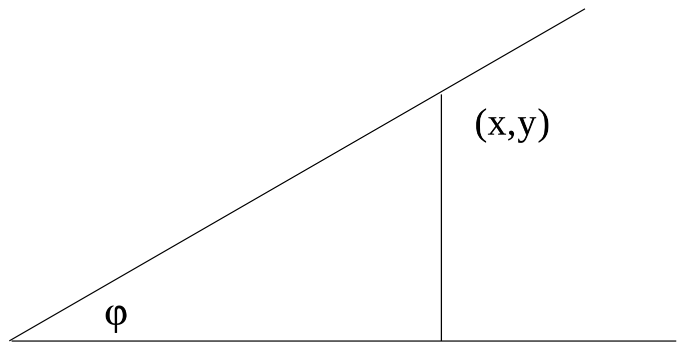
An angle is normally measured by the length of the arc it subtends. This length can be measured in two different units: degrees or radians. Degrees are defined by the condition that the total circumference have length $360°$ and radians by the condition that the total circumference have length $2\pi$, thus by the condition that the radius have length 1. We shall use radians as our measure. A right angle contains $90°$ or $\pi/2$ radians.




The point $(x, y)$ is at a distance $r = \sqrt{x^2 + y^2}$ from the origin, about which we are rotating. There are two triangles in the figure similar to the triangle with vertices $(0, 0, (x, 0), (x, y)$, a triangle which itself is not shown. It is right-angled with hypotenuse $r$, vertical side $y$ and horizontal side $x$. Of the two triangles, one has hypotenuse $r\cos(\theta)$. The other has hypotenuse $r\sin(\theta)$. The one whose sides are not given has its vertical side equal to $x\sin(\theta)$ and its horizontal side equal to $y\sin(\theta)$.

We are trying to find the coordinates $(x', y')$. They are seen to be

$$x' = x\cos(\theta) - y\sin(\theta),$$
$$y' = x\sin(\theta) + y\cos(\theta).$$




This is the triangle with vertices $(0,0)$, $(x,0)$, $(x,y)$. Let $\varphi$ be the indicated angle. Since

$$x = r\cos(\varphi), \qquad y = r\sin(\varphi),$$

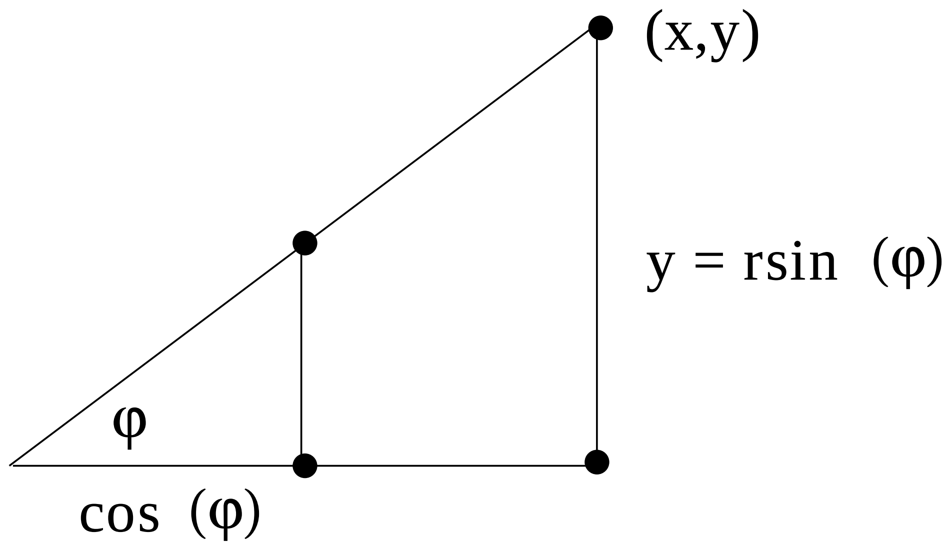
while

$$x' = r\cos(\varphi+\theta), \qquad y = r\sin(\varphi+\theta),$$

the formulas on the previous page yield formulas that will be familiar to most of you, but which I recall.

$$\cos(\varphi+\theta) = \cos(\varphi)\cos(\theta) - \sin(\varphi)\sin(\theta),$$
$$\sin(\varphi+\theta) = \cos(\varphi)\sin(\theta) + \sin(\varphi)\cos(\theta).$$

They will be important for us.




rr = xx + yy

**Niccolo Tartaglia (c. 1500-1557)**

**Gerolamo Cardano (1501-1576)**

**François Viète (1526-1573)**

$$ax^3 + bx^2 + cx^2 + d = 0$$

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

## Girard Desargues (c. 1591-1661)

## René Descartes (1596-1661)

## Pierre de Fermat (1601-1665)

# DISCOURS

# DE LA METHODE

## POUR BIEN CONDUIRE SA RAISON ET CHERCHER LA VERITÉ DANS LES SCIENCES

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*Si ce diſcours ſemble trop long pour eſtre tout leu en vne fois, on le pourra diſtinguer en ſix parties. Et, en la premiere, on trouuera diuerſes conſiderations touchant les ſciences. En la ſeconde, les principales regles de la Methode que l'Autheur a cherchée. En la 3, quelques vnes de celles de la Morale qu'il a tirée de cete Methode. En la 4, les raiſons par leſquelles il prouue l'exiſtence de Dieu & de l'ame humaine, qui ſont les fondemens de ſa Metaphyſique. En la 5, l'ordre des queſtions de Phyſique qu'il a cherchées, & particulierement l'explication du mouuement du cœur & de quelques autres difficultez qui appartienent a la Medecine, puis auſſy la difference qui eſt entre noſtre ame & celle des beſtes. Et en la derniere, quelles choſes il croit eſtre requiſes pour aller plus auant en la recherche de la Nature qu'il n'a eſté, & quelles raiſons l'ont fait eſcrire.*

Premiere partie.

Le bon ſens eſt la choſe du monde la mieux partagée : car chaſcun penſe en eſtre ſi bien pouruû, que ceux mefme qui font les plus difficiles a contenter en toute autre chofe, n'ont point couftume d'en defirer plus qu'ils en ont. En quoy il n'eft pas vrayfemblable que tous fe trompent; mais plutoft cela tefmoigne que la puiffance de bien iuger, & diftinguer le vray d'auec le faux, qui eft proprement ce qu'on nomme le bon fens ou la raifon, eft naturellement efgale en tous les hommes; et ainfi que la diuerfité de nos opinions ne vient pas de ce que les vns font plus raifonnables que les | autres, mais feulement de ce que nous conduifons nos penfées par diuerfes voyes, & ne confiderons pas les mefmes chofes. Car ce n'eft pas affez d'auoir l'efprit bon, mais le principal eft de l'appliquer bien. Les plus grandes ames font capables des plus grans vices, auffy bien que des plus grandes vertus; et ceux qui ne marchent que fort lentement, peuuent auancer beaucoup dauantage, s'ils fuiuent toufiours le droit chemin, que ne font ceux qui courent, & qui s'en efloignent.

Pour moy, ie n'ay iamais prefumé que mon efprit fuft en rien plus parfait que ceux du commun; mefme i'ay fouuent fouhaité d'auoir la penfée auffy prompte, ou l'imagination auffy nette & diftincte, ou la memoire auffy ample, ou auffy prefente, que quelques autres. Et ie ne fçache point de qualitez que celles cy, qui feruent a la perfection de l'efprit : car pour la raifon, ou le fens, d'autant qu'elle eft la feule chofe qui nous rend hommes, & nous diftingue des beftes, ie veux croyre qu'elle eft toute entiere en vn chafcun, & fuiure en cecy l'opinion commune des Philofophes, qui difent qu'il n'y a du plus & du moins qu'entre les *accidens*, & non point entre les *formes*, ou natures, des *indiuidus* d'vne mesme *espece*.

Mais ie ne craindray pas de dire que ie pense auoir eu beaucoup d'heur, de m'estre rencontré dés ma ieunesse en certains chemins, qui m'ont conduit a des considerations & des maximes, dont i'ay formé vne Methode, par laquelle il me semble que i'ay moyen d'augmenter par degrez ma connoissance, & de l'esleuer peu a peu au plus haut point, auquel la mediocrité de mon esprit & la cour|te durée de ma vie luy pourront permettre d'atteindre. Car i'en ay desia recueilly de tels fruits, qu'encore qu'aux iugemens que ie fais de moymesme, ie tasche tousiours de pencher vers le costé de la defiance, plutost que vers celuy de la presomption; & que, regardant d'vn œil de Philosophe les diuerses actions & entreprises de tous les hommes, il n'y en ait quasi aucune qui ne me semble vaine & inutile; ie ne laisse pas de receuoir vne extreme satisfaction du progrés que ie pense auoir desia fait en la recherche de la verité, & de conceuoir de telles esperances pour l'auenir, que si, entre les occupations des hommes purement hommes, il y en a quelqu'vne qui soit solidement bonne & importante, i'ose croyre que c'est celle que i'ay choisie.

Toutefois il se peut faire que ie me trompe, & ce n'est peutestre qu'vn peu de cuiure & de verre que ie prens pour de l'or & des diamans. Ie sçay combien nous sommes suiets a nous méprendre en ce qui nous touche, & combien aussy les iugemens de nos amis nous doiuent estre suspects, lorsqu'ils sont en nostre faueur. Mais ie seray bien ayse de faire voir, en ce dis-

# LA GEOMETRIE

## LIVRE PREMIER.

*Des probleſmes qu'on peut conſtruire ſans y employer que des cercles & des lignes droites.*

Tous les Probleſmes de Geometrie ſe peuuent facilement reduire a tels termes, qu'il n'eſt beſoin, par aprés, que de connoiſtre la longeur de quelques lignes droites, pour les conſtruire.

