LECTURE SERIES at the INSTITUTE FOR ADVANCED STUDY

Robert P. Langlands

The Practice of Mathematics

Ars longa, vita brevis

There are several central mathematical problems, or complexes of problems, that every mathematician who is eager to acquire some broad competence in the subject would like to understand, even if he has no ambition to attack them all. That would be out of the question! Those with the most intellectual and aesthetic appeal to me are in number theory, classical applied mathematics and mathematical physics. In spite of forty years as a mathematician, I have difficulty describing these problems, even to myself, in a simple, cogent and concise manner that makes it clear what is wanted and why. As a possible, but only partial, remedy I thought I might undertake to explain them to a lay audience.

I shall try for a light touch including, in particular, some historical background. Nevertheless the lectures are to be about mathematics. In the first set, there will be geometrical constructions, simple algebraic equations, prime numbers, and perhaps an occasional integral. Every attempt will be made to explain the necessary notions clearly and simply, taking very little for granted except the good will of the audience.

Starting in the easiest place for me, I shall give, during the academic year 1999/2000 about eight lectures on pure mathematics and number theory with the motto *beautiful lofty things*. Beginning with the Pythagorean theorem and the geometric construction of the Pythagorean pentagram, I shall discuss the algebraic analysis of geometric constructions and especially the proof by Gauss in 1796 of the possibility of constructing with ruler and compass the regular heptadecagon. This was a very great intellectual achievement of modern mathematics that can, I believe, be understood by anyone without a great aversion to high-school algebra. Then I will pass on to Galois's notions of mathematical structure, Kummer's ideal numbers, and perhaps even the relations between ideal numbers and the zeta-function of Riemann. This material will be a little more difficult, but I see no reason that it cannot be communicated. It brings us to the very threshold of current research.

Since this attempt is an experiment, the structure and nature of the lectures will depend on the response of the audience and on my success in revealing the fabric of mathematics. If it works out, I would like to continue in following years on classical fluid mechanics and turbulence, with motto *l'eau mêlée à la lumière*, and then, with the somewhat trite motto с того берега, on the analytical problems suggested by renormalization in statistical mechanics and quantum field theory.

West Building Lecture Hall, Institute for Advanced Study

Tuesdays at 4:30 pm

October 26, November 2, 9, 16 and February 1, 8, 15, 22

On the use of these notes.

The texts and diagrams collected below can be read independently or used as supplements to the video-tapes of the lectures. They include not only copies of the transparencies used during the course of the lectures, but also copies of various texts cited, together with translations, as well as the lecturer's own preparatory notes.

There was no clean division of the lectures into eight, or if the whole year is taken into account sixteen, hours, so that it would be futile to divide these notes into separate sections. The lectures overlapped, the end of one running into the beginning of the next, and transparencies were sometimes displayed more than once. Nonetheless, for the convenience of the viewer, for each lecture a precise location is indicated at which he can conveniently begin to read or to peruse the material pertinent to it.

The lyf so short, the craft so long to lerne,
Th'assay so hard, so sharp the conquering.

The decision to deliver these lectures arose from two sources. First of all, the Faculty of the Institute is being encouraged by our employers to leave, at least for brief moments, the confines of our disciplines and to present ourselves to the public at large. This is, in my view, an excellent idea, but the format chosen, one-hour public lectures, is not congenial to all of us, certainly not to me. It is never clear what to present. In particular, my own elucubrations lead at best infrequently to something that is worth communicating even to specialists, and even then they are not easily persuaded, if at all. So I hesitate to impose any discovery of mine on an innocent public.

On the other hand, I, like many other mathematicians, have spent, even perhaps wasted, much time on a large variety of problems on which I could make no inroads, so that my efforts have left no trace. Now, it is seldom the habit of mathematicians when they attack a problem to study systematically its history or the literature surrounding it, at least this has not been my habit. The youthful impulse is rather to prowl about the problem for a while, looking perhaps not for an open window or a door with a weak lock, for if these were available, some earlier malefactor would already have discovered them, but for some wall that can be scaled or some unsuspected underground access. The upshot is that if nothing is discovered, one comes away from the effort empty-handed, having learned little, except what to avoid.

With waxing age and waning energies a different impulse manifests itself, not the desire to overwhelm this or that outstanding problem by force or cunning but rather the desire to understand its sources and to formulate clearly its meaning and significance. Nevertheless, without some external compulsion or, at least, encouragement, this impulse would in all likelihood come to naught for what a mathematician, even an elderly one, really wants to do is discover new theories, new techniques and new methods and to solve the old problems. In an attempt to exploit to my own profit my new obligation to come out of the closet, and to kill two birds with one stone, I announced these lectures, from which I hope both you and I will be able to learn some genuine mathematics. I have already learnt a good deal that I did not know before. In preparing the lectures I have kept in mind especially those among my colleagues in the humanities who have frequently assured me of their desire to acquire some understanding of the subject. I am not sure how sincere they are, but I count on them, and on all of you, to let me know if my instructional efforts are failing and not simply by failing to mention.

Indeed, there are many reefs on which this undertaking can run aground. My inexperience with much of the material may turn out to be an advantage, but my pedagogical inadequacies are a handicap. Moreover, although these lectures are aimed at an audience whose experience with mathematics may have ended with high-school the bulk of you are undoubtedly professional mathematicians whose expectations may or may not be deceived. A number of the mathematicians, like me, will have been educated in inadequate North American schools and may, therefore, have had little experience with classical introductory mathematics, so the beginnings may amuse them. Even so, the moral pressure on me to move at too fast a pace will be great, and I am not entirely confident of my courage to resist it. Questions and observations that slow me down will be much appreciated.

Although the plans as announced are on the whole are rather grandiose, I still have only the vaguest ideas what I will do in the series on fluid mechanics or on statistical mechanics and renormalization promised for subsequent years. I thought it would be best to let the future take care of itself. I am already uneasy enough about the second term's set of lectures. For the first term, the plans are fairly clear, although the timing is uncertain. Four weeks may not be enough and I may have to run over.

FALL TERM

(I) Introductory geometrical material – the Pythagorean theorem.

(II) Geometrical construction of regular pentagon.

(III) Analytic geometry and complex numbers.

(IV) Gauss's construction of regular heptadecagon.

TWO MAJOR WORKS

Euclid's Elements and Gauss's Arithmetical Investigations

Comment on plan

It may be useful to explain briefly the structure of this set of lectures. The purpose of the Pythagorean theorem in the context of the construction of the regular pentagon is to construct a certain quadratic irrational or surd, a notion to be explained. The first impression of some may be that this is simply a rehash of high-school geometry, but quadratic irrationals remain of major interest even today and the nature of other irrationals (icosahedral for example) is a central problem. They were, as will be observed in passing, the source of a crisis in Greek mathematics that lasted more than a century and that was only resolved by a new understanding of the notion of number. If I were better informed I would spend more time discussing this crisis and its resolution.

The regular pentagon has, as I shall recall, an evident five-fold geometric symmetry. It also has, although this is far from evident, a four-fold symmetry that is the clue to its geometric construction, but of which the Greeks were unaware. This symmetry is revealed by an algebraic analysis that can only be carried out with the help of complex numbers, so that some time has to be spent introducing them to you and explaining their role in analytic geometry. Complex numbers are second nature to those with any mathematical training, but not to others. Since it is the others to whom these lectures are addressed, I shall spend the necessary time on them.

The hidden four-fold symmetry of the pentagon understood, we shall be in a position to understand the hidden sixteen-fold symmetry of the regular seventeen-sided polygon that permits it to be constructed geometrically. In contrast the heptagon, with seven sides and a hidden six-fold symmetry cannot be constructed geometrically, that is – to be more explicit about what is here intended by the adverb *geometrically* – with the aid of nothing but a ruler and a compass. The difference between 5, 7 and 17 is that

$$4 = 2 \cdot 2,\ 16 = 2 \cdot 2 \cdot 2 \cdot 2 \text{ but } 6 = 2 \cdot 3.$$

We want to understand why this difference in the factorization of the three numbers has such a striking geometric consequence.

That it did was discovered by Gauss as a lad of 18 in 1796. This is often presented as a curious juvenile achievement of little import in comparison with his other early achievements, accomplished when he was scarcely older. It would be better if I were able to say, when the time comes, more about Gauss. I fear that in the past I instinctively avoided acknowledging what it meant to be a real mathematician. So I cannot now give you any information beyond the familiar. He was extremely precocious, extremely powerful and inventive, with an apparently innate mathematical curiosity that I now appreciate is rare. Mathematical talent is perhaps more common than mathematical curiosity. Although born to parents of little or no means, his gifts were noticed early, and he was educated at the expense of the Duke of Brunswick, presumably with the expectation that he would become a functionary of some sort. Although the talents encouraged were apparently linguistic not mathematical he found himself at schools with good mathematics libraries and seems to have made himself familiar with some of the important mathematical research of the eighteenth century, including that of Euler and Lagrange, not only mastering it but also deepening it.

The construction of the regular heptadecagon could appear, if so presented, as a spontaneous contribution of an adolescent, but seems rather to have been rendered possible by Gauss's facility with the complex numbers developed in the course of the century and by his familiarity with Lagrange's attempts to analyze the solution of equations by radicals. What may have been spontaneous in his achievement was the return after more than two millenia to an ancient geometric problem, the construction of regular polygons, that had been abandoned because of the difficulties appearing for heptagons. But I do not know. Here, as elsewhere in the preparation of these lectures, I am brought face to face with my ignorance.

The plan and its structure are now clear, as is one glaring defect. One hour for the material in each of the four sections is not enough. An hour is not enough to make the material comprehensible and it is not enough to make the presentation fun. Once started, for example, on the construction of the pentagon, I found the geometry irresistible.

My feeling for the Greeks as mathematicians is every bit as inadequate as that for the youthful Gauss. I do not know whence came their curiosity and depth. Perhaps no-one does. We live in a highly structured environment dedicated to research. We earn our living by it and we pin our hopes of recognition on it, but the questions we ask and the problems we solve are determined more by tradition, more by our colleagues than by our own natural and spontaneous curiosity. We are seldom playful; our efforts are never simply for our own amusement. A brief romp with Greek mathematics in which we examine the construction of the pentagon at length may be an occasion to capture briefly the ludible spirit of the Greeks.

An hour is also not enough for an adequate understanding of analytic geometric and complex numbers nor for a presentation of the algebra required for Gauss's construction. The complex numbers are an enormously effective tool that swallows the geometry, but it will be good to ask ourselves how. Moreover the four-fold or sixteen-fold algebraic symmetry is far more subtle than the five-fold or seventeen-fold geometric symmetry. Since it will reappear again and in spades when, and if, we discuss Galois and Kummer, it is best to get used to it now.

The upshot is that four hours is scarcely enough. My plan is, therefore, simply to go on, probably for another four weeks, so that I will not have finished this first set until sometime in December. All being well, this will leave me some listeners and enough time to prepare for the second set in February.

$$5 - 1 = 4 = 2 \times 2$$

$$17 - 1 = 16 = 2 \times 2 \times 2 \times 2$$

BUT

$$7 - 1 = 6 = 2 \times 3$$

Rainer Maria Rilke
Der Schauende

Ich sehe den Bäumen die Stürme an,
die aus laugewordenen Tagen
an meine ängstlichen Fenster schlagen,
und höre die Fernen Dinge sagen,
die ich nicht ohne Freund ertragen,
nicht ohne Schwester lieben kann.

Dageht der Sturm, ein Umgestalter,
geht durch den Wald und durch die Zeit,
und alles ist wie ohne Alter:
die Landschaft, wie ein Vers im Psalter,
ist Ernst und Wucht und Ewigkeit.

Wie ist das klein, womit wir ringen,
was mit uns ringt, wie ist das groß;
Ließen wir, ähnlicher den Dingen,
uns so vom großen Sturm bezwingen –
wir würden weit und namenlos.

Was wir besiegen, ist das Kleine,
und der Erfolg selbst macht uns klein.
Das Ewige und Ungemeine
will nicht von uns gebogen sein.
Das ist der Engel, der den Ringern
des Alten Testaments erschien;
wenn seiner Wildersacher Sehnen
im Kampfe sich metallen dehnen,
fühlt er sie unter seinen Fingern
wie Saiten tiefer Melodien.

Wen dieser Engel überwand,
welcher so oft auf Kampf verzichtet,
der geht gerecht und aufgerichtet
und groß aus jener harten Hand,
die sich, wie formend, an ihn schmiegte.
Die Siege laden ihn nicht ein.
Sein Wachstum ist: Der Tiefbesiegte
von immer Größerem zu sein.

СОЗЕРЦАНИЕ

Деревья складками коры
Мне говорят об ураганах,
И я их сообщений странных
Не в силах слышать средь нежданных
Невзгод, в скитаньях постоянных,
Один, без друга и сестры.
Сквозь рощу рвется непогода,
Сквозь изгороди и дома.
И вновь без возраста природа,
И дни, и вещи обихода,
И даль пространств — как стих псалма.
Как мелки с жизнью наши споры,
Как крупно то, что против нас!
Когда б мы поддались напору
Стихии, ищущей простора,
Мы выросли бы во сто раз.
Все, что мы побеждаем, — малость,
Нас унижает наш успех.
Необычайность, небывалость
Зовет борцов совсем не тех.
Так ангел Ветхого завета
Нашел соперника под стать.
Как арфу, он сжимал атлета,
Которого любая жила
Струною ангелу служила,
Чтоб схваткой гимн на нем сыграть.
Кого тот ангел победил,
Тот правым, не гордясь собою,
Выходит из такого боя
В сознаньи и расцвете сил.
Не станет он искать побед.
Он ждет, чтоб высшее начало
Его все чаще побеждало,
Чтобы расти ему в ответ.

Habe Furcht vor dem großen Sturm,
und gib acht auf den kleinen Wind.

Transitional remarks

As a start, I return just for a minute to the beginnings of my unsuccessful effort over more than four decades to master a difficult trade. Through a kind of fluke I found myself at university at quite an early age – not quite seventeen – but with no preparation, whereupon on an impulse that I think I now understand I decided, with no notion whatsoever of mathematics or physics and no notion whatsoever of academic life, that I wanted to be a mathematician or perhaps a physicist. Although physics turned out finally to be too difficult for me, I did come across not long after this decision a copy of the Einstein volume in Schilpp's series of hefty tomes *The Library of Living Philosophers*. It was in the windows of a stationer's shop in the town near to the hamlet in which I grew up. (For those who are familiar with the local topography, it was on the corner of Columbia Ave. and 6th St. in New Westminster.)

I still have it although I never made much of it, but there was a brief intellectual biography by Einstein in which he describes two memorable intellectual experiences of his childhood: his introduction to a compass at the age of five and his discovery of Euclid at the age of twelve. Although an uncle appears to have introduced him to euclidean geometry and the pythagorean theorem even earlier, at that age he was presented with what was then a widely used text *Lehrbuch der Geometrie zum Gebrauch an höheren Lehranstalten* that thanks to our librarian, Momota Ganguli, I was able to have a look at. Although obscure in places, it appears a book rich in content that well deserved its success. Einstein cites explicitly the theorem that especially impressed him, oddly enough a theorem that is not to be found in the Elements, although it was presumably known at the time and to Euclid. It is an elegant fact with an elegant proof and quite simple, so that, as a warm-up, I begin with it.