**Comment le calcul d'Arithmetique ſe rapporte aux operations de Geometrie.**

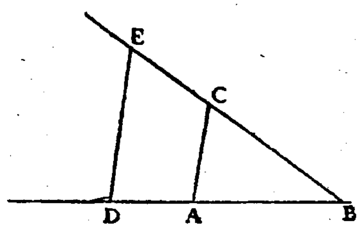
Et comme toute l'Arithmetique n'eſt compoſée que de quatre ou cinq operations, qui ſont : l'Addition, la Souſtraction, la Multiplication, la Diuiſion, & l'Extraction des racines, qu'on peut prendre pour vne eſpece de Diuiſion *; ainſi n'a-t-on autre choſe a faire, en Geometrie, touchant les lignes qu'on cherche, pour les preparer a eſtre connuës, que leur en adiouſter d'autres, ou en oſter; ou bien, en ayant vne

* *Nous indiquons, par des étoiles, les endroits auxquels se rapportent les commentaires de Schooten dans ses éditions latines de la* Geometrie (1649 et 1659). *La lettre de renvoi correspondante est, pour cette page,* A.

que ie nommeray l'vnité* pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement eſtre priſe a diſcretion*, puis en ayant encore deux autres, en trouuer vne quatrieſme, qui ſoit a l'vne de ces deux comme l'autre eſt a l'vnité, ce qui eſt le meſme que la Multiplication*; ou bien en trouuer vne quatrieſme, qui ſoit a l'vne de ces deux comme l'vnité eſt a l'autre, ce qui eſt le meſme que la Diuiſion*; ou enfin trouuer vne, ou deux, ou pluſieurs moyennes proportionnelles entre l'vnité & quelque autre ligne, ce qui eſt le meſme que tirer la racine quarrée, ou cubique, &c. Et ie ne craindray pas d'introduire ces termes d'Arithmetique en la Geometrie, affin de me rendre plus intelligible.

La Multiplication

Soit, par exemple, AB l'vnité, & qu'il faille multiplier BD par BC; ie n'ay qu'a ioindre les poins A & C, puis tirer DE parallele a CA, & BE eſt le produit de cete Multiplication.

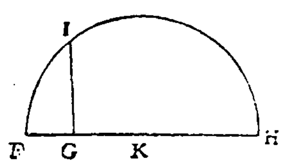



La Diuiſion.

Ou bien, s'il faut diuiſer BE par BD, ayant ioint les poins E & D, ie tire AC parallele a DE, & BC eſt le produit de cete Diuiſion.

L'Extraction de la racine quarrée.

Ou, s'il faut tirer la racine quarrée de GH, ie luy adiouſte en ligne droite FG, qui eſt l'vnité, & diuiſant FH en deux parties eſgales au point K, du centre K ie tire le cercle FIH; puis, eſleuant du point G vne ligne droite iuſques a I a angles droits ſur FH, c'eſt




* B. — C. — D. — E.

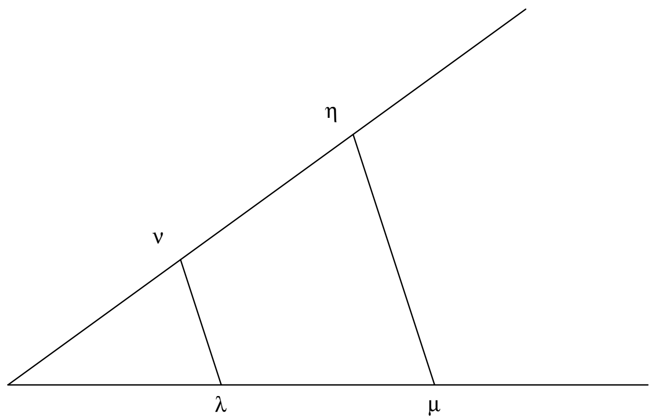
GI, la racine cherchée. Ie ne dis rien icy de la racine cubique ny des autres, a cause que i'en parleray plus commodement cy aprés.

Comment on peut vser de chiffres en Geometrie.

Mais souuent on n'a pas besoin de tracer ainsi ces lignes sur le papier, & il suffist de les designer par quelques lettres, chascune par vne seule. Comme, pour adiouster la ligne BD a GH, ie nomme l'vne $a$ & l'autre $b$, & escris $a+b$; et $a-b$, pour soustraire $b$ d'$a$; et $ab$, pour les multiplier l'vne par l'autre; et $\frac{a}{b}$, pour diuiser $a$ par $b$; et $aa$ ou $a^2$, pour multiplier $a$ par soy mesme; et $a^3$, pour le multiplier encore vne fois par $a$, & ainsi a l'infini; et $\sqrt{a^2+b^2}$, pour tirer la racine quarrée d'$a^2+b^2$; et $\sqrt{C.\,a^3-b^3+abb}$, pour tirer la racine cubique d'$a^3-b^3+abb$, & ainsi des autres.

Où il est a remarquer que, par $a^2$ ou $b^3$ ou semblables, ie ne conçoy ordinairement que des lignes toutes simples, encore que, pour me seruir des noms vsités en l'Algebre, ie les nomme des quarrés, ou des cubes, &c.

Il est aussy a remarquer que toutes les parties d'vne mesme ligne se doiuent ordinairement exprimer par autant de dimensions l'vne que l'autre, lorsque l'vnité n'est point determinée en la question : comme icy $a^3$ en contient autant qu'$abb$ ou $b^3$, dont se compose la ligne que i'ay nommée $\sqrt{C.\,a^3-b^3+abb}$; mais que ce n'est pas de mesme lorsque l'vnité est determinée, a cause qu'elle peut estre sousentendue partout où il y a trop ou trop peu de dimensions; comme, s'il faut tirer la racine cubique de $aabb-b$, il faut penser que la quantité $aabb$ est diuisée vne fois par l'vnité, & que




By similarity of triangles,

$$\frac{\eta}{\nu} = \frac{\mu}{\lambda}.$$

Thus

$$\frac{\eta}{\nu} \times \frac{\nu}{\lambda} = \frac{\mu}{\lambda} \times \frac{\nu}{\lambda}$$

or

$$\frac{\eta}{\lambda} = \frac{\mu}{\lambda} \times \frac{\nu}{\lambda}$$

## Warning

I observe that multiplication and division require that we draw a line through a point parallel to a given line. We have not given the proof that this is possible, but like the construction of the perpendicular bisector of the line joining two points, it requires taking the intersection of two circles.

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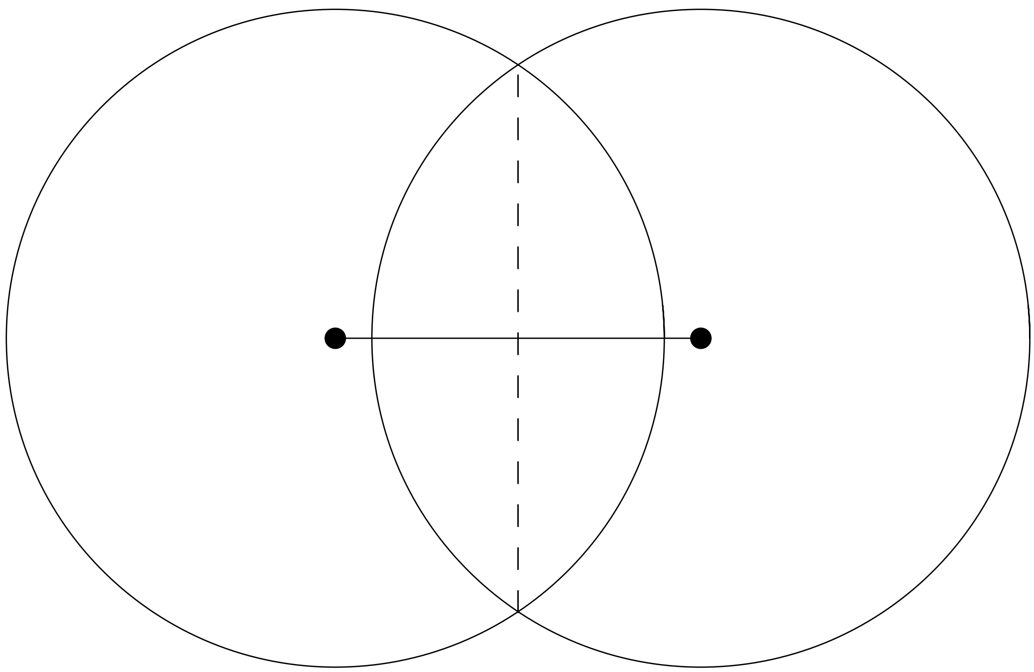
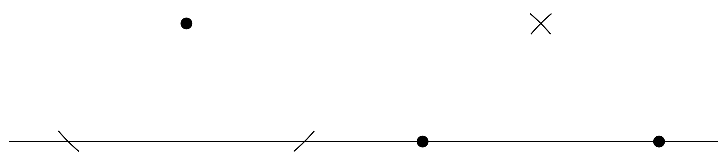
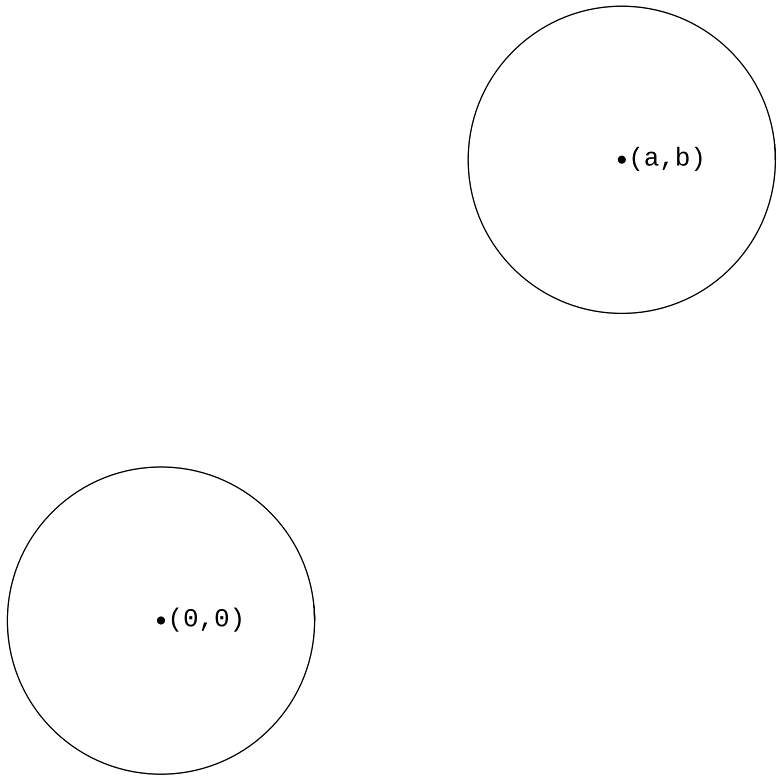
It appears therefore that we do not need to take the intersection of two circles because square roots can be constructed by intersecting a line with a circle. However, as we already noted, multiplication and division require the intersection of circles, so that the construction is by no means redundant.