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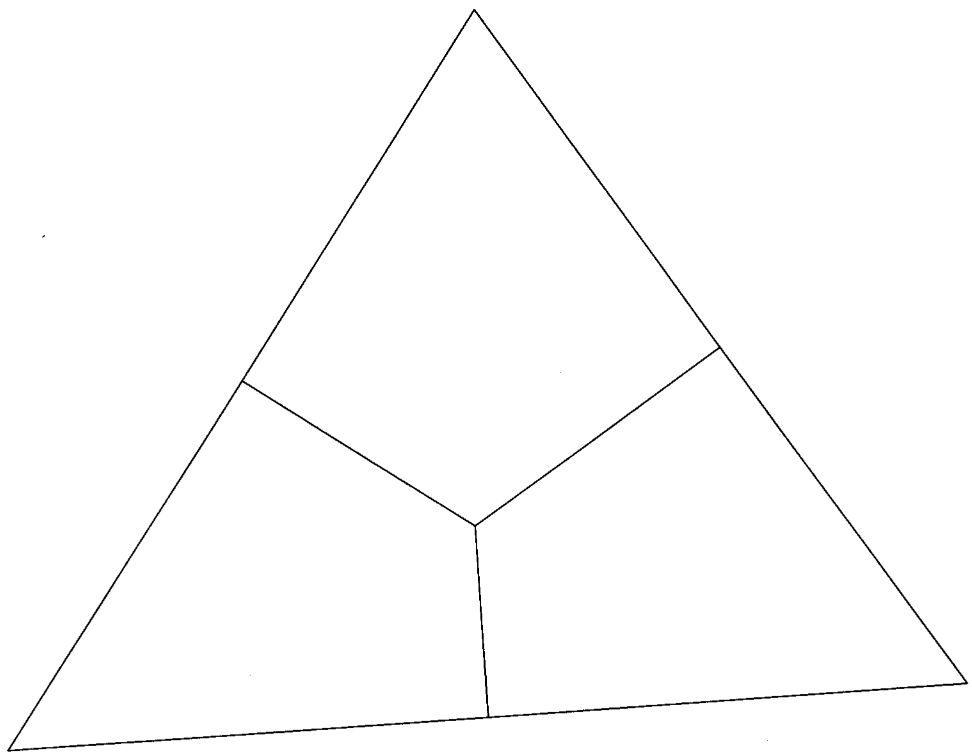
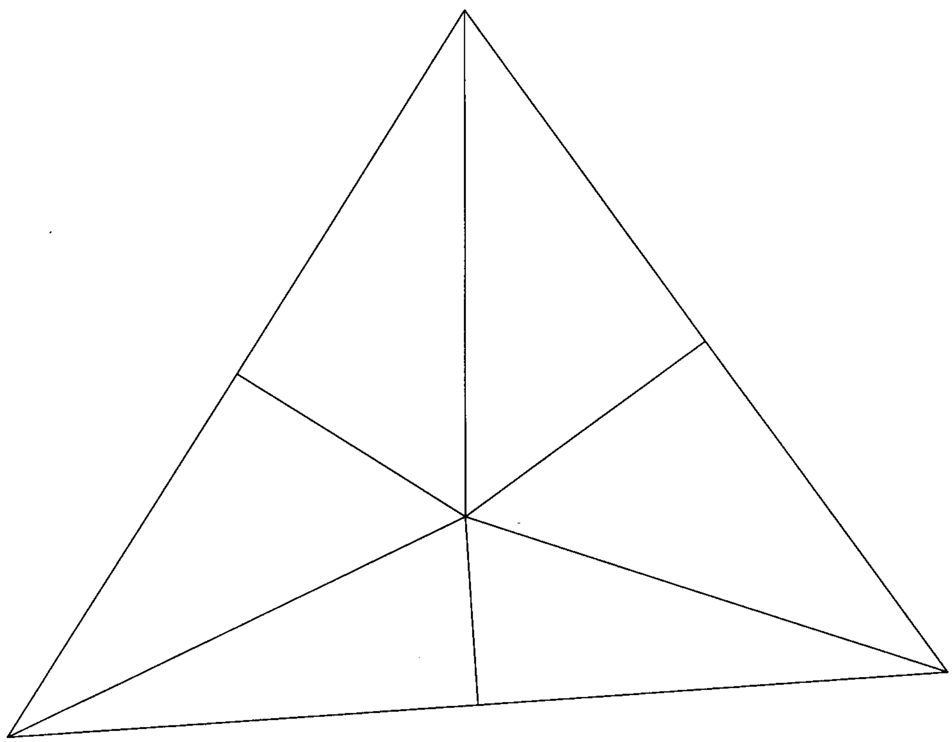
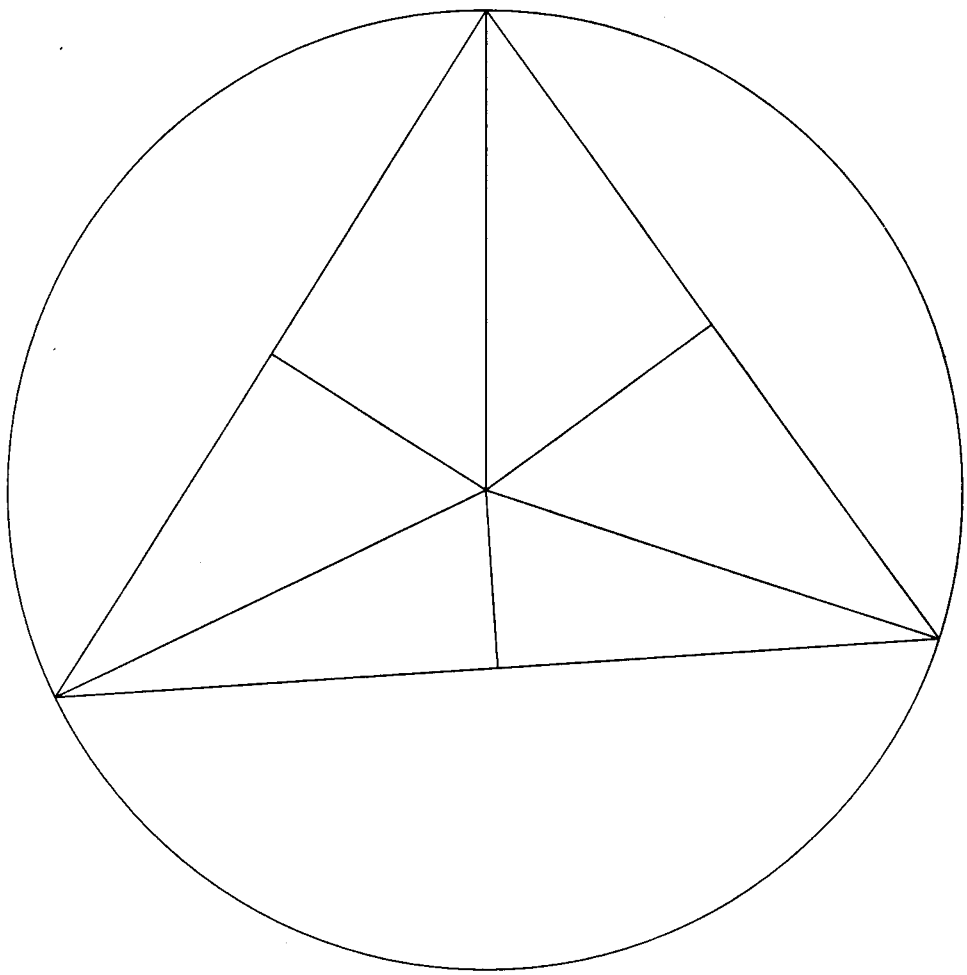
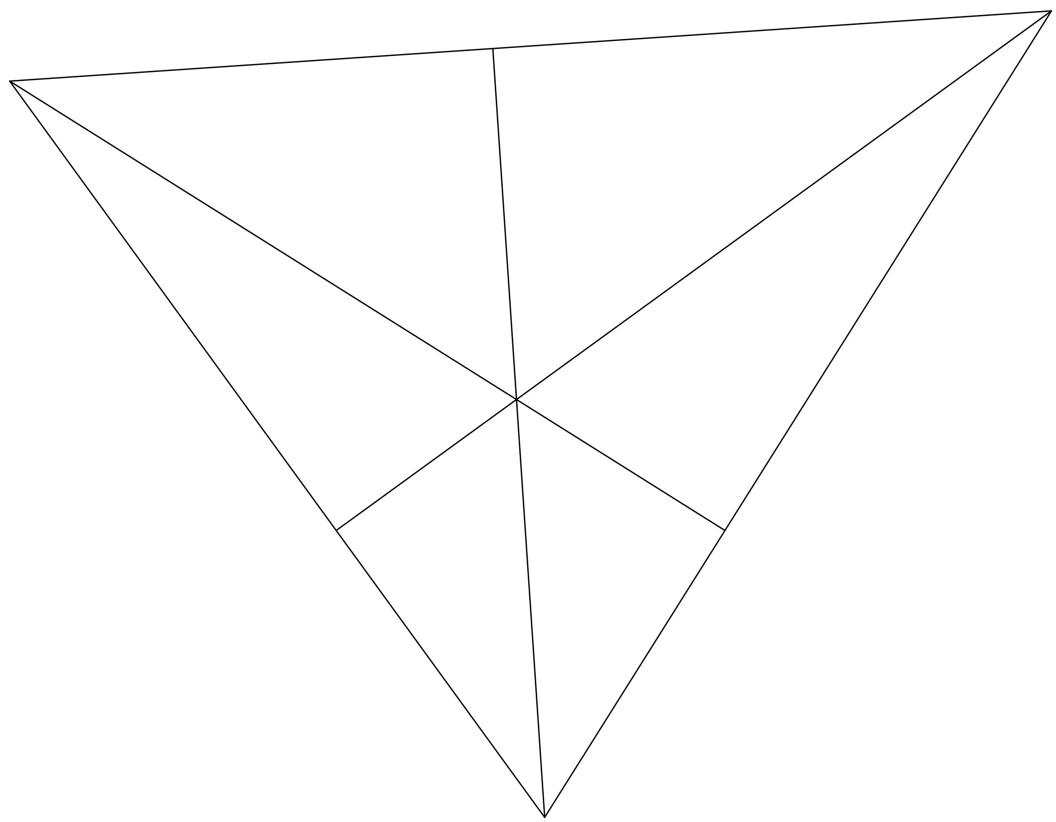
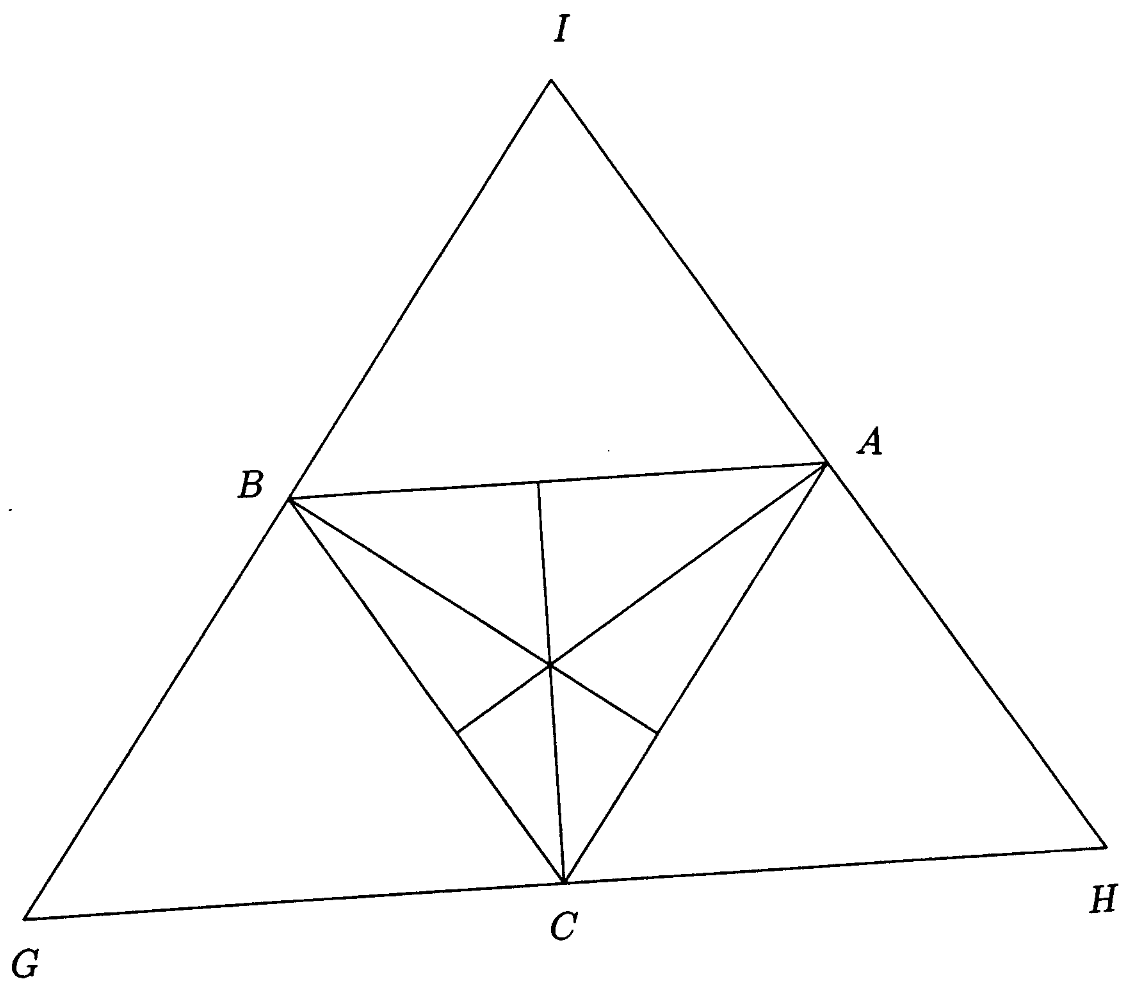
Anyhow, without any background in mathematics myself, and not knowing where to begin, I followed Einstein's implicit recommendation and acquired a copy of the Element's (Todhunter's edition in the Everyman collection, a very popular, very cheap collection of my youth and still, in my view, very useful). I did not make much headway with it. There were several reasons, not least, but not only, my uncertain intellectual taste. I already knew about analytic geometry and it was clear that most, perhaps all, the geometrical propositions of Euclid, which were the ones I was looking at, could be proved much more quickly algebraically. Since I had a lot of time to make up, I very quickly abandoned the geometry as such. In retrospect it is clear that the Elements is a much more complex and sophisticated text, both scientifically and historically, than I was able to deal with at the time.











$$\left.\begin{array}{l} BC = AH \\ BC = AI \end{array}\right\} \Longrightarrow AH = AI$$

HEATH'S INTRODUCTION TO THE EVERYMAN EDITION

It was with the greatest interest that I heard that the publishers thought of adding to Everyman's Library an edition of the *Elements* of Euclid; for what book in the world could be more suitable for inclusion in the Library than this, the greatest textbook of elementary mathematics that there has ever been or is likely to be, a book which, ever since it was written twenty-two centuries ago, has been read and appealed to as authoritative by mathematicians great and small, from Archimedes and Apollonius of Perga onwards? No textbook, presumably, can ever be without flaw (especially in a subject like geometry, where some first principles, postulates or axioms, have to be assumed without proof, and any number of alternative systems are possible), and flaws there are in Euclid; but it is safe to say that no alternative to the *Elements* has yet been produced which is open to fewer or less serious objections. The only general criticism of it which is deserving of consideration is that it is unsuitable as a textbook for very young boys and girls who are just beginning to learn the first things about geometry. This can be admitted without detracting in the least from the greatness or the permanent value of the book. The simple truth is that it was not written for schoolboys or schoolgirls, but for the grown man who would have the necessary knowledge and judgment to appreciate the highly contentious matters which have to be grappled with in any attempt to set out the essentials of Euclidean geometry as a strictly logical system, and, in particular, the difficulty of making the best selection of unproved postulates or axioms to form the foundation of the subject. My advice would, therefore, be: if you must spoon-feed the very young, do so; but when they have shown a taste for the subject and attained the standard necessary for passing honours examinations, let them *then* be introduced to Euclid in his original form as an antidote to the more or less feeble echoes of him that are to be found in the ordinary school textbooks of 'geometry.' I should be surprised if such qualified readers, making the acquaintance of Euclid for the first time, did not find it fascinating, a book to be read in bed or on a holiday, a book as difficult as any detective story to lay down when once begun. I know of one actual case, that of an undergraduate at Cambridge suddenly presented with a copy of Euclid, where this happened. This is the true test of such a book. Nor does the reading of it require the 'higher mathematics.' Any intelligent person with a fair recollection of school work in elementary geometry would find it (progressing as it does by gradual and nicely contrived steps) easy reading, and should feel a real thrill in following its development, always assuming that enjoyment of the book is not marred by any prospect of having to pass an examination in it! This is why I applaud the addition of this great classic to Everyman's Library; for everybody ought to read it who *can*, that is, all educated persons except the very few who are constitutionally incapable of mathematics.

Euclid and Harish-Chandra

In the early sixties, the mathematician Harish-Chandra joined the Faculty of the Institute, At that time his papers were much admired but little read. They were regarded as too difficult. Their purpose was to be correct, they took no pity on the reader, but moved forward relentlessly, lemma after lemma, theorem after theorem, paper after paper with little attempt to distinguish the real obstacles from the more straighforward development. Euclid, to whom I now come long after reading much of Harish-Chandra, has a similar style, so that I am struck with admiration for any school-boy of an earlier era who drew his intellectual nourishment directly from Euclid. I recall that Einstein did not.

Pythagoras 530 BC

Plato 380 BC

Eudoxus 360 BC

Aristotle 340 BC

Euclid 290 BC

It will be clear both from the announcement and the exculpatory remarks that my intention is to clarify both for myself and for the audience some problems of contemporary mathematics, and that any historical preliminaries are principally for the sake of clarifying the mathematical issues, not the historical issues. I want therefore to stress once again that when it comes to mathematics in antiquity or even during the Renaissance, I am almost immediately in over my head. In particular, since I make a few remarks concerning Pythagoras and the pythagorean theorem that may belong more to the realm of myth than of history, it may be best as a warning to the audience to cite some phrases from the book of Otto Neugebauer, an historian of science who, I recall for the younger members of the audience, spent the last years of his life attached to the Institute.

Some quotations from

O. Neugebauer's

The exact sciences in antiquity

1. The above example of the determination of the diagonal of the square from its side is sufficient proof that the "Pythagorean" theorem was known more than a thousand years before Pythagoras.

2. It seems to me evident, however, that the traditional stories of discoveries made by Thales or Pythagoras must be discarded as totally unhistorical.

3. We know today that all the factual mathematical knowledge which is ascribed to the early Greek philosophers was known many centuries before, though without the accompanying evidence of any formal method which the mathematicians of the fourth century would have called a proof. For us, there is nothing to do but to admit that we have no idea of the role which the traditional heroes of Greek science played.

4. The Greeks themselves had many theories about the origins of mathematics. ... A much more sophisticated attitude is represented by Aristotle, who considers the existence of a "leisure class", to use a modern term, a necessary condition for scientific work. Our factual knowledge about the development of scientific thought and of the social position of the men who were responsible for it is so utterly fragmentary, however, that it seems to me completely impossible to test any such hypothesis, however plausible it may appear to a modern man.