**Line through a point parallel to a given line**



## Equations for circles in Cartesian geometry




Consider, first of all, a circle about the origin. By the pythagorean theorem, the equation is

$$x^2 + y^2 = r^2,$$

if $r$ is the radius of the circle and $(x, y)$ a point on it. If we start with another circle with center $(a, b)$, then we translate it to a circle with center at the origin.

$$(x, y) \to (x - a, y - b).$$

Since the new point is on the circle of radius 1 it satisfies

$$(x - a)^2 + (y - b)^2 = r^2 \tag{A}$$

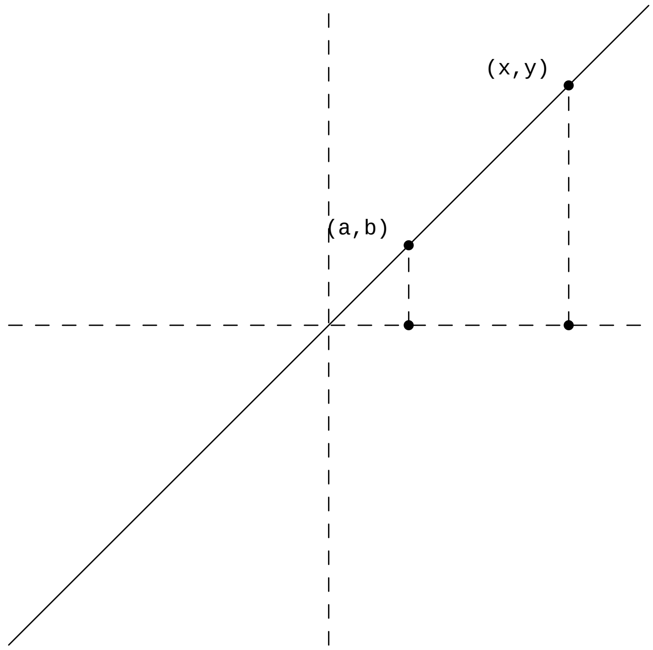
This is the equation of a general circle. If we expand all the powers it becomes

$$x^2 + y^2 + 2ax + 2by + d = 0, \tag{B}$$

where $d = a^2 + b^2 - r^2$. Recall that for any two numbers $x$ and $a$

$$(x + a)^2 = x^2 + xa + ax + a^2 = x^2 + 2ax + a^2.$$

## Equations for lines in Cartesian geometry




Consider, first of all, a line through the origin. Suppose this is the line through $(0,0)$ and $(a,b)$ and that we want to determine what equation is satisfied by a general point $(x,y)$ on it. Since the triangle with vertices $(0,0)$, $(x,0)$ and $(x,y)$ is similar to the triangle with vertices $(0,0)$, $(a,0)$ and $((a,b)$ we have

$$y : b = x : a, \quad \frac{y}{b} = \frac{x}{a}, \quad ay = bx, \quad bx - ay = 0.$$

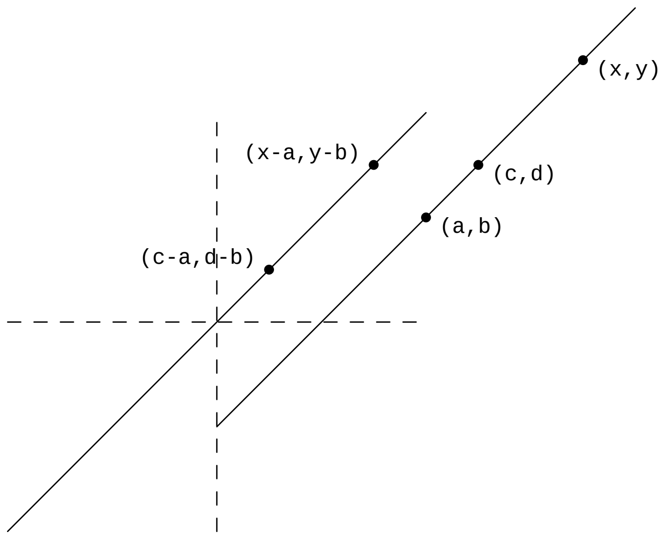
Next suppose we have an arbitrary line through the points $(a,b)$ and $(c,d)$ and that $(x,y)$ is a point on it. Then translating through $(-c,-d)$ we obtain a line through the origin $(0,0)$ on which $(a-c, b-d)$ and $(x-c, y-d)$ lie. thus

$$(b-d)(x-c) - (a-c)(y-d) = 0.$$

Setting $e = (b-d)$, $f = -(a-c)$, $g = -(b-d)c + (a-c)d$ we can change this equation to

$$ex + fy + g = 0.$$

This is the general equation of a line. It contains $x$ and $y$ to the first power and three constants.




Observe that the coefficients $e$, $f$ and $g$ that appear in the equation of the line can be expressed algebraically in terms of the coefficients $(a, b)$ and $(c, d)$ of any two points on it.

## Intersection of lines

In the equation for a line $ex + fy + g = 0$ that we obtained, the point $(-f, e)$ was a point on the line through the origin obtained by translation. If we started from a parallel line to obtain $e'x + f'y + g'$ then the translated line would be the same and $(-f', e')$ would lie on it. Thus by similarity of triangles

$$e : e' = -f : -f' = f : f' \quad \frac{e}{e'} = \frac{f}{f'}, \quad ef' = fe', \quad ef' - f'e = 0.$$

The final equation is then the condition that the two equations

$$ex + fy + g = 0, \qquad e'x + f'y = g' = 0$$

define parallel lines.

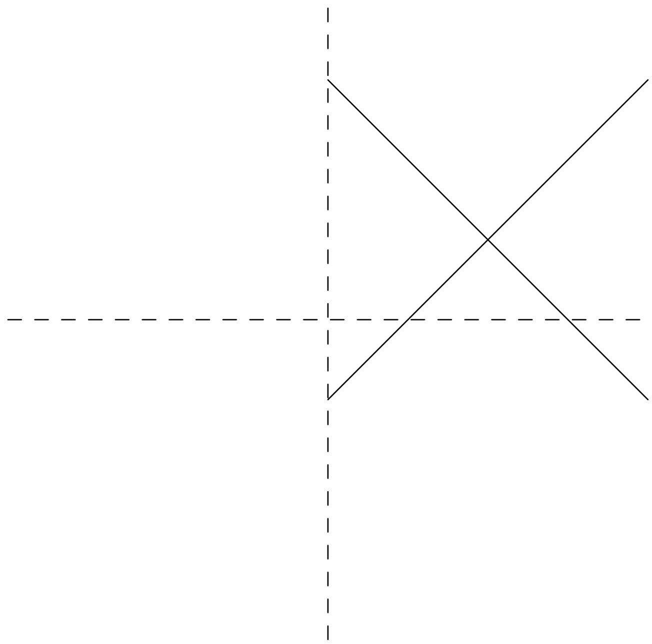
If two lines are not parallel they must have a point in common, thus a point that satisfies the two equations

$$\begin{aligned} ex + fy + g &= 0, \\ e'x + f'y + g' &= 0. \end{aligned}$$

We solve first for $x$, multiplying the first equation by $f'$ and the second by $f$ and subtracting.

$$(f'e - fe')x + f'g - fg' = 0, \quad x = -\frac{f'g - fg'}{f'e - e'f}.$$

Since $f'e - fe'$ is not 0, This yields $x$. we find $y$ in a similar fashion. Thus from the point of view of geometrical constructions, the intersection of lines is not so interesting. It yields a point that we construct with addition, multiplication and division from the coefficients of the equations of the lines, and thus simply from the coordinates of points on the lines.



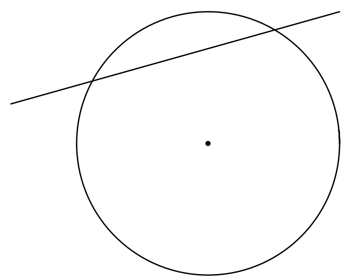
$$x - y = 1, \qquad x + y = 3$$

$$x = 2, \quad y = 1$$

**Intersection of a line and a circle**

First of all, translate the the circle so that its center is at $(0,0)$ and translate the line in the same way, so that the points of intersection are also translated. If $(a,b)$ is the *known* center of circle, this just means subtracting $a$ or $b$ from the coordinates of all points. The translation can be undone at the end by adding $a$ and $b$ back. The circle will then have an equation

$$x^2+y^2=r^2.$$



Suppose the line is given by its equation

$$ax+by+c=0. \qquad \text{New } a,\ b$$

Either $a$ or $b$ will not be 0. I treat the case that $a$ is not 0. Then

$$x=-\frac{by+c}{a}.$$

Thus

$$\frac{(by+c)^2}{a^2}+y^2=r^2, \qquad (by+c)^2+a^2y^2=a^2r^2$$

or

$$(b^2+a^2)y^2+2bcy+c^2-a^2r^2=0.$$

Solve this equation by the usual formula – which then has to be simplified algebraically – to obtain

$$y=-\frac{bc}{a^2+b^2}\pm\frac{\sqrt{b^2c^2-(b^2+a^2)(c^2-a^2r^2)}}{a^2+b^2}.$$

The conclusion is that the points of intersection can be found by calculating the square root of a number formed from known numbers: the three numbers $a$, $b$ and $c$ that are determined by two points on the line and the radius $r$ of the circle.