The exact sciences in antiquity

Extract

This is confirmed by a small tablet, now in the Yale Babylonian Collection. On it is drawn a square with its two diagonals. The side shows the number 30, the diagonal the numbers $1, 24, 51, 10$ and $42, 25, 35$. The meaning of these numbers becomes clear if we multiply $1, 24, 51, 10$ by 30, an operation which can easily be performed by dividing $1, 24, 51, 10$ by 2 because 2 and 30 are reciprocals of each other. The result is $42, 25, 35$. Thus we have obtained from $a = 30$ the diagonal $42; 25, 35$ by using

$$\sqrt{2} = 1; 24, 51, 10.$$

The accuracy of this approximation can be checked by squaring $1; 24, 51, 10$. One finds

$$1; 59, 59, 59, 38, 1, 40$$

corresponding to an error of less than $22/60^4$.

Comments

$$1; 24, 51, 10 \div 2 = 0; 30 + 0; 12 + 0; 0, 25 + 0; 0, 0, 30 + 0; 0, 0, 5 = 0; 42, 25, 35$$

$$1 + \tfrac{24}{60} + \tfrac{51}{60^2} + \tfrac{10}{60^3} = 1.41421296\ldots$$

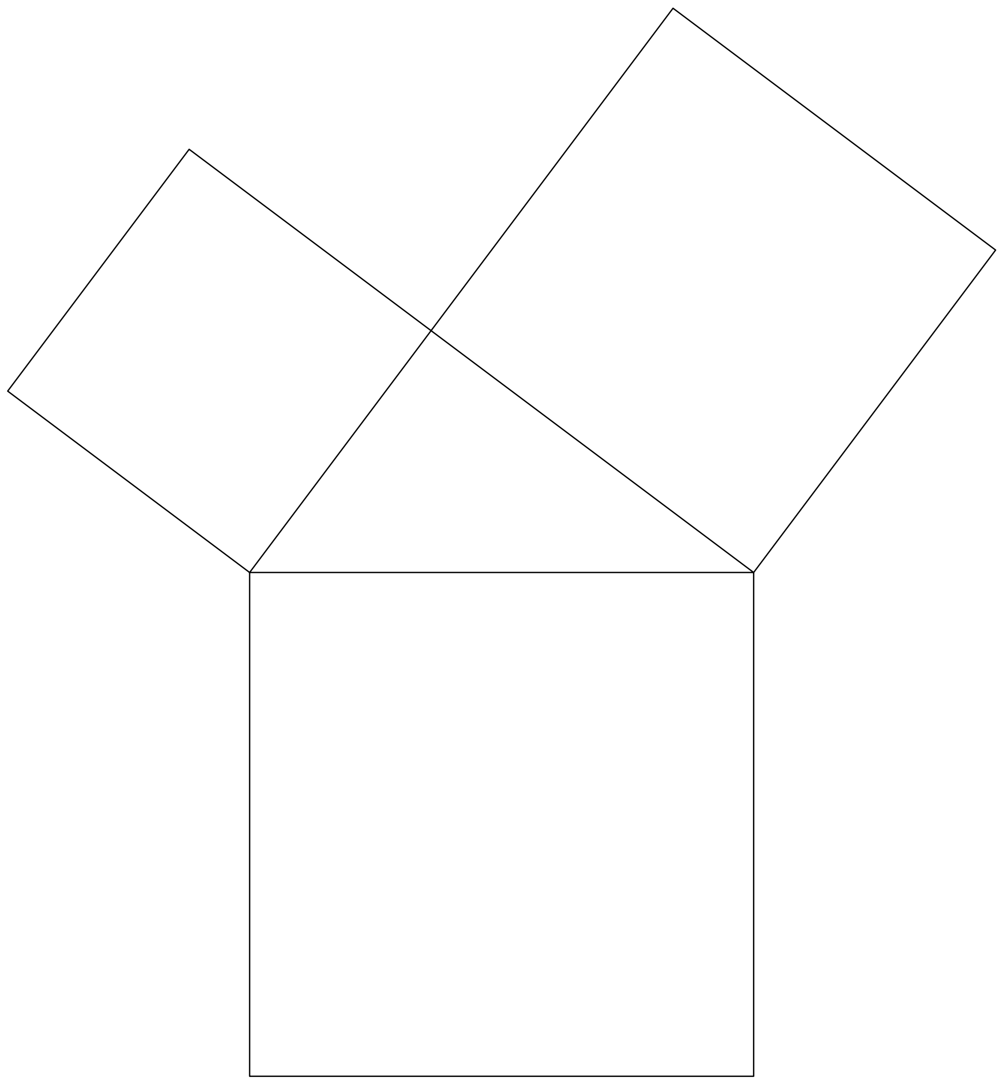
$$\begin{gathered} 2 \times 24 = 48; \quad 2 \times 51 = 1, 42; \quad 2 \times 10 = 20; \\ 24 \times 24 = 9, 36; \quad 51 \times 51 = 43, 21; \quad 10 \times 10 = 1, 40; \\ 24 \times 51 = 20, 24; \quad 24 \times 10 = 4, 0; \quad 51 \times 10 = 8, 30; \\ \text{Doubled:} \quad 40, 48; \quad 8, 0; \quad 17, 0; \end{gathered}$$

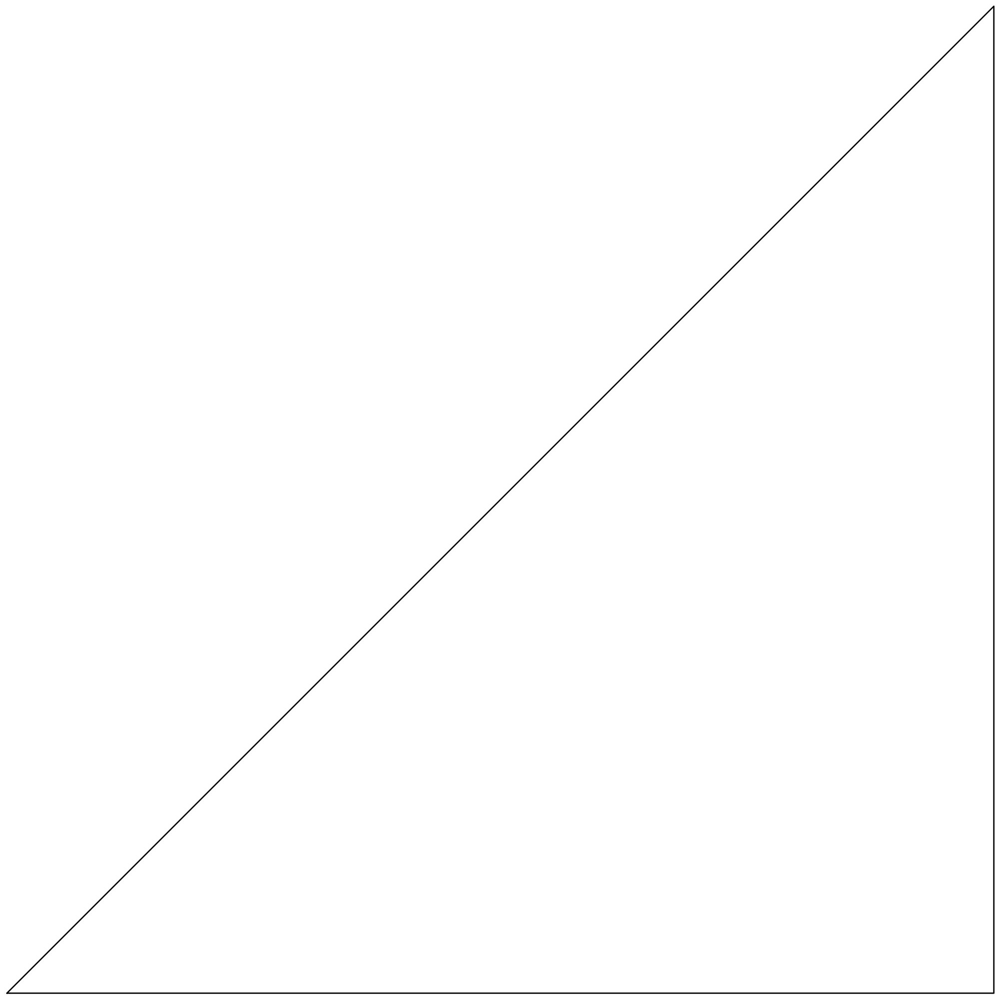
Thus $1; 24, 51, 10$ squared is

$$1; +; 9, 36+; 0, 0, 43, 21+; 0, 0, 0, 0, 1, 40+; 48+; 1, 42+; 0, 0, 20+; 0, 40, 48+; 0, 0, 8+; 0, 0, 0, 17$$

or

$$1; 59, 59, 59, 38, 1, 40$$



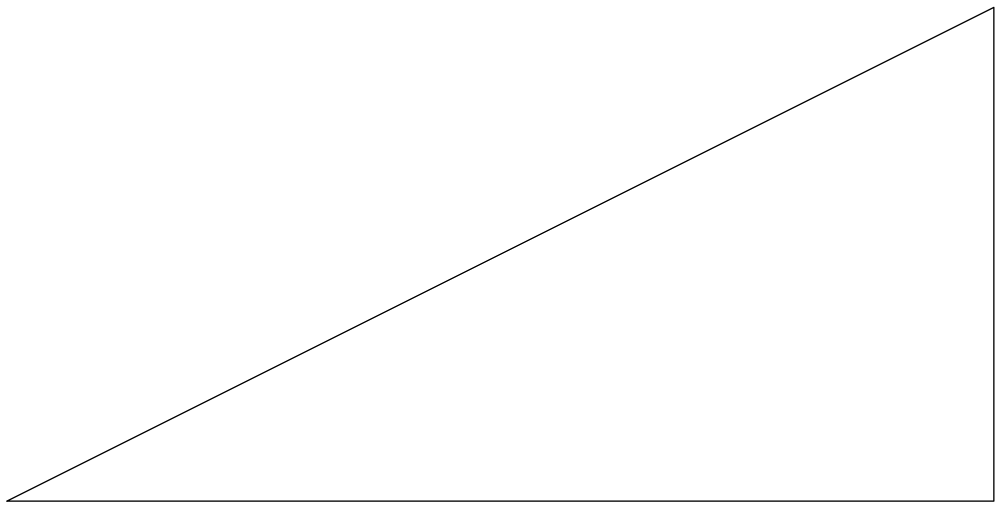


$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2$$

$$a = 2c, \quad 2b^2 = 4c^2, \quad b^2 = 2c^2$$

Thus a and b both even.



$$5 = \frac{a^2}{b^2}$$

$$5b^2 = a^2$$

$$a = 5c, \quad 5b^2 = 25c^2, \quad b^2 = 5c^2$$

Thus a and b both divisible by 5.

Aristotle: Analytica Priora

For all who effect an argument *per impossibile* infer syllogistically when something impossible results from the assumption of its contradictory; e.g. that the diagonal of the square is incommensurate with the side, because odd numbers are equal to evens if it is supposed to be commensurate. One infers syllogistically that odd numbers come out equal to evens, and one proves hypothetically the incommensurability of the diagonal, since a falsehood results through contradicting this.

Plato: Theaetetus

Socrates. But I am more interested in our own Athenian youth, and I would rather know who among them are likely to do well. Tell me then, if you have met with any who is at all remarkable.
Theororus. Yes, Socrates, I have become acquainted with one very remarkable Athenian youth, whom I commend to you as well worthy of your attention. If he had been a beauty I should have been afraid to praise him, ...
Socrates. Herein lies the difficulty which I can never solve to my satisfaction – What is knowledge?
Theodorus. I would rather that you would ask one of the young fellows.
Theatetus. Theodorus was writing out for us something about roots, such as the sides of squares three or five feet in area showing that they are incommensurable by the unit: he took the other examples up to seventeen, but there for some reason he stopped. Now as there are innumerable such roots, the notion occurred to us of attempting to find some common description which can be applied to them all.

The comments at the end of the first talk that were not about Rilke were about square roots, and in particular about the proof that $\sqrt{2}$ and $\sqrt{5}$ are irrational. So I want to make a couple of additional remarks.

I give first of all, two standard quotations, one from Aristotle and one from Plato. The one from Aristotle makes clear that the arithmetic proof of the irrationality of $\sqrt{2}$ that I presented was not too much of an anachronism, even if it was not the first proof discovered by the Greeks. This is not certain. There is a second proof, a geometric proof. This can be construed as a proof by paper-folding and was shown to me by one member of the audience. It has been suggested, for various reasons, that the first proof ever given may have been geometric, but there is apparently no solid, universally accepted evidence or argument one way or the other.

Although the dialogue Theaetetus is not primarily concerned with mathematics but with other matters, I cite selectively from it to obtain a dialogue appropriate to our purposes. Theodorus, apparently a celebrated geometer but known largely through this text was the teacher of Theatetus. That he stopped at 17 has been taken as evidence that his proof must have been difficult of execution, for example geometric, and not along the line of the one I gave for $\sqrt{5}$. Hardy and Wright in their *Introduction to the theory of numbers* are not at all persuaded by this line of argument. In any case the issue with Theodorus is not $\sqrt{2}$ but the square roots of larger numbers, 3, 5 and so on up to 17, with of course 4, 9, and 16 omitted. Hardy and Wright offer, however, both arithmetic and geometric proofs, and it is worthwhile having a look at their discussion.

I observe in passing that Theodorus seems to have had an anxious administration, fearful of sexual harassment charges, looking over his shoulder.

The importance of the discovery of the irrationality of $\sqrt{2}$ and of the crisis it caused in Greek mathematics is germane to the story I want to relate only in so far as we want to arrive at some understanding of the contemporary problems posed by irrationality, and indeed by irrationality of a special kind, by algebraic irrationality, a notion that it will take us some time to reach. Nonetheless, a few succint quotations from Neugebauer and from Heath may not be out of place, especially since as was observed to me by a friend, Eudoxus, the great mathematician who resolved the crisis, may be quite unfamiliar to mathematicians. I cannot say that his name had been familiar to me.

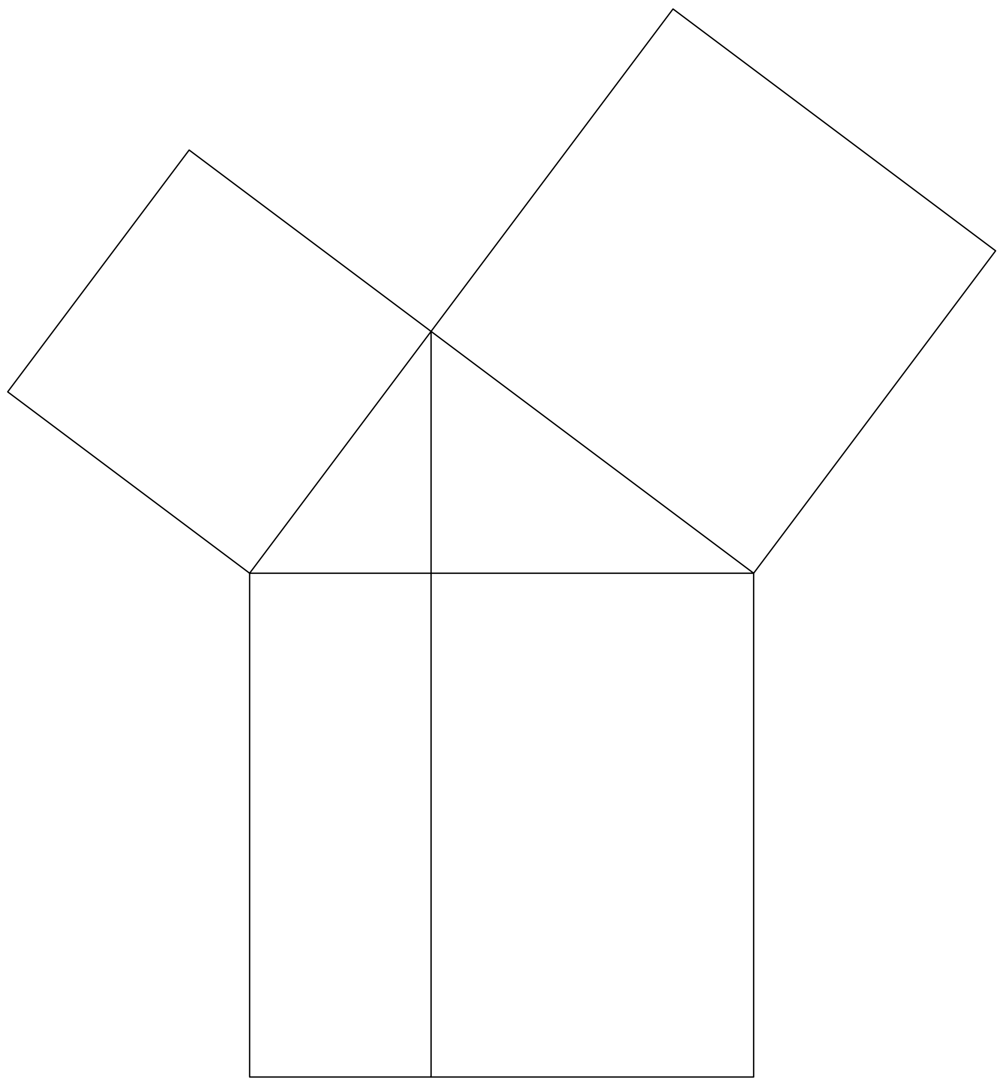
Observe (perhaps!) that one things that emerges from the comments of Neugebauer is that the geometrical treatment which I follow was the result of the Greek's geometrical treatment of number, somewhat of a historical accident and apparently peculiar to the Greeks, as other mathematicians of antiquity without the Greek's respect for logic proceeded differently. Since the geometrical treatment is, as we shall see, much less straightforward than an algebraic treatment, it may be that the historical approach obscures as much as it enlightens.