**An example**

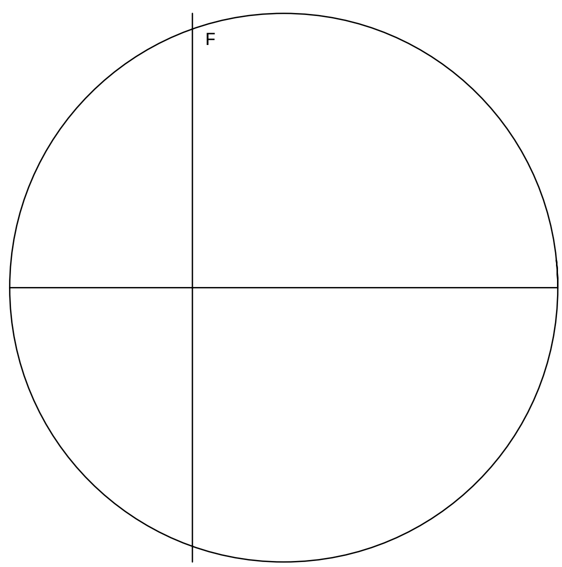
Suppose we want to find the square root of a number $D$. Take a circle with center at the origin and diameter $1+D$, thus with radius $(1+D)/2$. Take the vertical line through the point $((-D+1)/2, 0)$. It will have an equation

$$x + \frac{D-1}{2} = 0.$$

Thus we have to substitute $a =$, $b = 0$, $c = (D-1)/2$, $r = (D+1)/2$ in the previous formula. The first term on the right is 0 and the second becomes $\sqrt{D}$ because

$$r^2 - c^2 = \frac{(D+1)^2}{4} - \frac{(D-1)^2}{4} = D.$$

Thus the second coordinate of the point on the figure is the square root of the number $D$, so that intersecting an appropriate line with an appropriate circle, we find the square root of any positive number. This was an observation of Descartes.



## Intersection of two circles

It turns out that the intersection of two circles can also be determined by the extraction of square roots.

Once again we translate so that one of the circles has its center at the origin, or perhaps better we choose the origin to be at the center of one of the circles. Then we choose the axis of abscissas to be the line passing through the center of the two circles, and the unit distance to be the distance between the two centers. Then the first circle has an equation

$$x^2 + y^2 = r^2 \Longrightarrow y^2 = r^2 - x^2$$

and the second center at $(1, 0)$

$$(x-1)^2 + y^2 = R^2 \Longrightarrow (x-1)^2 + r^2 - x^2 = R^2,$$

because its center is at $(1, 0)$. Simplify to obtain

$$2x = r^2 + 1 - R^2 \Longrightarrow x = \frac{r^2 + 1 - R^2}{2}$$

Thus $y$ is equal to

$$\sqrt{r^2 - \frac{(r^2 + 1 - R^2)^2}{4}} = \sqrt{-\frac{(r^2 - R^2)^2}{4} + \frac{r^2 + R^2}{2} - \frac{1}{4}}$$

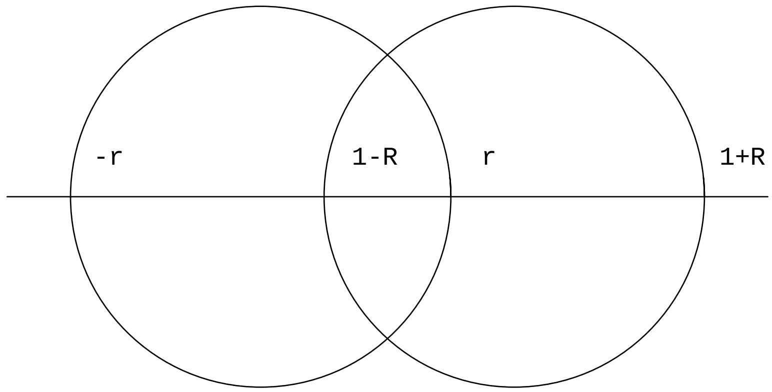
To verify this, try $r = 1/2$, $R = 1/2$, when the intersection consists of a single point with $y = 0$. As a further verification, note that if $x \geq 0$, thus if $r^2 + 1 \geq R^2$ then for $y$ to be a real number, the expression under the square-root sign has to be positive or at least not negative, thus $r \geq x$ or

$$2r \geq r^2 + 1 - R^2 \Longleftrightarrow R^2 \geq (r-1)^2.$$

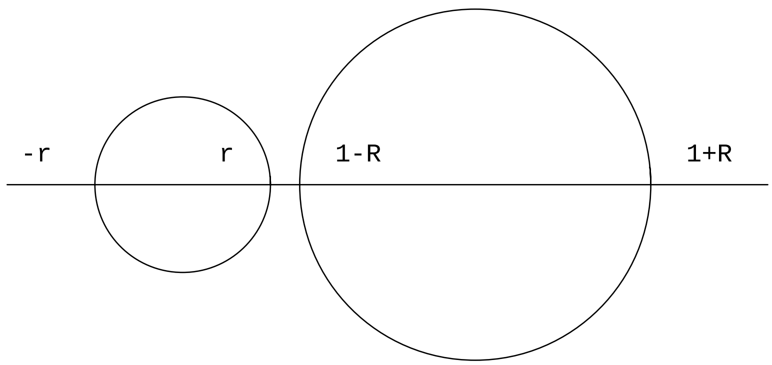
If $r \geq 1$ this means that $R \geq r - 1$ and if $r < 1$ it means that $R + r \geq 1$. If $x < 0$ thus if $r^2 + 1 < R^2$ then $2r$ has to be larger than or equal to $R^2 - r^2 - 1$. Thus $(r+1)^2 \geq R^2$ or $r + 1 > R$ which is the same as $1 - R > -r$. I give some examples.

## Examples

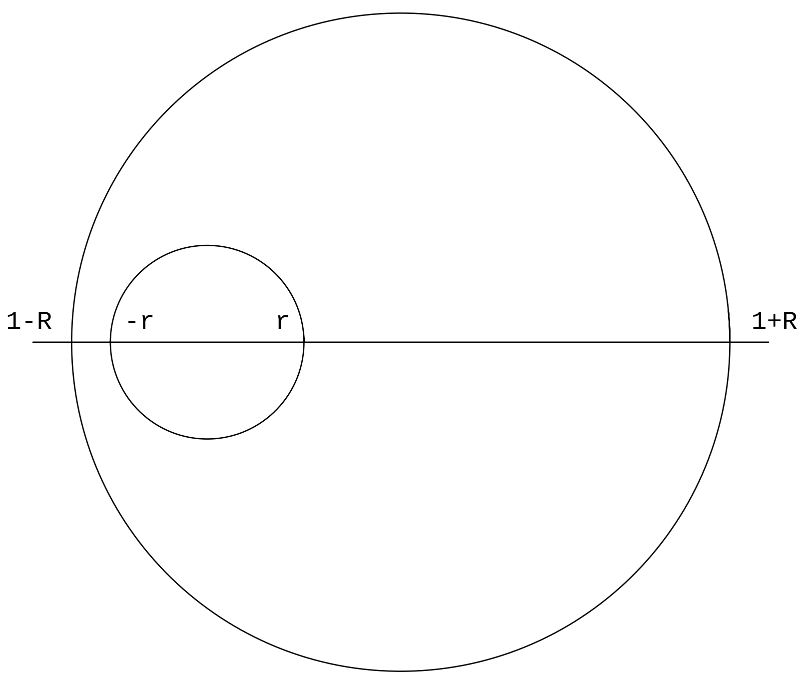
For the two circles with centers at $(0,0)$ and $(1,0)$ and radii $r$ and $R$ to meet the points on the axis of abscissas with abscissa $-r$, $1-R$, $r$, $1+R$ must lie in this order. This is illustrated in the examples.




$r = R = .75$




$r = .3, \quad R = .6$




$r = .5, \quad R = 1.7$

**Some texts consulted**


HEATH'S EUCLID

DESCARTES: DISCOURS DE LA METHODE

MORRIS KLINE:
MATHEMATICAL THOUGHT FROM ANCIENT TO MODERN TIMES

CARL BOYER: HISTORY OF ANALYTICAL GEOMETRY

GAUSS: ARITHMETICAL INVESTIGATIONS

FELIX KLEIN: FAMOUS PROBLEMS OF ELEMENTARY GEOMETRY

## On Descartes

Although it has been and remains my intention to talk about mathematics and not to allow myself any digressions, which would quickly reveal enormous gaps in my general culture, even, and perhaps especially, in that related to matters directly pertinent to mathematics and its history, I was last week questioned about the relation of Descartes's method to his mathematics and about the influence of his mathematics on that which followed. Even a brief glance at Descartes's works and his correspondence with other savants of the period renders evident that no serious answer can be offered without a knowledge of the European intellectual environment in the first decades of the seventeenth century that I do not have.

It is, however, easy to say one or two things based almost entirely on the *Discours* itself, since Descartes is quite explicit about the role of mathematics in his proposed methodology, a methodology that seems to have matured over the course of 17 years, from 1619, when he was twenty-three years old, to 1637, when he was forty-one. In the intervening years, he had seen much of the world, at least of Europe, beginning with service at the battle of the White Mountain in Bohemia, an early, decisive battle of the Thirty Years War. The last nine years before the appearance of the *Discours* were spent in the Netherlands. Although, so far as I can tell, the region was still embroiled in the aftermath of a struggle for independence and in religious conflict, Descartes life appears to have been untroubled.

The method as such appears, according to Descartes's account, to have been formulated early. Descartes introduces its precepts with the remark that "he had studied a little when young among other parts of philosophy, logic, and among other parts of mathematics, the analysis of the geometers and algebra, three arts or sciences that seemed capable of contributing to his plan." On closer examination, logic served more to explain the known than to learn the new. As for the analysis of the ancients and the algebra of the moderns, they both seemed too abstract, and the first too constrained to the use of figures and the second to various rules and signs out of which an art had been constructed that confused the mind rather than cultivating it.

He himself proposes an art with a few simple precepts that I summarize briefly: never to accept anything as true except that which he recognizes as clearly such; to divide each difficulty that he meets into manageable pieces; to proceed in his thinking, stage by stage, from the simple to the complex; to review his thinking so carefully that he was sure that nothing had been omitted. I believe that these precepts are the essence of his method.

Then oddly enough, in spite of his previous strictures, he passes back to mathematics. I cite the text, with a more or less literal translation. " These long chains of reasonings, all simple and easy, that geometers customarily use to arrive at their most difficult proofs, gave me occasion to imagine that everything that could be known by men, followed in the same fashion; and that provided only that one abstained from accepting anything for true that was not, and that one kept always to the necessary order in deducing one from another, there were none so abstruse that one would not ultimately arrive at them, nor so hidden that they would never be discovered. I had no difficulty in finding the correct place to begin, because I knew already that it was with the simplest and the easiest to know; and considering that, among all those who had already searched for the truth in the sciences, it was only the mathematicians who had been able to discover some proofs, that is to say, some certain and evident reasoning, I did not at all doubt that these would be the things they had examined; even though I hoped for no other profit than that they might accustom my wit to nourish itself with truths and not to content itself with false arguments. But I had no intention, for all that, of trying to learn all the particular sciences commonly called mathematical; and seeing that although their objects were different they nevertheless were in accord and that they did not treat of other things except for various pertinent ratios or proportions,

*One might suppose that Descartes is here thinking of Book V of Euclid because the theory of proportions contained therein can be applied to lengths, areas, volumes or numbers!*

I thought that it would be more profitable to examine these proportions in general, and only in those subjects that would make a knowledge of them easier for me, but also without restricting them in any manner, in order to be able to apply them later to the other subjects for which they were appropriate. Then, having observed that to understand them I would have sometimes to consider them separately and sometimes to imagine or understand them several at a time, I thought that in order to consider them better separately, I should imagine them as lines, because I found nothing that was more simple, or that I could more distinctly represent to my imagination and my senses;

*This is pertinent to the choice of problems in La Géométrie*

but that in order to hold them in my mind or to understand several at once, it was necessary to explain them with some signs (ciphers), the shortest possible; and in this way I took the best from geometrical analysis and from algebra, correcting the faults of one by the other."

Of course, Descartes was by no means primarily a mathematician, and it may not be clear from these remarks that he was a mathematician at all. In fact he had two quite different mathematical talents: he was able to discover new facts, which mathematicians normally call theorems or, nowadays, results and perhaps even to prove them; and he could formulate new concepts. Of course, I cannot say with any authority how new – that would demand a knowledge of sixteenth and seventeenth century science that I do not have, but secondary sources suggest that the mathematical results of *La géométrie* are pretty much his own. The prettiest, mentioned in passing, without proof is Descartes's rule of signs, that appears, difficult as this is to believe, to have been first proved by Gauss. It would be a simple exercise for any mathematician in the room, even those to whose attention it had never been drawn. The proof that suggests itself uses differentiation and mathematical induction. I explain the statement briefly. Although it is not directly germane to our purposes, the explanation will make later concepts easier. Besides it is useful now and again, in lectures of this kind, to offer simple, comprehensible mathematical assertions, that can be certified as genuinely elegant, if for no other reason than to give the rest of the world some feeling for what this notion means to the mathematician.

## Descartes's rule of signs

THE NUMBER OF TRUE (POSITIVE) ROOTS IS AT MOST EQUAL TO THE NUMBER OF SIGN CHANGES. THE NUMBER OF FALSE (NEGATIVE) ROOTS IS AT MOST EQUAL TO THE NUMBER OF SIGN DOUBLETS.

$$(x-1)(x-2)(x-3) = x^3 - 6x^2 - 11x - 6 = 0$$

Positive roots are: $x = 1, 2, 3$. Three sign changes; no doublets.

$$(x+1)(x+2)(x-3) = x^3 - 7x - 6 = 0$$

Positive roots are: $x = 3$. Negative: $x = -1, -2$. Two sign changes; three doublets.

$$(x+1)(x-2)(x-3) = x^3 - 4x^2 + x + 6 = 0$$

Positive roots are: $x = 2, 3$. Negative: $-1$. Two sign changes; one doublet.

$$(x^2+1)(x-3) = x^3 - 3x^2 + x - 3 = 0$$

Positive roots are: $x = 3$. No negative roots. Three sign changes; no doublets.

$$(x+1)(x+2)(x+3) = x^3 + 6x^2 + 11x + 6 = 0$$

No positive roots. Negative roots: $-1, -2, -3$. No sign changes. Three doublets.

$$(x-1)^2(x-2) = x^3 - 4x^2 + 5x - 2 = 0$$

Roots: $1, 1, 2$. Three sign changes. No doublets.

NOTE: Every equation of degree $n$ (dimension for Descartes) has exactly $n$ roots, but they can be complex and they can be repeated!.

I recall that Descartes introduction of Cartesian geometry appears in one of the three appendices to *Discours de la méthode*, the others being on optics and on rainbows. It is not yet clear to me what the relation between the *Discours* and the appendices is. The appendices are more than appendices and the body more than an introduction to the appendices.

Although the appendix on geometry is largely devoted to construction problems, they are by and large not the construction problems that concern us, thus those that can be effected with ruler and compass. Our problems are in Descartes terminology planar problems although he sometimes to them as two-dimensional problems, but two dimensional has nothing to do with the plane. Descartes simply means that in the final algebraic equation arising from the geometric problem the unknown appears as a square. He deals with these problems briefly in the first few pages and does come back to them repeatedly, but they are simply the first in a sequence of problems, followed by solid problems or problems of dimension three and four, and then by "supersolid" problems of dimension 6.

There are a number of points that he makes, that are worth recalling here. The first is that the use of ruler and compass alone is an artificial restriction. The compass is a mechanical device, as is the ruler, and other mechanical devices could be considered, of which Descartes suggests one. It constructs cube roots, fourth roots and indeed roots of any order. The restriction to ruler and compass has, however, a great deal of historical importance, because that is what we find in Euclid, as well as great theoretical significance, because it had eventually to be asked what the algebraic and arithmetic significance of the restriction was.

Descartes wanted to play hardball. I cannot assure you that he succeeded. It would I believe take a great deal of study of earlier authors and of Descartes's contemporaries to determine exactly what his contributions were.

Since he wanted to demonstrate that the use of algebraic equations and coordinates permitted the discovery and demonstration of new theorems, most of the appendix is devoted to indeterminate construction problems, thus to problems that define a curve, or to problems that require more than a ruler and compass for their solution.

The free passage back and forth from the geometry to the algebra, allows him to convert one kind of geometric problem to another. For example to duplicate a cube, one needs to extract the cube root of 2. If the side of a cube is $a$, so that its volume is $a^3$ then a cube of side $\sqrt[3]{2}a$ has double its volume, namely $2a$. Descartes showed that all cubic equations, in particular

$$x^3 = 2$$

could be solved by intersecting circles with a conic section, in particular with a parabola. Indeed he shows that all cubic and quartic equations can be solved geometrically in this way.

In other words, cubic and quartic irrationalities are all constructed by intersecting circles and parabolas. The converse is also true.

Eventually he reaches irrationalities of degree, or as he says dimension, 5 and 6, but here the construction of solutions requires the use of more complicated mechanical devices, requiring moving parabolas.

There is among Descartes's often Delphic remarks one that, to me at least, anticipates in a curious way the use of complex numbers. He observes, for example, that the geometric construction of the solutions of cubic equations entails being able to perform exactly two constructions, that of the cube root of a positive number, thus of constructing two mean proportionals to two given lengths, and trisection of an arbitrary angle. We will come back to this remark later, but neither exactly what Descartes had in mind nor what he had learned from others is clear to me.