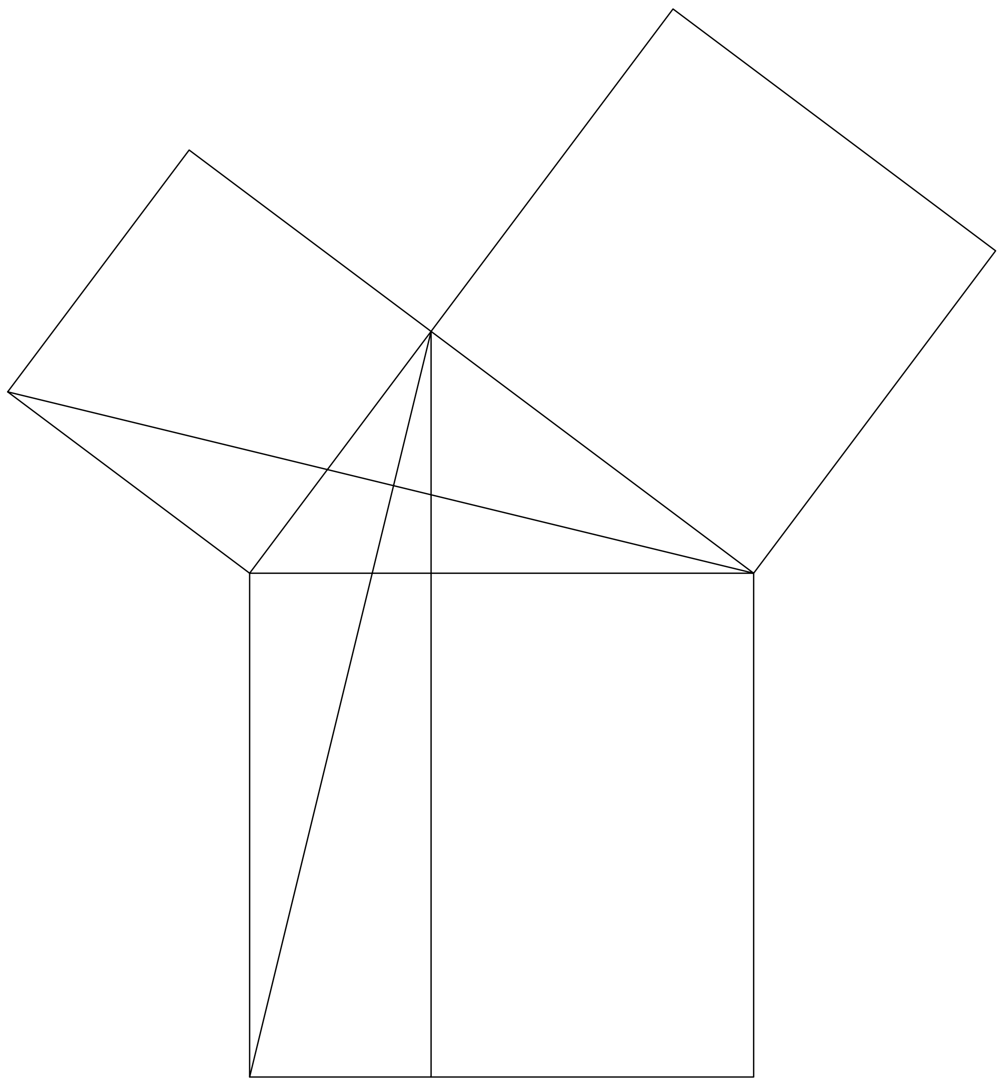
1.(Neugebauer) It is also generally accepted that the essential turn in the development came about through the discussion of the consequences of the arithmetical fact that no ratio of two integers could be found such that its square had the value 2. The geometrical corollary that the diagonal of the square could not be "measured" by its side obviously caused serious discussion about the relation between geometrical and arithmetical proof. ... The reaction of the mathematicians ... led to two major steps. ... this gave rise to the strictly axiomatic procedure. Secondly, it had become clear that one should consider the geometrical objects as the given entities such that the case of integer ratios appeared as a special case of only secondary interest.

2.(Heath) *Theory of proportion.* The anonymous author of a scholium to Euclid's Book V ... tells us ... that this Book, containing the general theory of proportion...'is the discovery of Eudoxus, the teacher of Plato'. There is no reason to doubt the truth of this statement. ...

The essence of the new theory was that it was applicable to incommensurable as well as commensurable quantities; and its importance cannot be overrated, for it enabled geometry to go forward again, after it had received the blow which paralyzed it for the time. This was the discovery of the irrational, at a time when geometry still depended on the Pythagorean theory of proportion, that is, the numerical theory which was of course applicable only to commensurables. ...

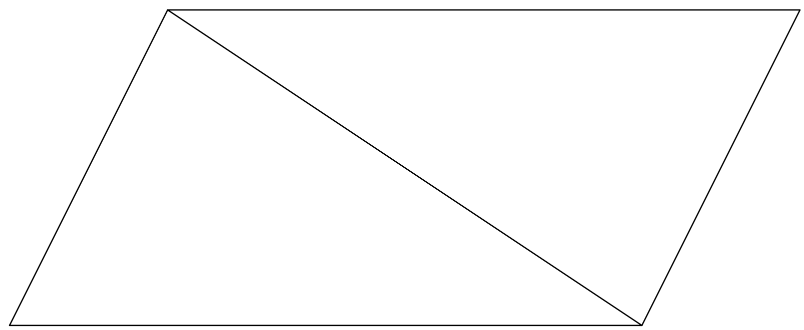
The greatness of the new theory itself needs no further argument when it is remembered that the definition of equal ratios in Eucl. V, Def. 5 corresponds exactly to the modern theory of irrationals due to Dedekind, and that it is word for word the same as Weierstraß's definition of equal numbers.

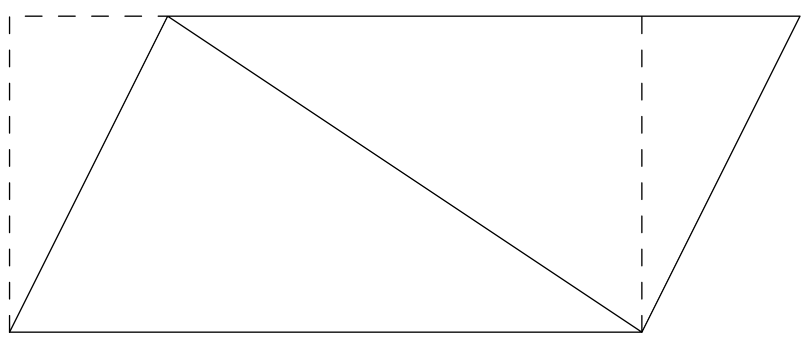




AREA OF TRIANGLE

$$\text{Area}=\frac{1}{2}\times\text{Base}\times\text{Height}$$





A difficulty

I introduce the formula for the area of a triangle, as it is familiar, but in the euclidean context an area is an area and a number is a number, and they are different things. We should rather, in Euclid's terms, speak of the area of a parallelogram being double of that of the triangle or simply of the parallelogram being double the triangle, but leave this aside for the moment.

Since a triangle has three different sides, each of which can be taken as the base, and correspondingly three different heights, the formula for the area is in fact three formulas, and one may ask why they should give the same result. In fact, this question is not addressed in Euclid. It appears, as my colleague Pierre Deligne pointed out to me, to be an implicit, not explicit, assumption on the part of Euclid that the area of a plane figure is a well defined notion and that it is additive, thus if one figure is decomposed into the sum of two, as a parallelogram is decomposed into the sum of two triangles, then the area of the larger will be the sum of the areas of the smaller.

This difficulty vitiates, as Deligne observed, Euclid's proof of the pythagorean theorem. I run through Euclid's proof and various alternate proofs, and then return to the difficulty.

****

Hilbert treats the definition of areas of polygons in his book *Grundlagen der Geometrie*, the basic problem being to show that the area is well-defined. He begins by showing that the three possible formulas for the area of a triangle all give the same result. For this he needs the theory of similar triangles, thus the theory of proportions or Book V of Euclid. In other words, from an even more rigorous point of view Euclid's efforts to avoid this theory in his early chapters have been in vain. Areas can only be adequately defined by introducing numbers, thus proportions, and by establishing some consequences of the theory of proportions, thus the theory of similar triangles.

Eudoxus and Grothendieck

Euclid's proof of the pythagorean theorem has been a puzzle to many people. Nietzche apparently found it *stelzbeinig* and *hinterlistig*, thus stilted and sly, and he was not far off the mark. Aldous Huxley, in a short story about thwarted mathematical genius *Young Archimedes* suggests, and he has the support of some historians of mathematics, that Pythagorus was likely to have used a simpler, more geometrically evident proof.

First alternate proof

It seems, however, that in spite of the appeal of this proof most historians are of the opinion that it does not have a Greek feel. Heath presents a proof that is possibly Greek, and possibly the one used by Pythagorus or his school.

Second alternate proof

The difficulty is that it uses the notion of similar triangles, and similar triangles are first discussed by Euclid in Book VI after he has developed, following Eudoxus, the theory of proportions in Book V. The theory of proportions is, if one likes, the theory of pure numbers, as opposed to lengths or areas, a notion whose difficulty appears when it is recognized that not all lengths or areas are multiples of one aliquot part, as the Greeks discovered as a consequence of the pythagorean theorem, so that there are pure numbers that are not representable as fractions a/b. Thus, it appears that the first proof of the pythagorean theorem was by a method which was revealed by further deductions from the theorem as flawed.

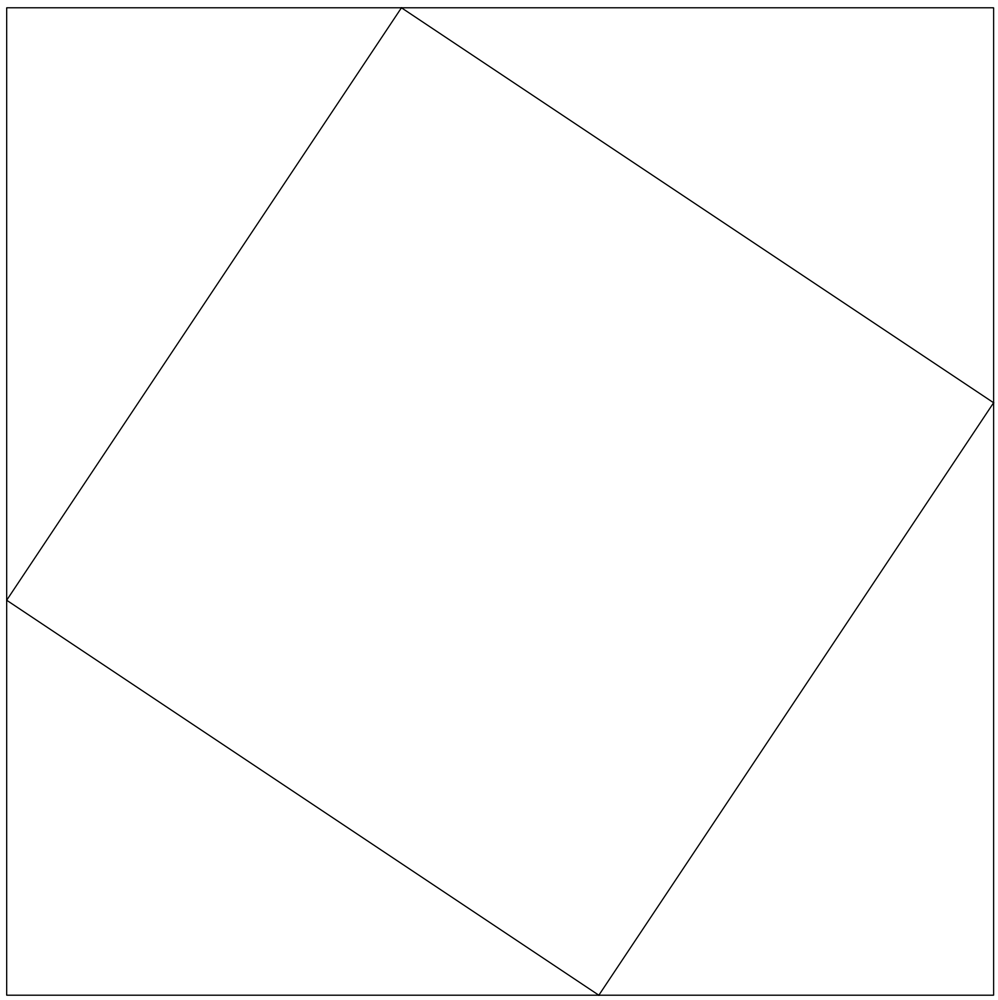
One aspect of the art of mathematics is the formulation of good definitions. In the second half of the twentieth century Grothendieck was the master of this. His definitions, with a breathtaking clarity, often transformed what had previously been regarded as an arcane, difficult theory, for example, that of complex multiplication, into self-evident consequences. A first reading of Euclid on proportions suggests that Eudoxus may have been the Greek Grothendieck. The proof of Proposition VI.1, accessible to all of you, is an elegant example of what can be done with a good definition. I recommend it as supplementary reading.

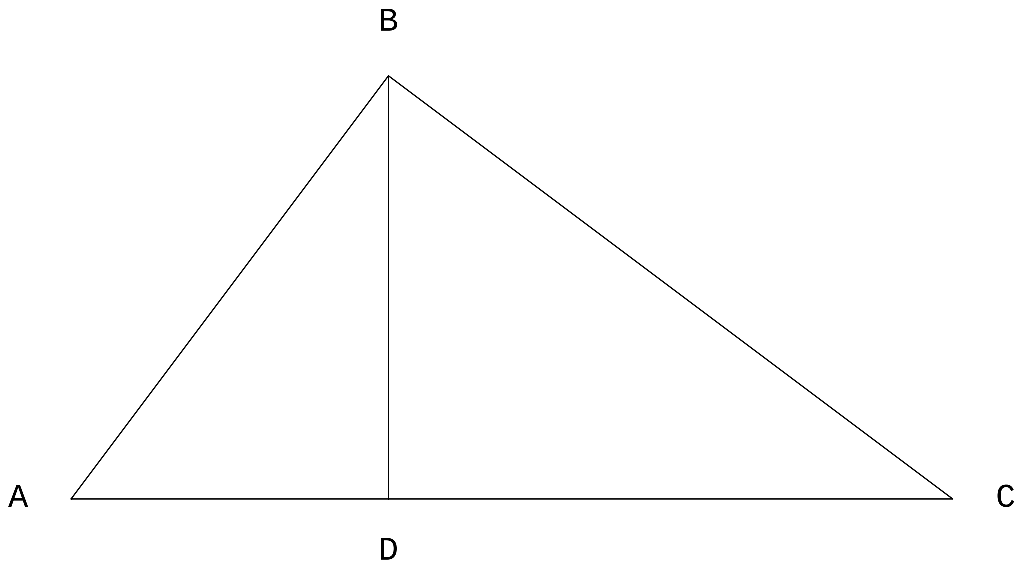
Proposition VI.1 *Triangles and parallelograms which are under the same height are to one another as their bases.*

At all events, Euclid's proof of Proposition I.47, the pythagorean theorem, appears, as Heath argues, to be Euclid's original response to an expository challenge. He needed the theorem early; yet he was not permitted to use the earlier proof because he had not yet developed the theory of proportions and the theory of similar triangles. So he found a new proof, not altogether different from the earlier proof yet without its flaw.

"There!" he said triumphantly, and straightened himself up to look at them. "Now I'll explain."

And he proceeded to prove the theorem of Pythagoras — not in Euclid's way, but by the simpler and more satisfying method which was, in all probability, employed by Pythagoras himself. He had drawn a square and dissected it, by a pair of crossed perpendiculars, into two squares and two equal rectangles. The equal rectangles he divided up by their diagonals into four equal right-angled triangles. The two squares are then seen to be the squares on the two sides of any of these triangles other than the hypothenuse. So much for the first diagram. In the next he took the four right-angled triangles into which the rectangles had been divided and rearranged them round the original square so that their right angles filled the corners of the square, the hypotenuses looked inwards, and the greater and lesser sides of the triangles were in continuation along the sides of the squares (which are equal to the sum of these sides). In this way the original square is redissected into four right-angled triangles and the square on the hypotenuse. The four triangles are equal to the two rectangles of the original dissection. Therefore the square on the hypotenuse is equal to the sum of the two squares — the squares on the two other sides — into which, with the rectangles, the original square was first dissected.