If

$$YA : YB = YB : YC = YC : YD$$

then

$$(\frac{YA}{YB})^3 = \frac{YA}{YB}\frac{YB}{YC}\frac{YC}{YD} = \frac{YA}{YD}$$

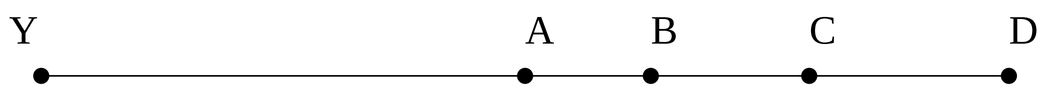
The first relations are referred to by saying that $YB$ and $YC$ are two mean proportionals between $YA$ and $YD$. If $YA : YD = 1 : 2$ then in effect we want

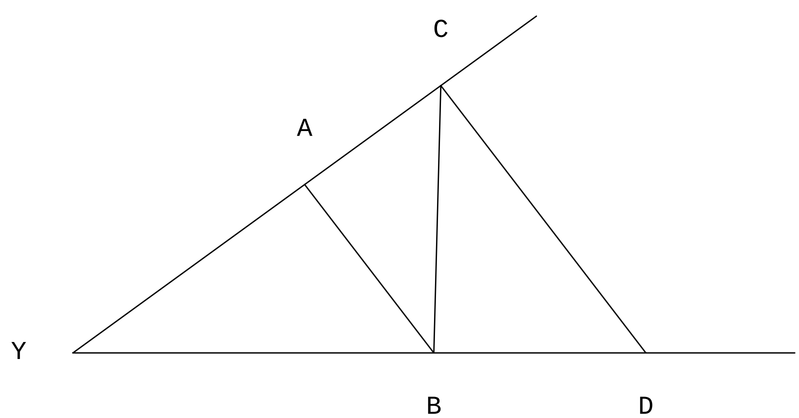
$$(\frac{YA}{YB})^3 = \frac{1}{2}$$

or

$$(\frac{YB}{YA})^3 = 2$$

Thus if $YA$ is the side of a cube then $YB$ will be the side of a cube of twice the volume.



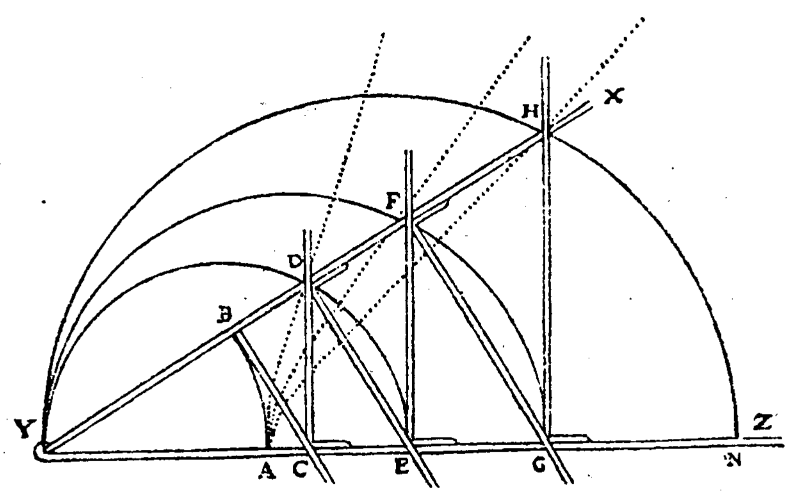
In the same way one can ask for four mean proportionals,

$$YA : YB = YB : YC = YC : YD = YD : YE = YE : YF$$

which implies that

$$(\frac{YA}{YB})^5 = \frac{YA}{YF}$$

Voyés les lignes AB, AD, AF & femblables, que ie fuppofe auoir efté defcrites par l'ayde de l'inftrument YZ[a], qui eft compofé de plufieurs reigles, tellement iointes que, celle qui eft marquée YZ eftant areftée fur la ligne AN, on peut ouurir & fermer l'angle XYZ, & que, lorfqu'il eft tout fermé, les poins B, C, D, <E>[b] F, G, H font tous affemblés au point A;




mais qu'a mefure qu'on|l'ouure, la reigle BC, qui eft iointe a angles droits auec XY au point B, pouffe vers Z la reigle CD, qui coule fur YZ en faifant toufiours des angles droits auec elle; & CD pouffe DE, qui coule tout de mefme fur YX en demeurant parallele a BC; DE pouffe EF; EF pouffe FG; celle cy pouffe GH; & on en peut conceuoir vne infinité d'autres, qui fe pouffent confequutiuement en mefme façon, & dont les vnes facent toufiours les mefmes angles auec YX, & les autres auec YZ. Or, pendant

a. XYZ *Schooten*.
b. E *a été ajouté par Schooten*.

ſi la quantité inconnuë n'a que trois dimenſions; ou bien a telle :

$$z^4 \infty * \cdot apzz \cdot aaqz \cdot a^3r,$$

ſi elle en a quatre; ou bien, en prenant $a$ pour l'vnité,

a telle : $z^3 \infty * \cdot pz \cdot q$
& a telle : $z^4 \infty * \cdot pzz \cdot qz \cdot r$ (*).

| Aprés cela, ſuppoſant que la Parabole FAG eſt deſia deſcrite, & que ſon aiſſieu eſt ACDKL, & que ſon coſté droit eſt $a$ ou 1 (*), dont AC eſt la moitié, & enfin que le point C eſt au dedans de cete Parabole, & que A en eſt le ſommet : il faut faire CD $\infty \frac{1}{2}p$, & la prendre du meſme coſté qu'eſt le point A au regard du point C[a], s'il y a $+p$ en l'Equation; mais, s'il y a $-p$, il faut la prendre de l'autre coſté. Et du point D, ou bien, ſi la quantité $p$ eſtoit nulle, du point C, il faut eſleuer vne ligne a angles droits iuſques a E, en ſorte qu'elle ſoit eſgale a $\frac{1}{2}q$. Et enfin,

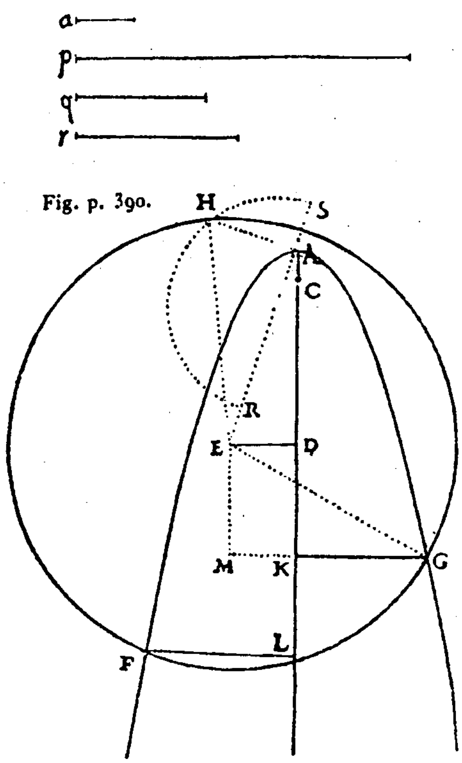



Fig. p. 390.

(*) T. — V.

a. Lire « qu'est le point C au regard du point A ».

du centre E, il faut deſcrire le cercle FG, dont|le demi-diametre ſoit AE, ſi l'Equation n'eſt que cubique, en ſorte que la quantité *r* ſoit nulle. Mais quand il y a $+r$, il faut, dans cete ligne AE prolongée, prendre d'vn coſté AR eſgale a *r*, & de l'autre AS eſgale au coſté droit de la Parabole, qui eſt 1 ; & ayant deſcrit vn cercle dont le diametre ſoit RS, il faut faire AH perpendiculaire ſur AE, laquelle AH rencontre ce cercle RHS au point H, qui eſt celuy par où l'autre cercle FHG doit paſſer. Et quand il y

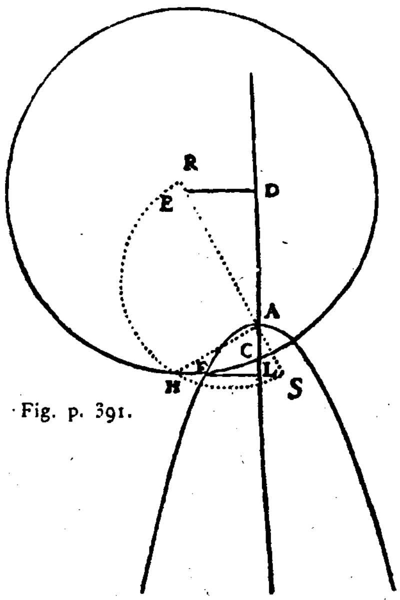
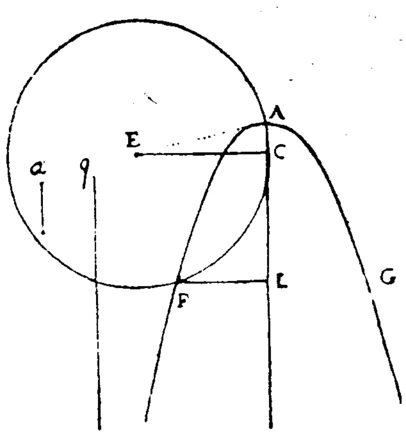
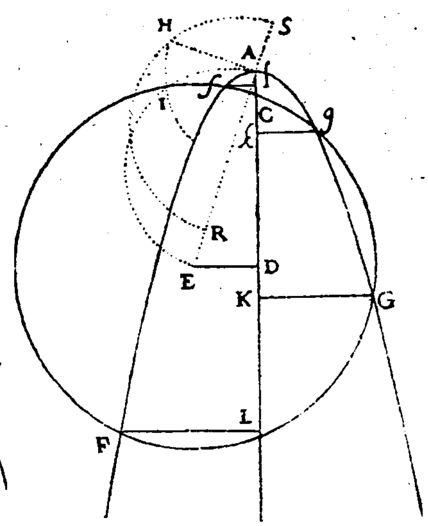



Fig. p. 391.

Fig. p. 392.




a $-r$, il faut, aprés auoir ainſi trouué la ligne AH,

## CONCLUSION

If a geometric construction requires in its analytic form nothing but addition, subtraction, multiplication, division, and the extraction of square roots, then it can be achieved with ruler and compass. These arithmetic operations are to be applied to the two coordinates of each point given by the construction problem. Conversely if it can be achieved with ruler and compass, then when represented analytically all points involved in the construction will have coordinates that can be obtained from those of the points initially given by these five arithmetic operations. The results may be very complicated. If $(a, b)$ and $(c, d)$ are two of the points given, one new coordinate might be

$$\sqrt{a^2 + \frac{ac^2}{\sqrt{b^2 + d^2}}} - \sqrt{7b^2 + c\sqrt{a^2 + \frac{2b^4}{\sqrt{c^2 + d^2}}}}$$

This point reached, we can now concentrate almost entirely on the algebra!