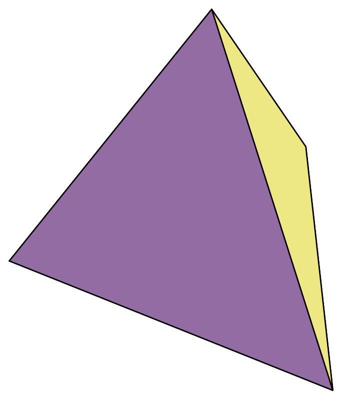


$$\frac{AD}{AB} = \frac{AB}{AC} \quad \implies \quad AD \cdot AC = AB^2$$

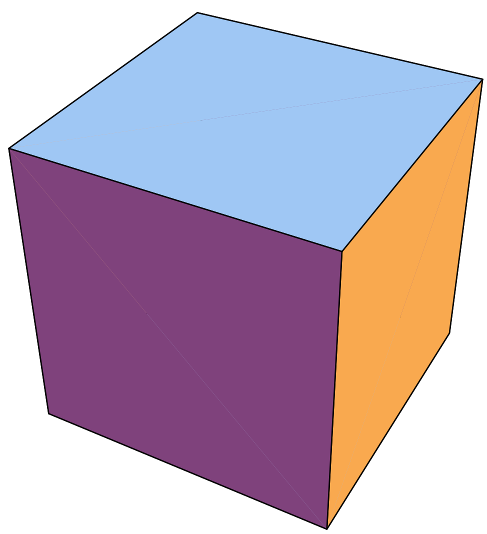
$$\frac{DC}{BC} = \frac{BC}{AC} \quad \implies \quad DC \cdot AC = BC^2$$

Thus

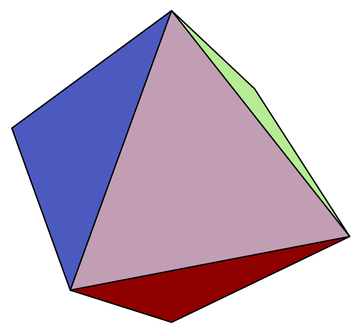
$$AB^2 + BC^2 = AD \cdot AC + DC \cdot AC = AC^2$$



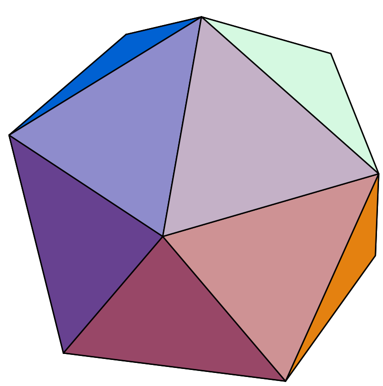
TETRAHEDRON/FIRE



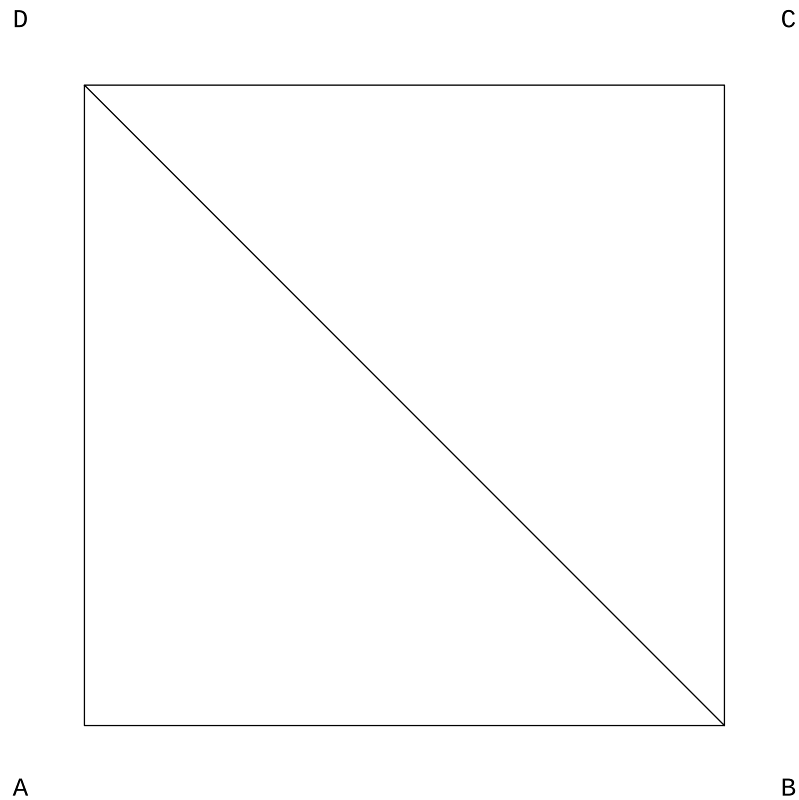
CUBE/EARTH



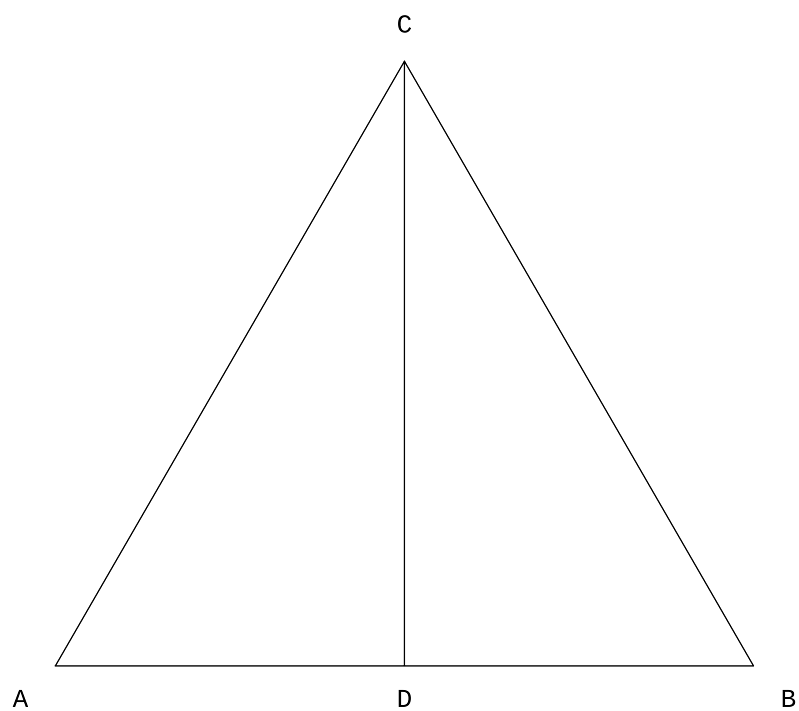
OCTAHEDRON/AIR



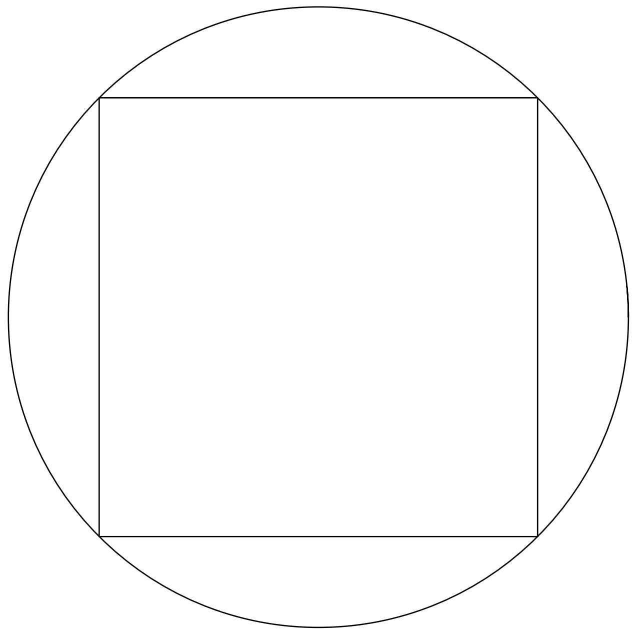
ICOSAHEDRON/WATER

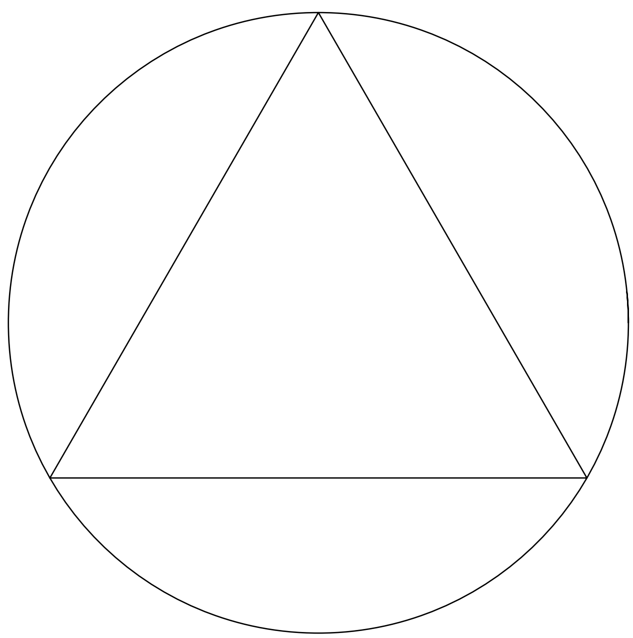


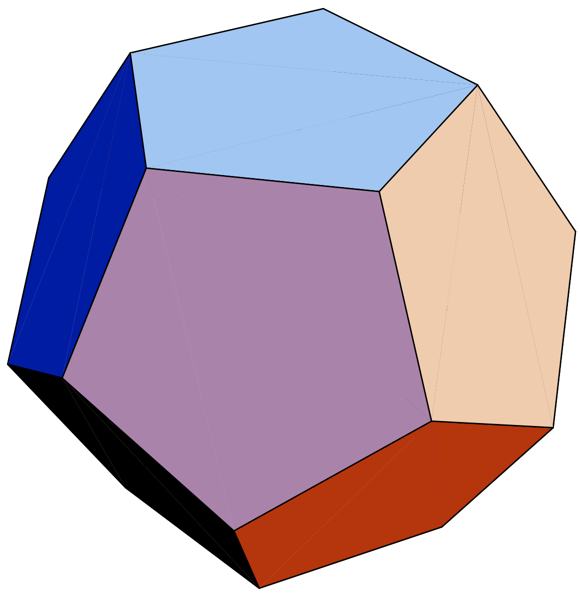
$$DB : AB = \sqrt{2} : 1$$



$$CD : AB : AD = \sqrt{3} : 2 : 1, \quad (\sqrt{3}\ )^2 + 1 = 2^2$$





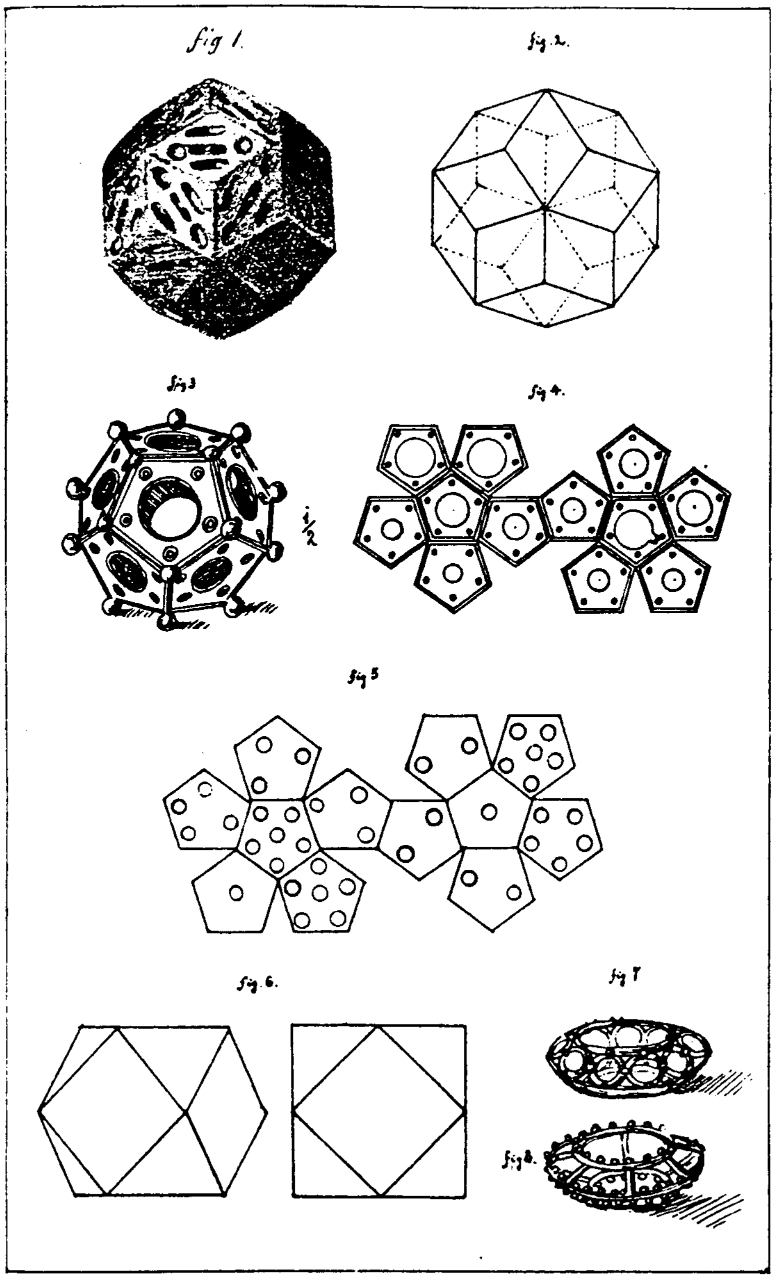


DODECAHEDRON

The Dodecahedron and the Celts

The use of the pentagon as an artistic motif was apparently extremely extremely rare, and it appears, although I have made no serious attempt to examine the literature, that the dodecahedron as an artistic motif was almost entirely confined to the Celts, and perhaps the Etruscans, during the first half of the first millenium before Christ. It has been proposed. with good reason, that it was suggested to the Celts, for whom the smelting of iron was very important, by the form of iron pyrite crystals, which is approximately but not exactly dodecahedronal. That would contradict the laws of crystallography. Hermann Weyl in his book on symmetry observes that radiolarians appear with dodecahedral symmetry, but since they are minute marine creatures, it is unlikely that they had come to the attention of the Celts.

It has also been suggested that Pythagorus who spent some time in Italy may have had some contact with the Celts or with Etruscans and may have been introduced by them to the dodecahedron.



1896. Math.-phys. Cl. 4.

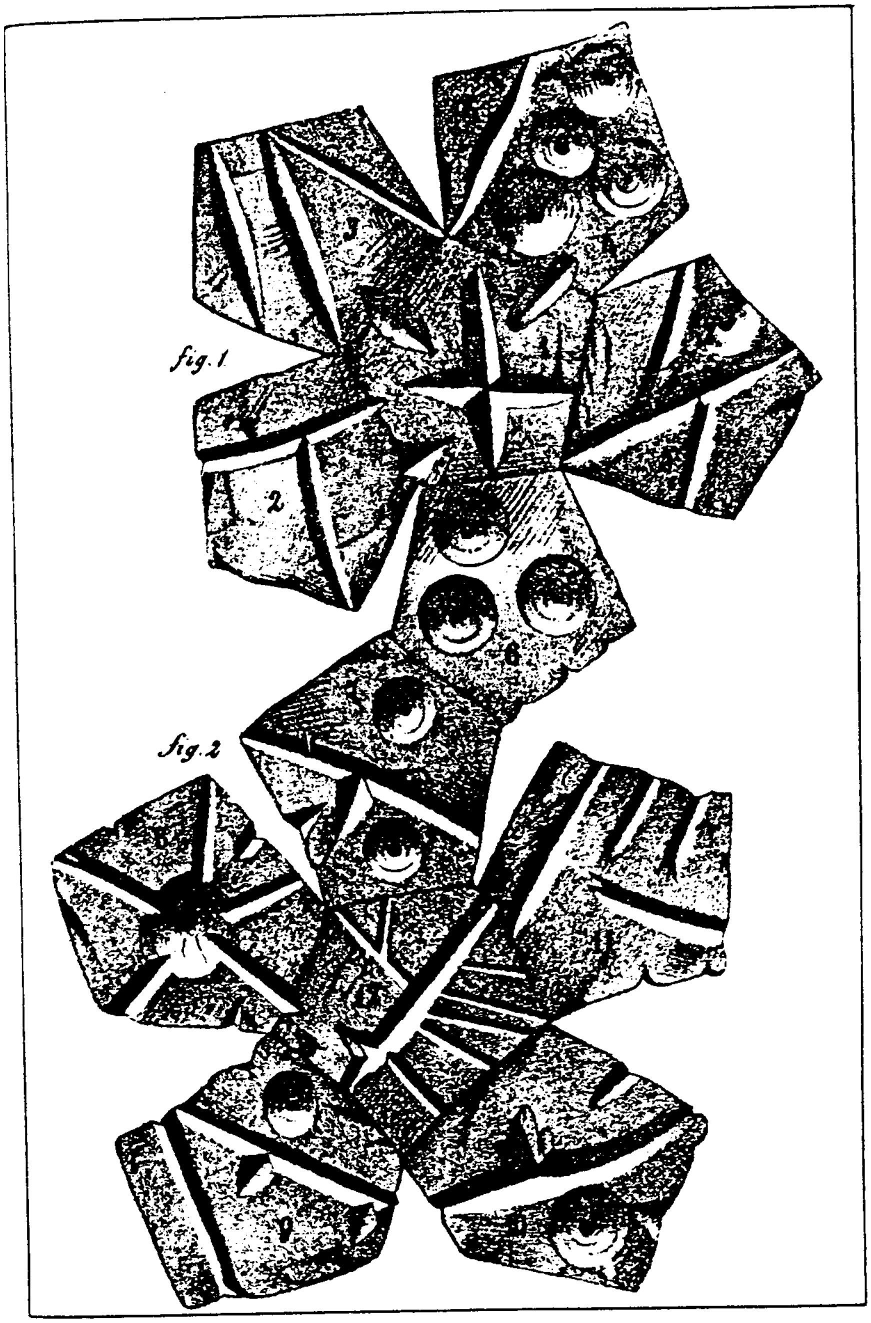
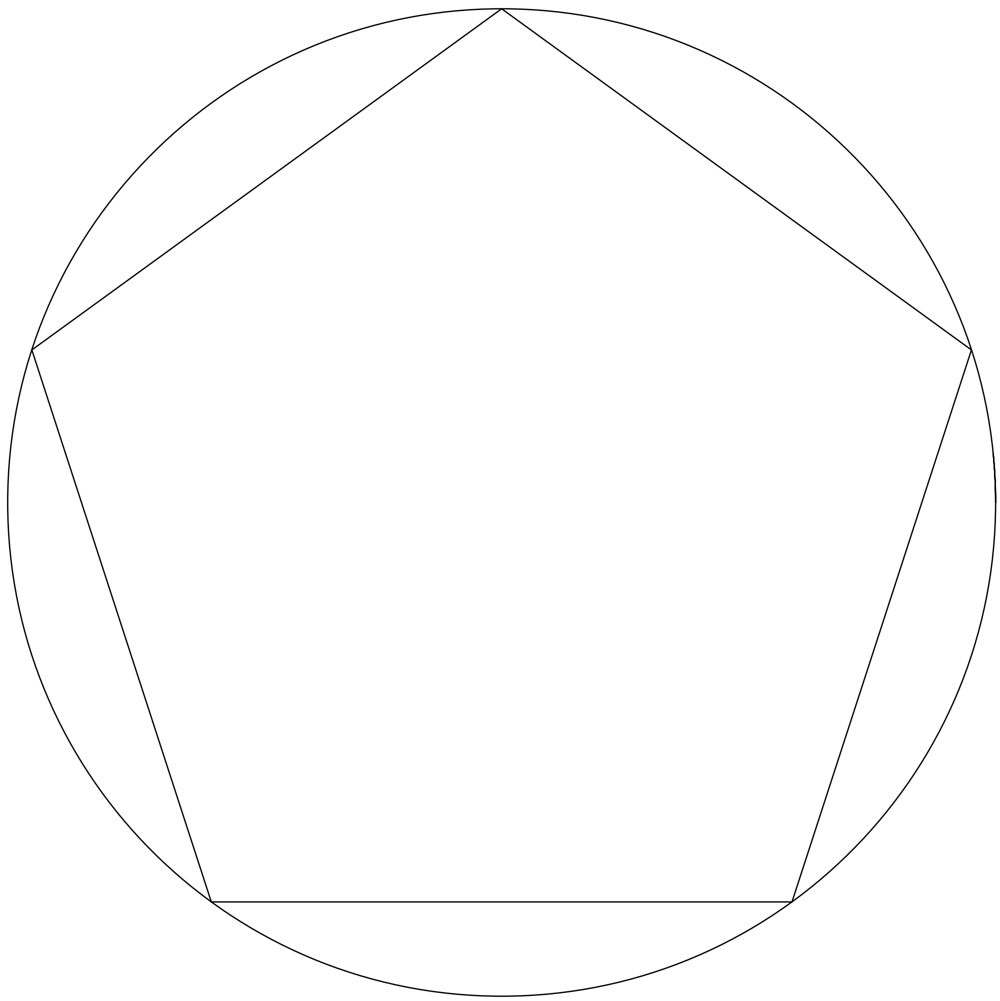
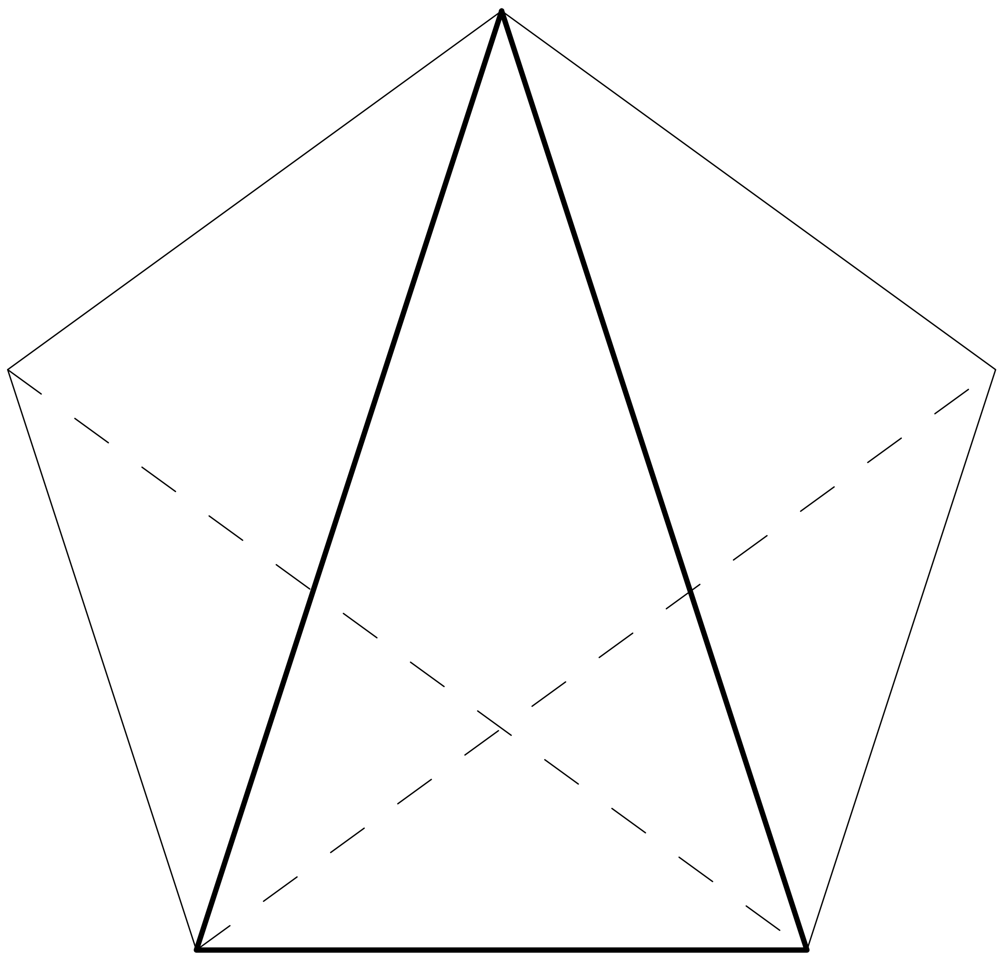


fig. 1
fig. 2





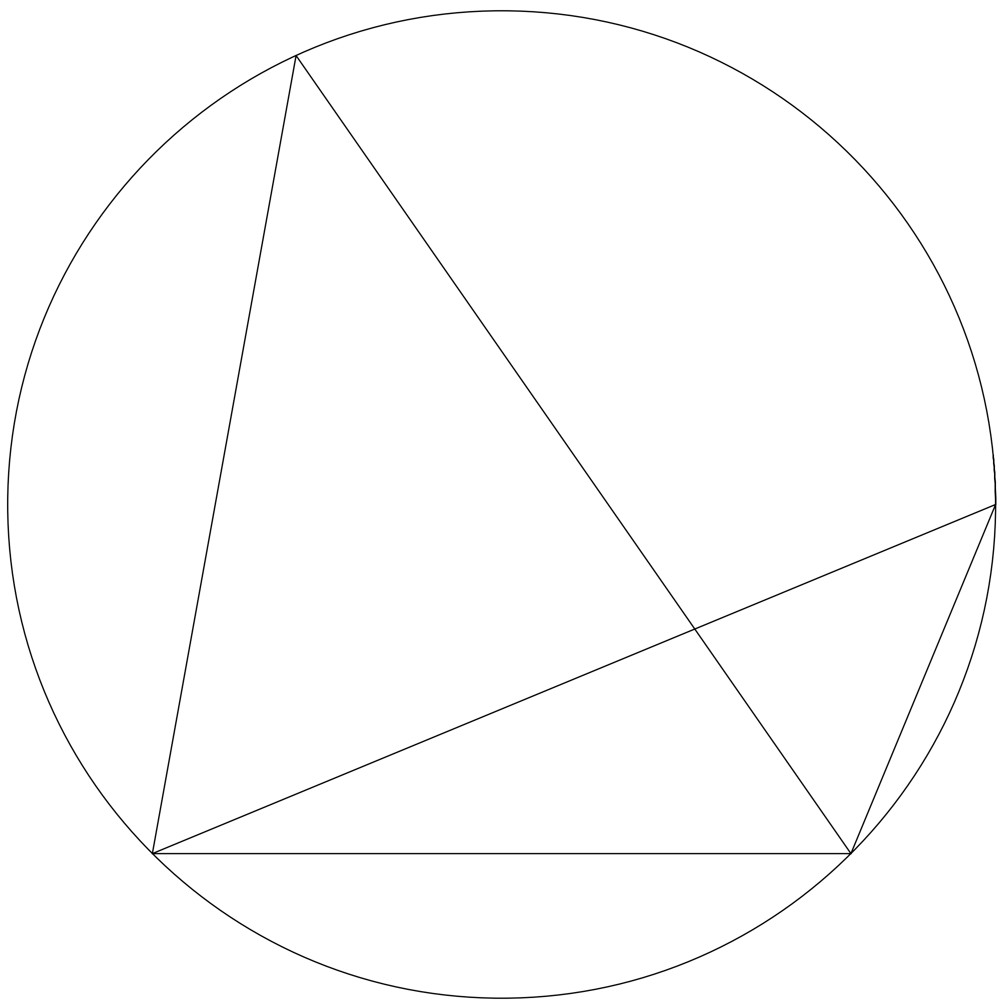
$$\theta + 2\theta + 2\theta = 180^\circ$$