## Complex numbers

We want to analyze in an algebraic manner, thus in terms of Cartesian geometry, the geometric construction of the regular pentagon. For this it is best to go beyond Descartes and to employ *complex* numbers. I begin with a rapid review of their definition.

Recall that the formula for the solution of a quadratic equation

$$ax^2 + bx + c = 0$$

is

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \tag{I}$$

Thus there are two solutions, different save in the exceptional case that $b^2 = 4ac$. If $b^2 - 4ac$ is negative, the square root does not exist in the usual sense and we are, in order to have a complete theory, forced to introduce purely formally the square roots of negative numbers. If $C$ is positive,

$$\sqrt{-C} = \sqrt{-1}\sqrt{C}, \tag{II}$$

so that once we have a square root of $-1$ that we are prepared to multiply with any *real* number we have the square root of any number. If we are prepared to add formally a real number to one of these purely imaginary numbers, in which we permit both the positive square root of $C$ and the negative square root, then we have all the numbers

$$A + \sqrt{-1}B = A + B\sqrt{-1}$$

that appear in (I), so that we can formally solve any quadratic equation. If $b^2 - 4ac < 0$, take

$$A = -\frac{b}{2a}, \quad B = \sqrt{4ac - b^2}$$

## The algebra of complex numbers

We need to be able to perform the usual arithmetic operations on complex numbers. Rather than constantly writing $\sqrt{-1}$, it is the mathematician's habit to write simply $i$. Thus $i$ is a number whose square is $-1$ and the only trick in operating with $i$ or with the square root of $-1$ is to replace $i^2$ with $-1$ whenever it occurs. Thus

$$(a+bi)\times(c+di)=ac+adi+bci+bdi^2=(ac-bd)+(ad+bc)i$$

More generally when adding, subtracting, and multiplying in any order any number of complex numbers the result is always expressed finally as the sum of a real number $a$ and another real number $b$ times $i$.

We add two complex numbers

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

Subtraction is the same.

Division is more difficult. Suppose $A=a+bi$ and $C=c+di$ are two complex numbers, the second of which is not $0=0+0i$ and we want to solve

$$\frac{A}{C}=E\Longleftrightarrow A=CE$$

Write $E$ out explicitly $E=e+fi$. Then

$$(a+bi)=(c+di)\times(e+fi)$$

Multiply both sides by $c-di$. Since

$$(c-di)\times(c+di)=c^2+d^2,$$

we have

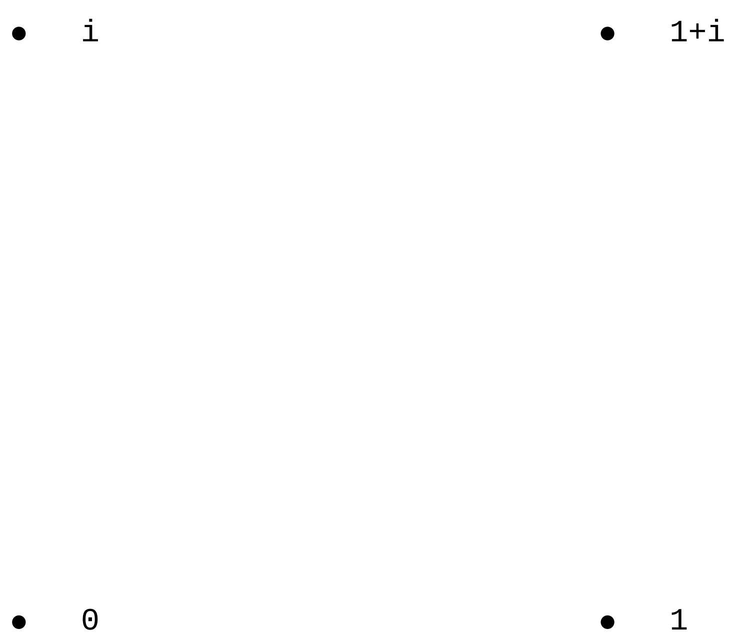
$$(a+bi)\times(c-di)=(c^2+d^2)\times(e+fi)$$

or

$$e+fi=(a+bi)\times\left(\frac{c}{c^2+d^2}-\frac{d}{c^2+d^2}i\right)$$

**Complex numbers and Cartesian geometry**

We usually think of complex numbers as being points in the Cartesian plane. The complex number $a + bi$ being associated, or even identified in our thinking, with the point $(a, b)$



## Division continued

The formula for division can be written as

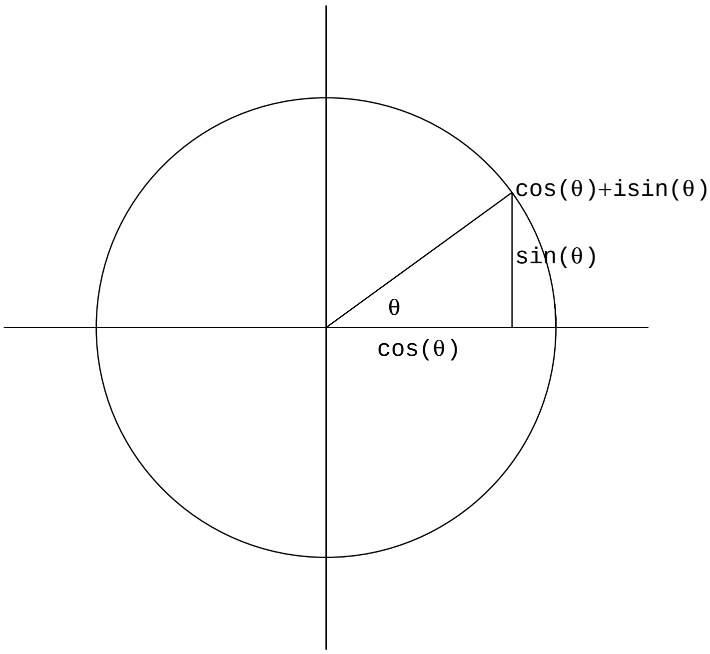
$$\frac{1}{c+di} = \frac{c}{c^2+d^2} - \frac{d}{c^2+d^2}i$$

Observe also in passing that

$$(c+di) \times (c-di) = c^2 + d^2$$

This is a positive number that is 0 only if the complex number is 0.

**Complex numbers on the unit circle and rotation**




Multiply $x + yi$ by $\cos(\theta) + i\sin(\theta)$.

$$(\cos(\theta) + i\sin(\theta)) \times (x + iy) = (\cos(\theta)x - \sin(\theta)y) + i(\sin(\theta)x + \cos(\theta)y)$$

According to an earlier formula the effect is to rotate the point $(x, y)$ through an angle $\theta$. In particular

$$(\cos(\theta) + i\sin(\theta))(\cos(\varphi) + i\sin(\varphi)) = (\cos(\theta + \varphi) + i\sin(\theta + \varphi))$$

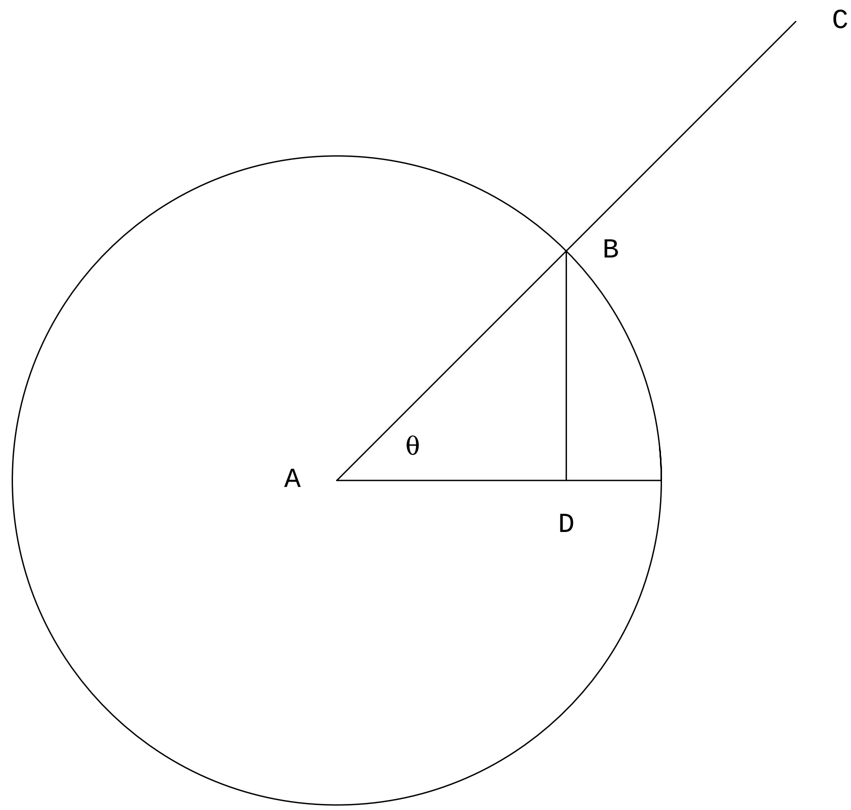
Taking $\varphi = \theta$ we obtain

$$(\cos(\theta) + i\sin(\theta))^2 = \cos(2\theta) + i\sin(2\theta),$$

and

$$(\cos(\theta/2) + i\sin(\theta/2))^2 = \cos(\theta) + i\sin(\theta),$$

## Square roots of complex numbers




$$C = (x, y) \qquad B = (\cos(\theta), \sin(\theta))$$

$$AC = \sqrt{x^2 + y^2} = r \qquad AB = 1$$

$$(x, y) = (r\cos(\theta), r\sin(\theta)), \qquad x + iy = r \times (\cos(\theta) + i\sin(\theta))$$

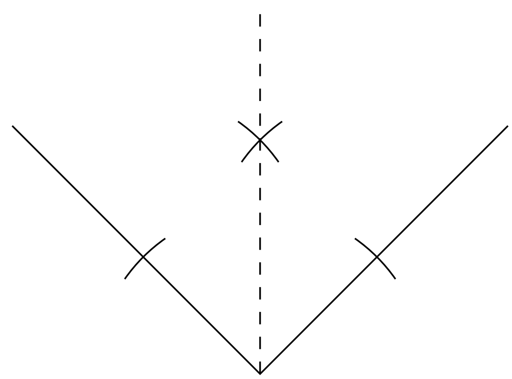
$$\sqrt{x + iy} = \sqrt{r}\sqrt{\cos(\theta) + i\sin(\theta)} = \sqrt{r}(\cos(\theta/2) + i\sin(\theta/2))$$

We know that $\sqrt{r}$ can be found with ruler and compass. So can

$$\sqrt{\cos(\theta) + i\sin(\theta)}$$

because it is simply a matter of bisecting the angle $\theta$.