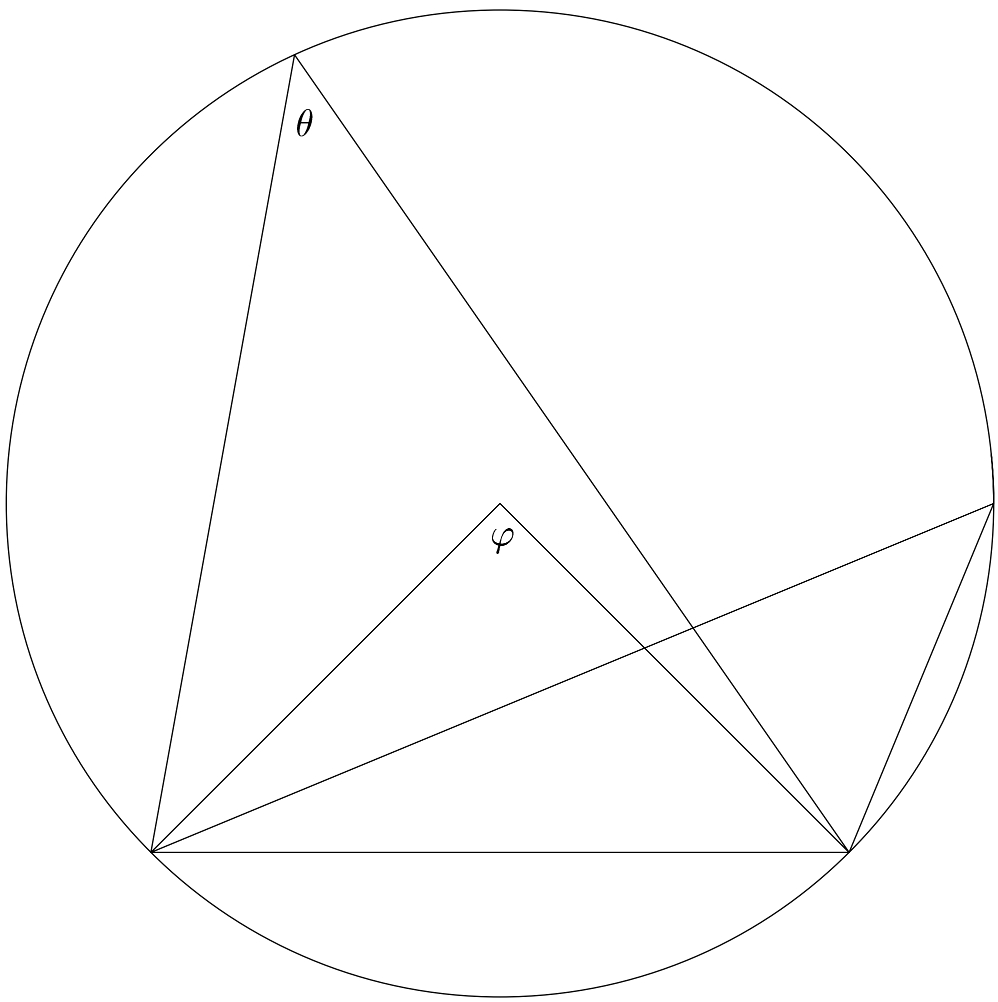
$$5\theta = 180^\circ$$

$$\theta = 36^\circ$$

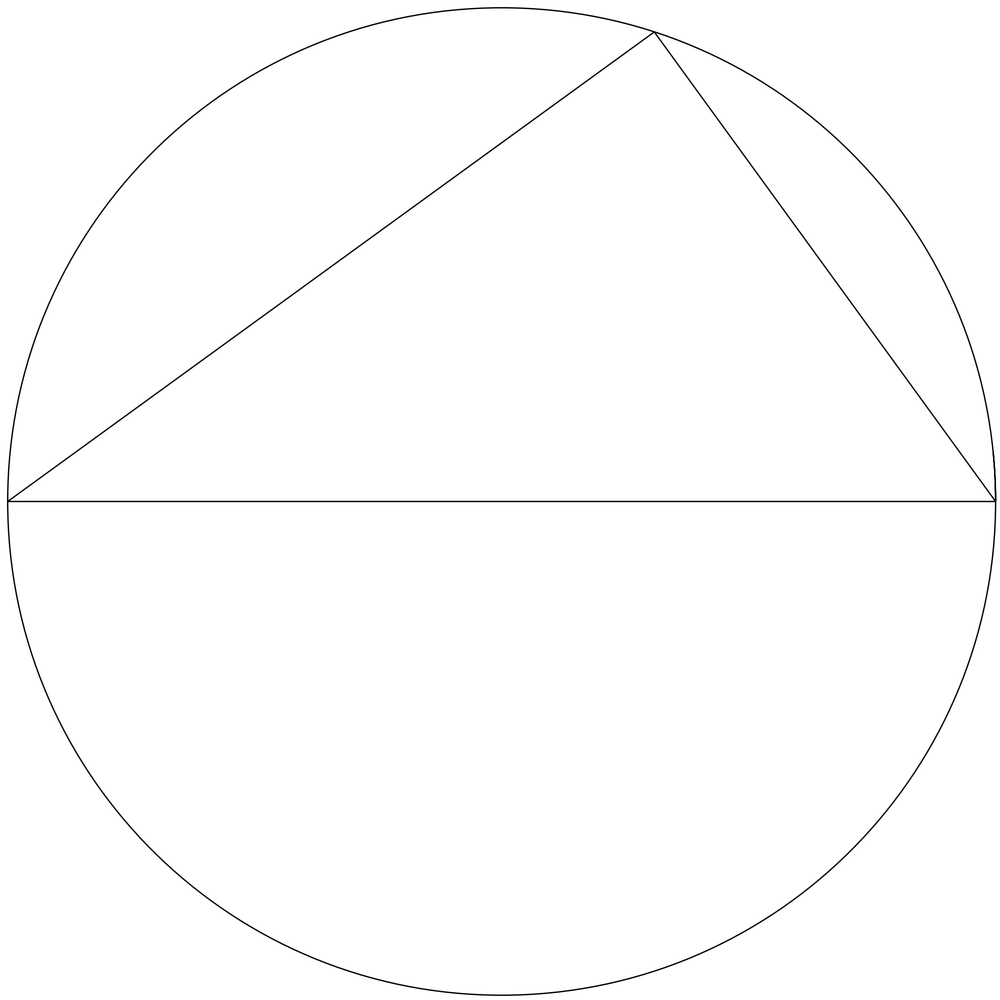
$$5\theta = \pi$$

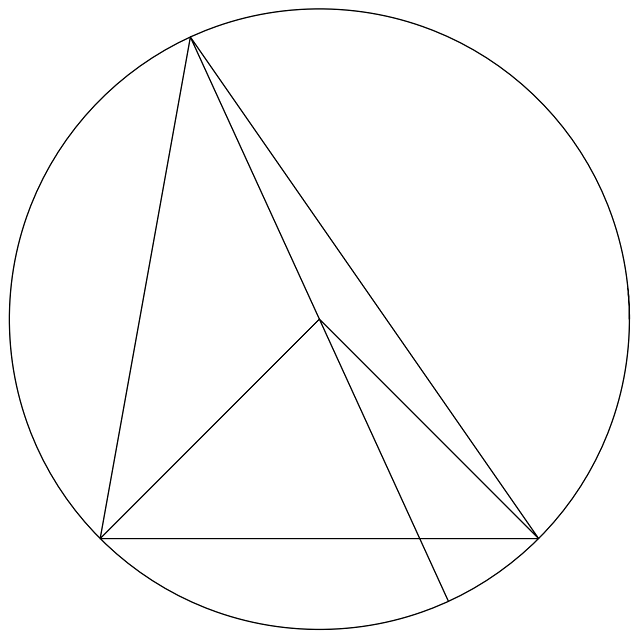
$$\theta = \frac{\pi}{5}$$

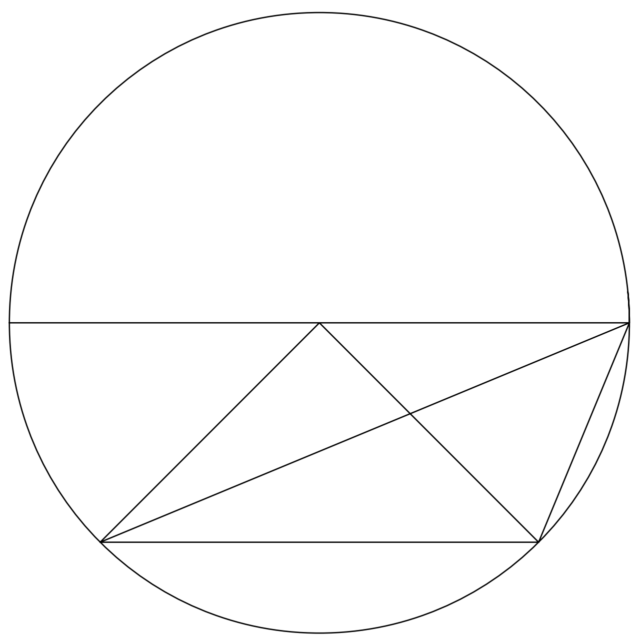




$$\varphi = 2\theta$$







$$2\angle EAB = \angle EAB + \angle EBA$$

$$\angle EAB + \angle EBA + \angle BEA = 2 \text{ right angles}$$

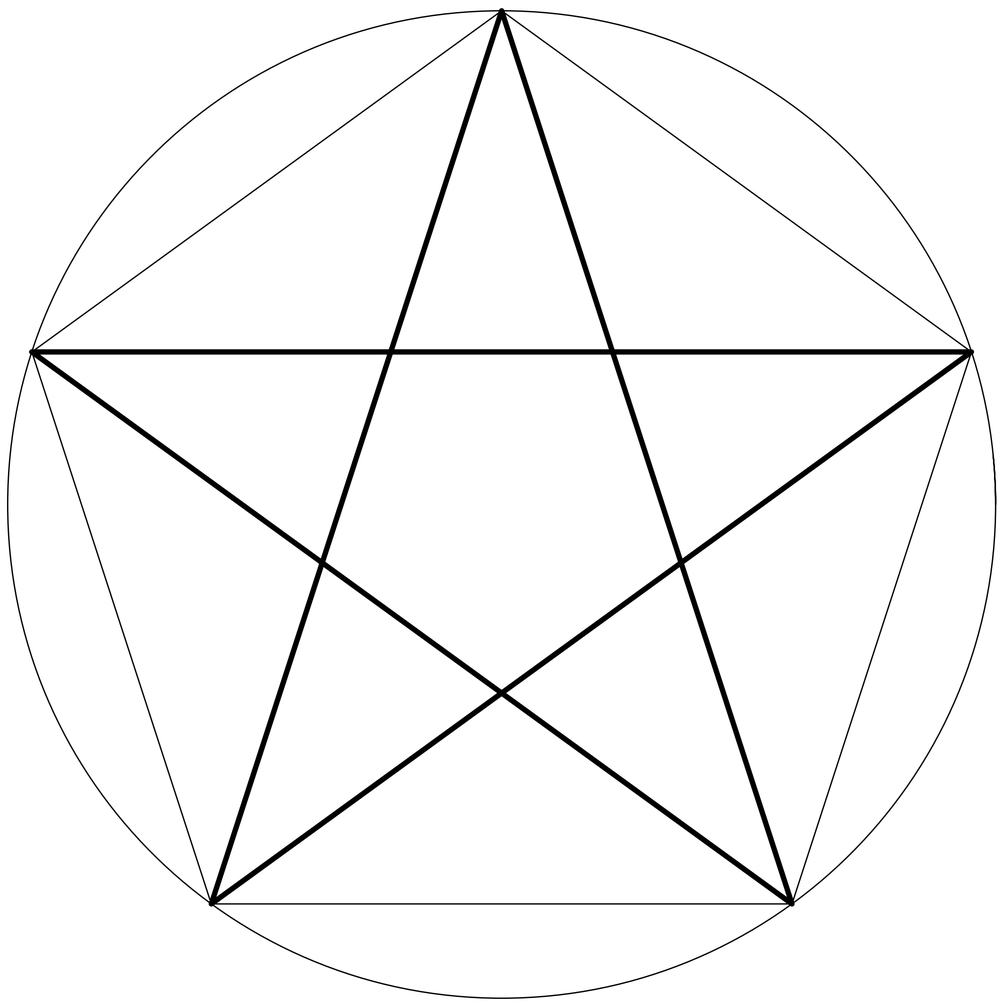
$$\angle FEB + \angle BEA = 2 \text{ right angles}$$

$$\implies \angle FEB = 2\angle EAB$$

$$\angle GEC = 2\angle GDC$$

$$\angle GEB = 2\angle GDB$$

$$\implies \angle BEC = 2\angle BDC$$



Reminder

Our ultimate purpose over the course of the year is to acquire a feeling for modern number theory, especially for a few of the many conjectural links between algebraic irrationalies on the one hand and, on the other, the Riemann zeta function and, implicitly, various similar functions. The rogue's yarn that will run through much of the material is the algebraic symmetry to which the name of Galois is attached and which I wanted to introduce in as concrete and appealing a way as possible, and in a way that linked it, in a certainly anachronous but not entirely factitious manner, with classical mathematics.

Apart from its intrinsic appeal, that is the reason for treating the construction of the pentagon, and our task today will be to acquire some feel for this construction. It is not easy. So we have to spend an hour on difficult mathematics. You should not be discouraged if you don't understand everything. What follows, namely the basic notions of analytic or Cartesian geometry, will be a little duller, but easier.

****

It is generally accepted that the subject of modern algebra was born during the Renaissance and analytic geometry, at least the treatment of Descartes, would have been unthinkable without the new algebraic methods. Among other things, it was understood during this period how to solve cubic equations and quartic equations. I could have used them as an introduction to the algebraic symmetry, but decided that a geometrical introduction would be more appealing.

THE CONSTRUCTION OF THE TRIANGLE

Proposition IV.10 *To construct an isosceles triangle having each of the angles at the base double of the remaining one.*

This proposition is a result of several others from which I single out the three most important.

Proposition II.11 *To cut a given line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.*

Proposition III.37 *If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle, the straight line which falls on it will touch the circle.*

Proposition III.32 *If a straight line touch a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle.*

With this proposition we are in the fourth book of the Elements. The pythagorean theorem is Proposition I.47, and is thus from the first book. There are thirteen books in all, but the propositions that we need come from the first four, which are all geometric in content. Book I contains the most familiar material, ending with the pythagorean theorem. Book II, to which we shall have to return for some propositions, deals with what is sometimes called geometrical algebra, thus with material that is somewhat perplexing to a modern reader, as familiar algebraic operations are clothed in very unfamiliar geometric garb. Book III deals by and large with properties of circles that have, even for us, an appealing geometric meaning. We shall need some of them, too, so that we shall acquire some familiarity with these two books at first hand. Book IV deals with the inscription of polygons in a circle, starting with the more elementary and familiar case of an arbitrary triangle and a square. I observe in passing the difference. An arbitrary triangle can be inscribed in a circle, but not all quadrilaterals can. Finally it is shown how to inscribe in a circle a regular pentagon, a regular hexagon, and a regular pentadecagon, with fifteen sides. To inscribe a hexagon is easy and the construction is probably known to all of you. and the regular pentadecagon is easily dealt with once the triangle and the pentagon are inscribed. Thus our aim is to reach the end of the fourth book. We shall go no farther in Euclid for the moment, but it may be useful to review briefly the contents of the remaining books.

The fifth book has quite a different character. It treats Eudoxus's theory of proportions. The sixth book treats, on the basis of Book V, largely of similar figures, especially of similar triangles, and thus contains material that is either intuitively familiar or familiar from school geometry, where it will have been treated without the explicit help of the theory of proportions. Books VII, VIII and IX treat of numbers and number theory, especially of prime numbers, so that we may have occasion to return to them, explicitly or implicitly.

From the point of view of our later algebraic analysis of Euclidean constructions, Book X is the most interesting. We shall show that Euclidean constructions amount, in algebraic terms, to repeatedly adding, subtracting, multiplying and dividing numbers and repeatedly forming their square roots. This book studies, entirely in geometrical terms, the numbers so obtained. I give some examples from Heath's notes to the book. Our purpose is not to study a large number of specific examples or to study the examples in exclusively geometric terms, but to understand why Euclid's constructions lead only to square roots, and what can be constructed using nothing but square roots, and no other surds or algebraic irrationalities.

Books XI, XII, and XIII are about three-dimensional geometry, but are quite different. Book XI treats lines and planes in three-space, thus what we would call the affine geometry of three-space, usually treated in courses on linear algebra, up to and including volumes of parallelograms. Book XII uses especially the method of exhaustion to treat areas and volumes of other, less simple plane figures and solid volumes, for example, circles (disks!) and spheres. Finally, Book XIII treats the construction and the properties of the five regular, or Platonic, solids. This is, among other things, a much deeper, or if you like more elaborate, analysis of various quadratic irrationalities, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, that we have already encountered. Thus it is not unrelated to Book X.

$$\sqrt{a\sqrt{B}}$$

$$\sqrt[4]{AB} = \sqrt{\sqrt{AB}}$$

$$\sqrt{a^2 - \frac{k^2a^2}{1+k^2}}$$

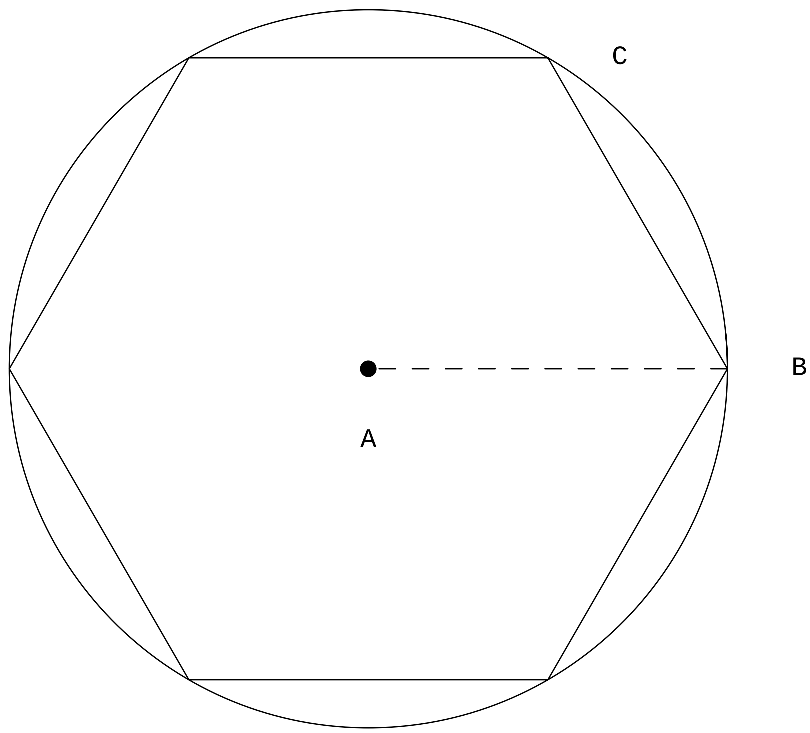
$$\frac{A - k^2A}{\sqrt[4]{A(A-k^2A)}}$$

$$\frac{\rho}{\sqrt{2}}\sqrt{1 - \frac{k}{\sqrt{1+k^2}}}$$

$$\frac{\rho}{\sqrt{2(1+k^2}}\sqrt{\sqrt{1+k^2}+k} + \frac{\rho}{\sqrt{2(1+k^2}}\sqrt{\sqrt{1+k^2}-k}$$

$$\frac{\rho\lambda^{1/4}}{\sqrt{2}}\sqrt{1+\frac{k}{\sqrt{1+k^2}}} + \frac{\rho\lambda^{1/4}}{\sqrt{2}}\sqrt{1-\frac{k}{\sqrt{1+k^2}}}$$

HEXAGON



$$AB = BC$$

Proposition II.11 *To cut a given line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.*

This reappears in Book VI as Proposition VI.30, but after the theory of proportions has been established in Book V.

Proposition VI.30 *To cut a given line in extreme and mean ratio.*

I recall the definition.

Definition VI.3 *A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less.*

Thus

$$\frac{1}{x} = \frac{x}{1-x} \Longrightarrow 1 - x = x^2 \Longrightarrow x^2 + x - 1 = 0 \Longrightarrow x = \frac{-1 \pm \sqrt{5}}{2}$$

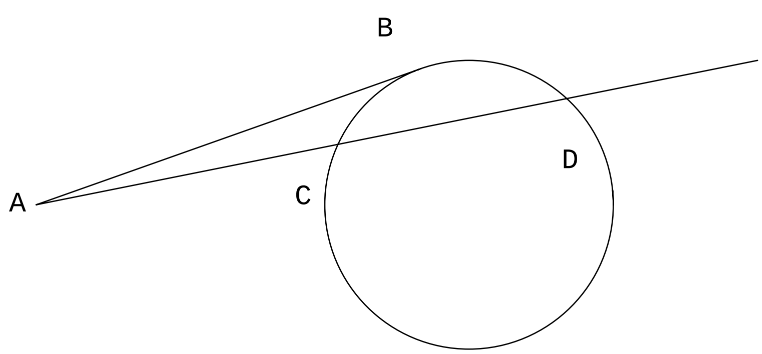
Since x is positive

$$x = \frac{\sqrt{5}-1}{2} \Longrightarrow \frac{x}{1-x} = \frac{\sqrt{5}+1}{2} \quad \text{Golden Section}$$

Proposition III.37 *If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle, the straight line which falls on it will touch the circle.*

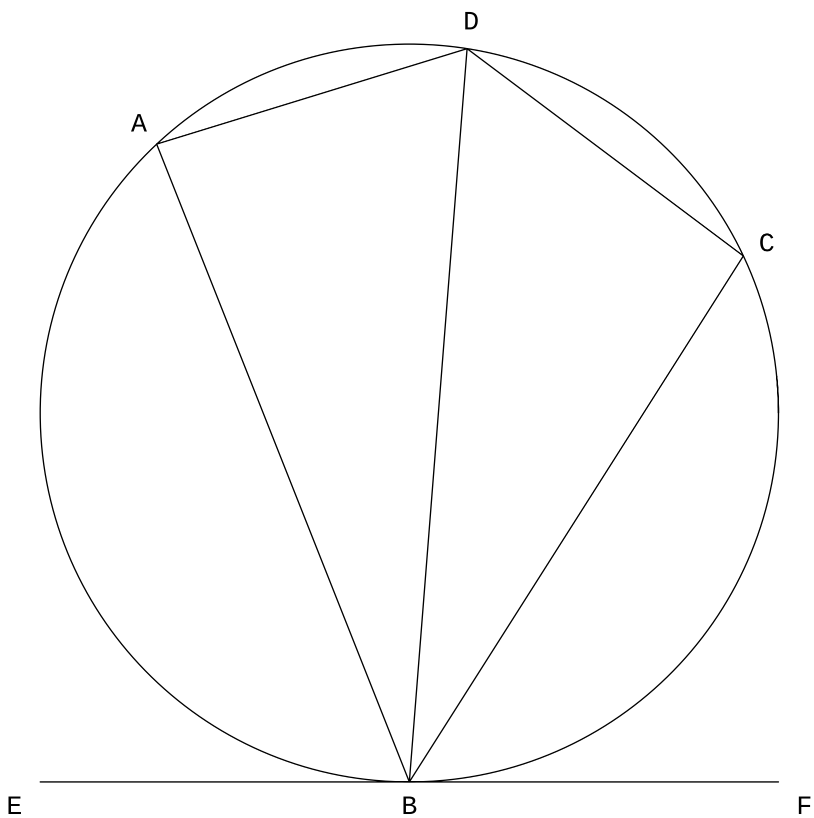
This proposition is the converse to the following one, the one I shall prove.