## Bisection of an angle



## Other roots of complex numbers

## Cube roots

$$
(I) \qquad \begin{aligned}(\cos(\theta)+i\sin(\theta))^3 &= (\cos(\theta)+i\sin(\theta))((\cos(\theta)+i\sin(\theta))^2\\ &= (\cos(\theta)+i\sin(\theta))(\cos(2\theta)+i\sin(2\theta))\\ &= \cos(3\theta)+i\sin(3\theta)\end{aligned}
$$

Thus

$$
(II) \qquad (\cos(\theta/3)+i\sin(\theta/3))^3 = \cos(\theta)+i\sin(\theta)
$$

and

$$
(III) \qquad \sqrt[3]{\cos(\theta)+i\sin(\theta)} = \cos(\theta/3)+i\sin(\theta/3)
$$

We apply equation (I) to $\theta = 0$, $\theta = 2\pi/3$ or $\theta = 4\pi/3$. Then $3\theta$ is 0, $2\pi$ or $4\pi$ so that

$$
\cos(3\theta)+i\sin(3\theta) = 1 + 0 \cdot i = 1.
$$

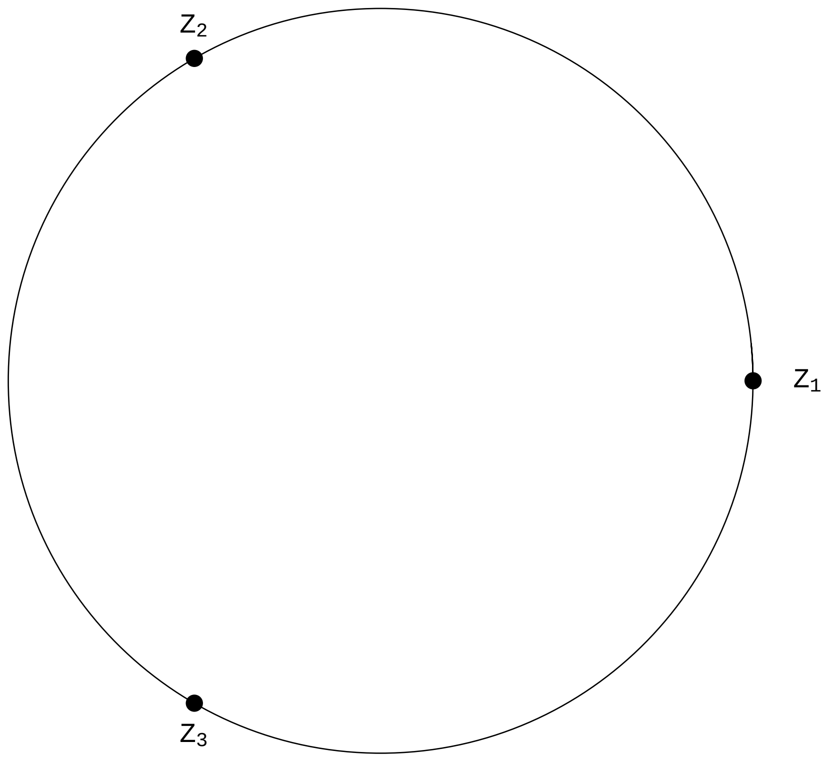
So we have found three square roots of 1. There are no more! The first is

$$
\cos(0)+i\sin(0) = 1,
$$

hardly a surprise. In order to display these three points graphically, we denote them

$$
Z_1 = 1, \quad Z_2 = \cos(2\pi/3)+i\sin(2\pi/3), \quad Z_3 = \cos(4\pi/3)+i\sin(4\pi/3)
$$

**Cube roots continued**




These three complex numbers satisfy the equation

$$z^3 - 1 = 0$$

We divide $z^3 - 1$ by $z - 1$. We use long division.

First step.

$$\begin{array}{lll} z^3 - 1 & & \\ \underline{z^3 - z^2} & & z^2 \times (z-1) \\ z^2 - 1 & & \end{array}$$

Second step.

$$\begin{array}{lll} z^2 - 1 & & \\ \underline{z^2 - z^1} & & z \times (z-1) \\ z^1 - 1 & & \end{array}$$

Third step.

$$\begin{array}{lll} z^1 - 1 & & \\ \underline{z^1 - 1} & & 1 \times (z-1) \\ 0 & & \end{array}$$

Thus the remainder is 0 and

$$\frac{z^3 - 1}{z - 1} = z^2 + z + 1$$

or

$$z^3 - 1 = (z-1)(z^2 + z + 1)$$

Substitute $z_2$. Then

$$0 = (z_2 - 1)(z_2^2 + z_2 + 1) \Longrightarrow z_2^2 + z_2 + 1 = 0$$

We solve this equation.

$$z_2 = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

Thus

$$z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

so that $\cos(2\pi/3) = 1/2$, $\sin(2\pi/3) = \sqrt{3}/2$. For similar reasons $z_3$ is the other root.

$$z_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

## One more example of long division

To divide $2z^3 + 3z^2 + 4z + 1$ by $z^2 + 3z + 2$

First step:

$$\begin{array}{ll} 2z^3 + 3z^2 + 4z + 1 & \\ \underline{2z^3 + 6z^2 + 4z} & \qquad 2z \times (z^2 + 3z + 2) \\ \quad -3z^2 + 1 & \end{array}$$

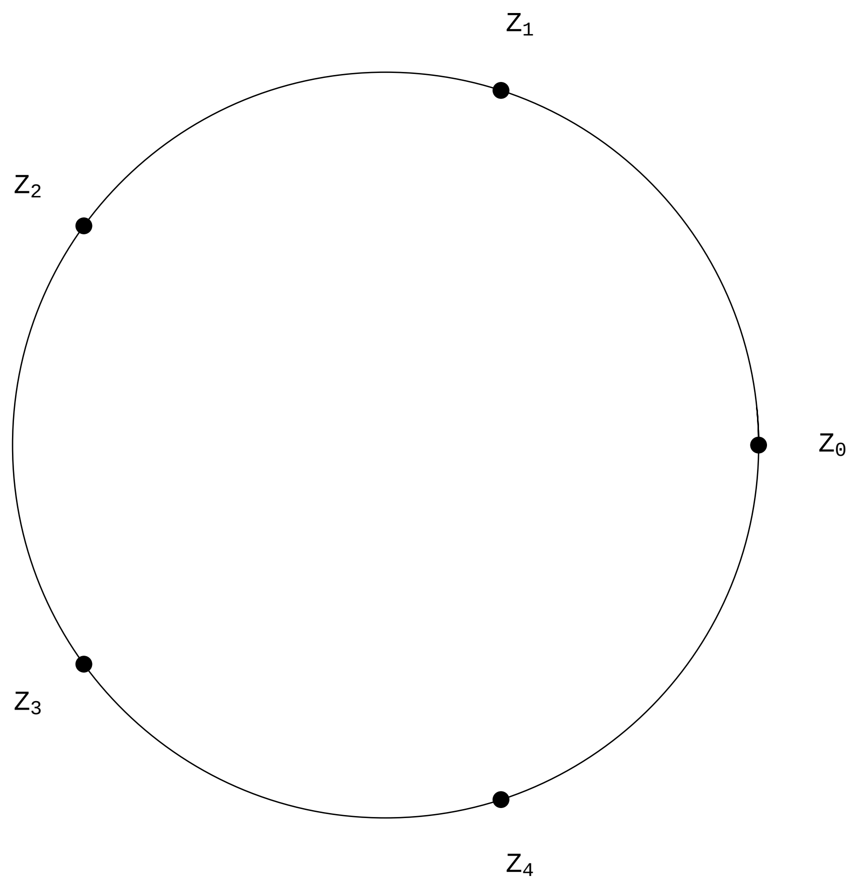
Second step:

$$\begin{array}{ll} -3z^2 + 1 & \\ \underline{-3z^2 - 9z - 6} & \qquad -3 \times (z^2 + 3z + 2) \\ \qquad 9z + 7 & \end{array}$$

We have divided by a polynomial of degree two and the remainder has smaller degree, namely one. The result is:

$$2z^3 + 3z^2 + 4z + 1 = (2z - 3) \times (z^2 + 3z + 2) + 9z + 7$$

## Fifth roots of unity




After the discussion of the cube roots of unity, it should come as no surprise that the the fifth roots of unity are the numbers

$$\cos(2k\pi/5) + i\sin(2k\pi/5), \qquad k = 0, 1, 2, 3, 4$$

They form the vertices of a regular pentagon. Thus if we can show that they can be obtained by repeatedly extracting square roots, we will have an algebraic proof of the possibility of constructing the regular pentagon with ruler and compass.