Proposition III 36 *If a point be taken outside a circle and from it there fall on the circle two straight lines, and if one of them cut the circle and the other touch it, the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference will be equal to the square on the tangent.*



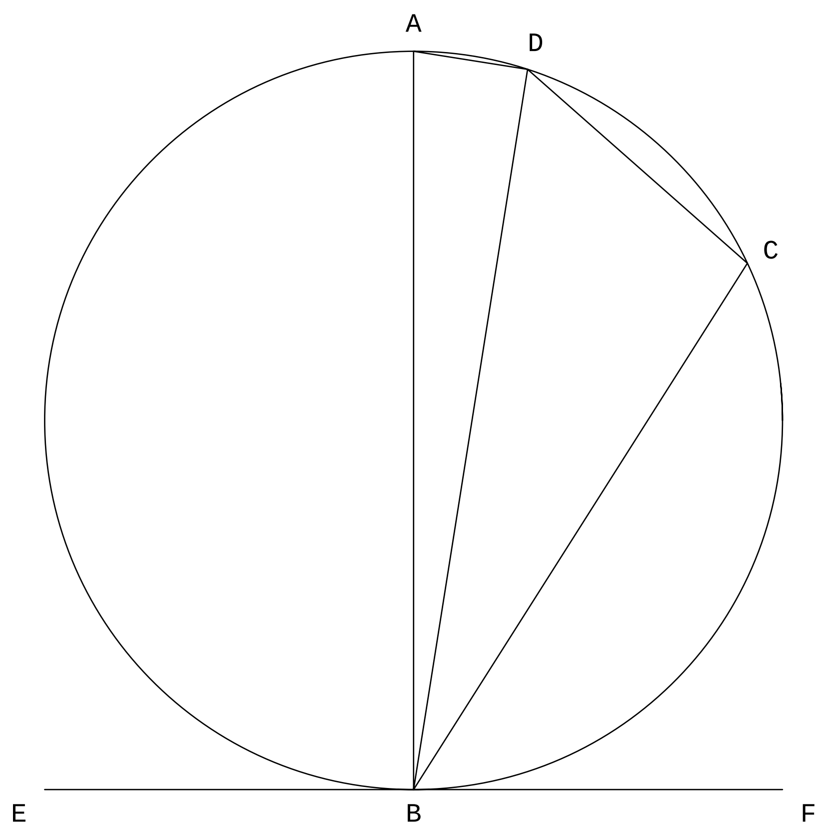
$$AC \cdot AD = AB^2$$

Proposition III.32 *If a straight line touch a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle.*



$$\angle FBD = \angle BAD, \quad \angle EBD = \angle BCD$$

PROOF



$$\angle ADB = \square \Longrightarrow \angle BAD + \angle ABD = \square$$

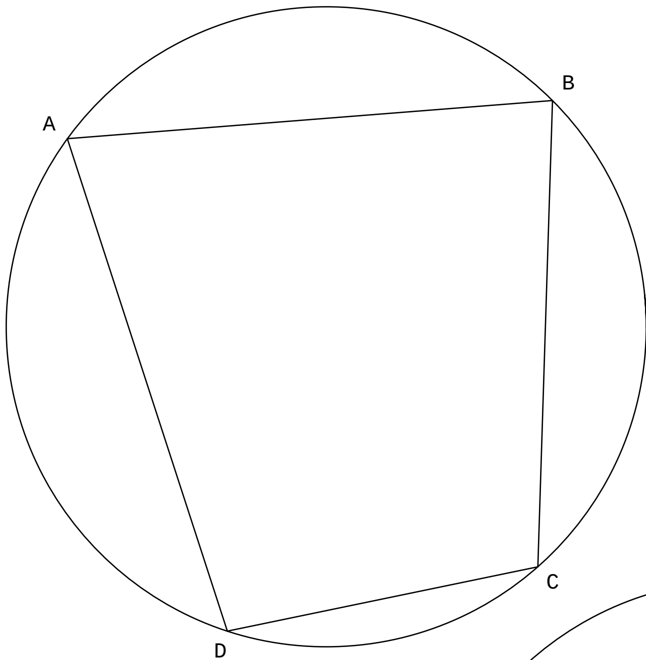
$$\angle ABF = \square \Longrightarrow \angle FBD + \angle DBA = \square$$

Thus

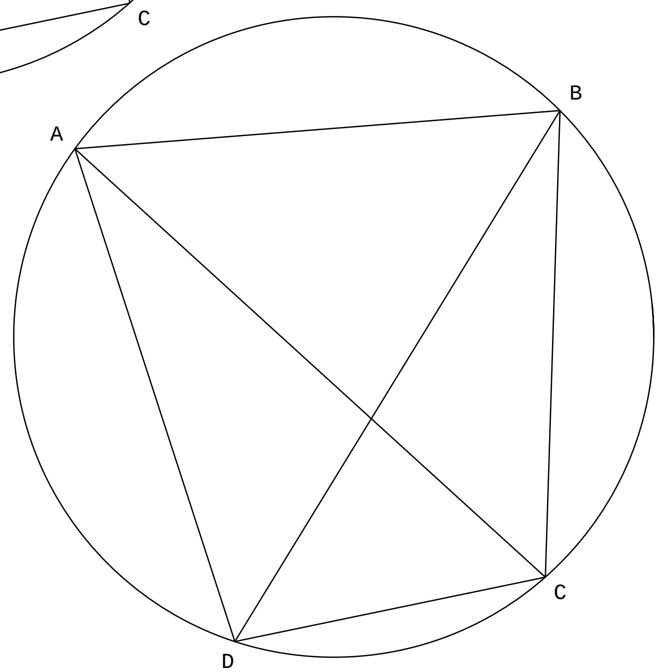
$$\angle BAD = \angle FBD \Longrightarrow \angle EBD = \angle BCD$$

SUPPLEMENT TO PROOF

Proposition III.22 *The opposite angles of quadrilaterals in circles are equal to two right angles*



Add two lines to the diagram.

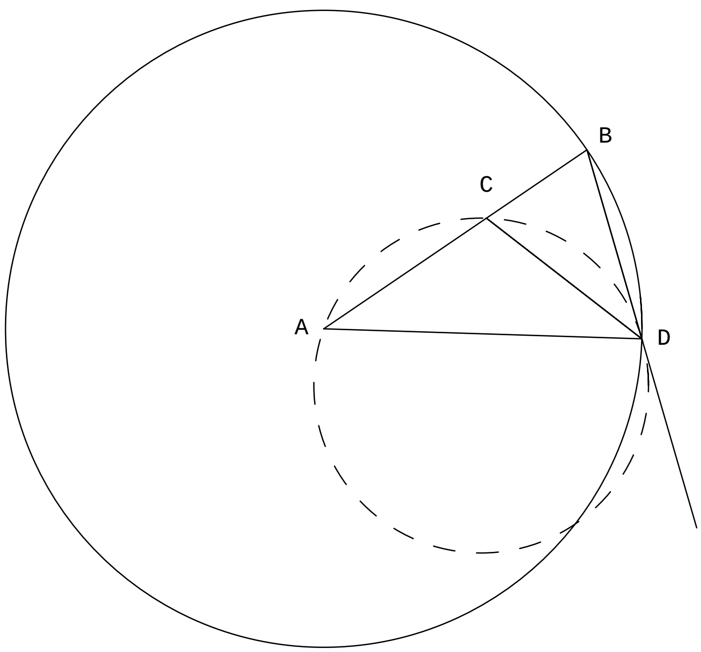


$\angle CAB = \angle BDC$
$\angle ACB = \angle ADB$

$\Longrightarrow \angle ADC = \angle BAC + \angle ACB$
$\Longrightarrow \angle ABC + \angle ADC =$
$\angle ABC + \angle BAC + \angle ACB = 2\square$

CONSTRUCTION

Take a segment AB and divide it by a point C so that $AB : AC = AC \div CB$. Draw a circle with center A passing through B and choose D so that $BD = AC$. The desired triangle is ABD



$$AC = BD, \quad AB \cdot BC = AC^2 \Longrightarrow AB \cdot BC = BD^2$$

Thus BD touches the circle ACD. Moreover $\angle BDC = \angle DAC$. Therefore

$$\angle CBD = \angle BDA = \angle BDC + \angle CDA = \angle DAC + \angle CDA = \angle BCD$$

Thus $CD = BC = AC$ so that $\angle CAD = \angle ADC$ and $\angle BCD = 2 \times \angle CAD$.

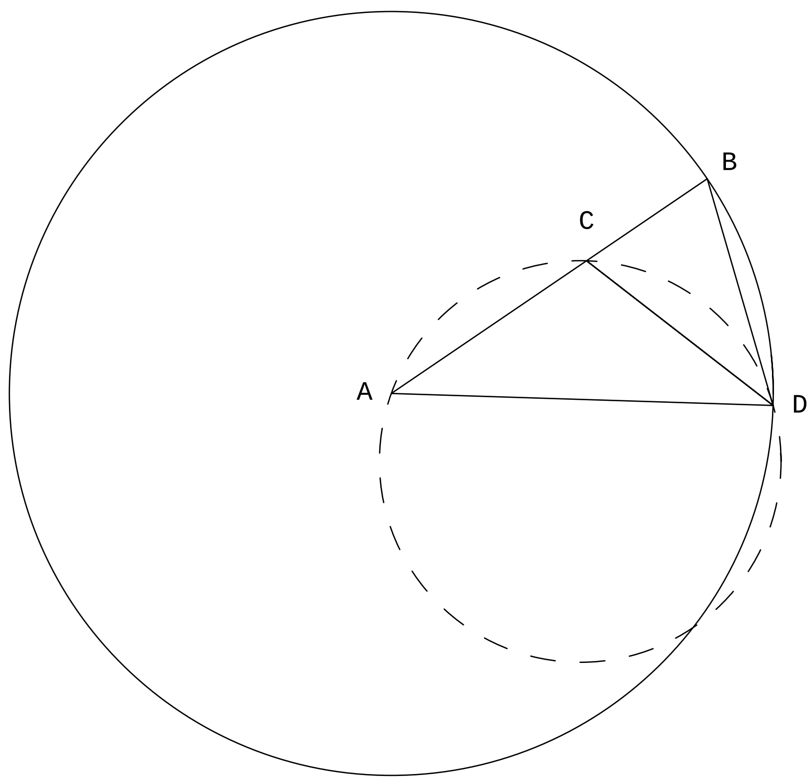
But $\angle BCD = \angle BDA = \angle ABD$.

THE CONSTRUCTION BACKWARD

Suppose the triangle ABD is given with the property that

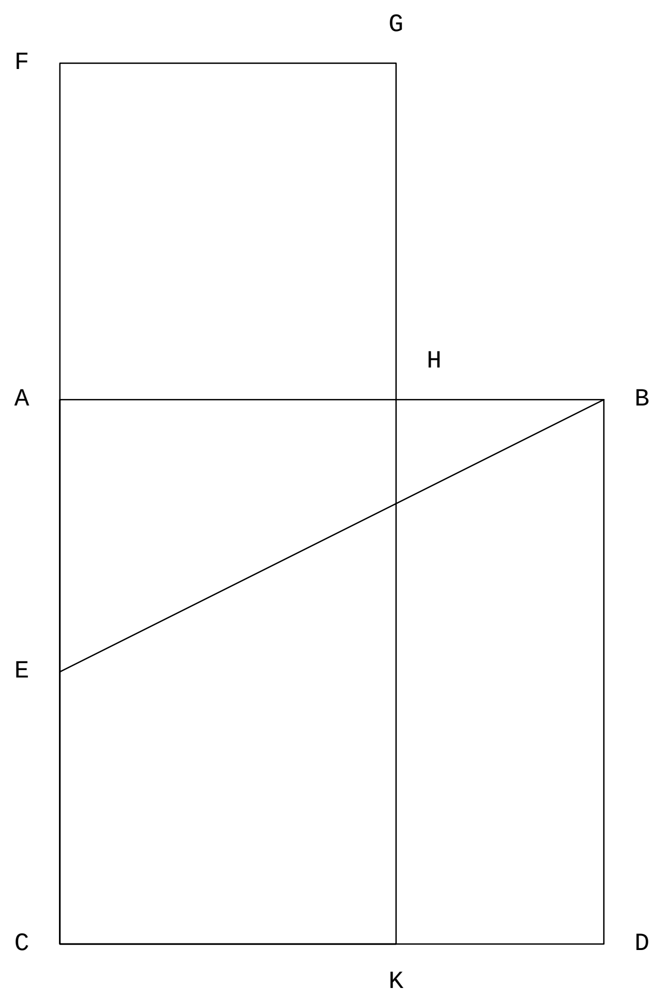
$$\angle BDA = \angle DBA = 2 \times \angle DAB$$

Bisect BDA. Then $BD = DC = CA$. BD must touch the circle ACD so that $AC^2 = BD^2 = BC \cdot BA$.



Proposition II.11 *To cut a given line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.*

Let AB be the given line. Describe the square $ABCD$ on AB. Bisect AC at E and join E to B. (Note that if $AB = 1$ then $EB = \sqrt{1^2 + 1/2^2} = \sqrt{5}/2$.) Let EF be made equal to BE. Draw the square FH on AF. (Then $AH = FA = \sqrt{5}/2 - 1/2$.)



As a first step we need that

$$CF \cdot FA + AE^2 = EF^2 \qquad \Big(\ (y+1) \cdot y + \frac{1}{2^2} = (y + \frac{1}{2})^2\ \Big)$$

This is Proposition II.6 and will be proved separately.

$$EF^2 = EB^2 = AE^2 + BA^2 \Longrightarrow CF \cdot FA = BA^2$$

$$FA = FG \Longrightarrow FK = AD \quad \text{(Areas)}$$

Subtract AK from each. Then $FH = HD$. The rectangle contained by AB, BH is equal to the square on HA.

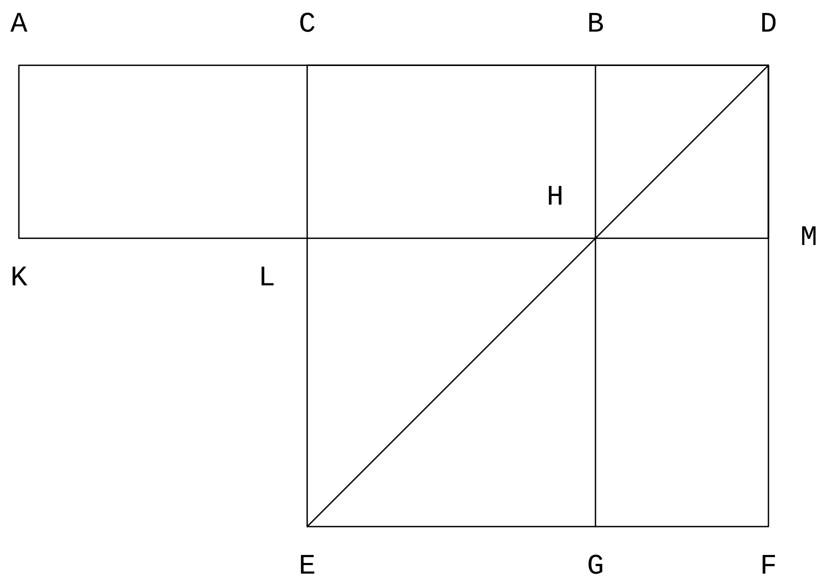
Proposition II.6 *If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.*

This is the relation

$$(x+y) \cdot y + \frac{x^2}{2^2} = (y + \frac{x}{2})^2$$

AB $(= x)$ is bisected at C and BD $(= y)$ is added to it in a straight line. To be shown that

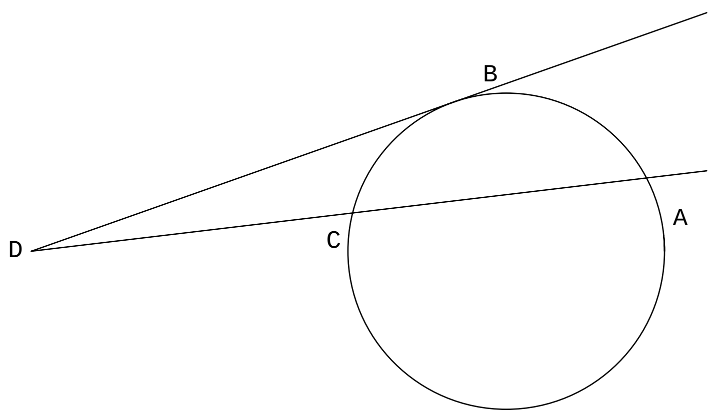
$$AD \cdot DB + CB^2 = CD^2$$



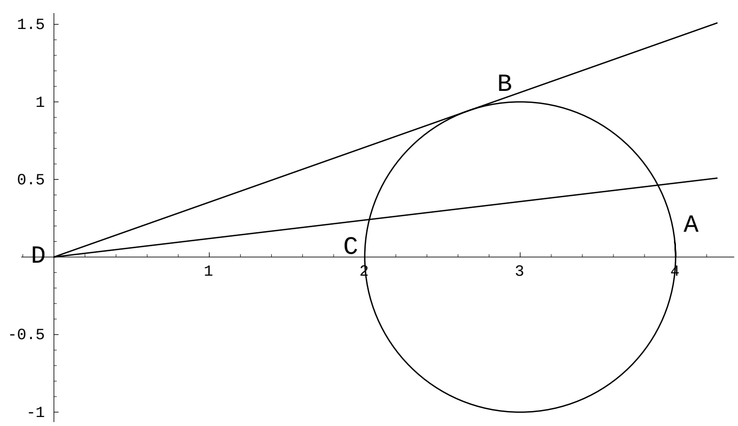
Draw the square $CEFD$; join DE and draw BG parallel to DF and KM through H parallel to AD. Thus $AL = CH = HF$. Adding CM, we have

$$AM = \text{Gnomon} CDF$$

Thus the rectangle AD, BD is equal to the gnomon CDF and the rectangle AD, BD together with the square on CB is equal to the gnomon plus LG, thus to the square on CD.



I introduce coordinates.



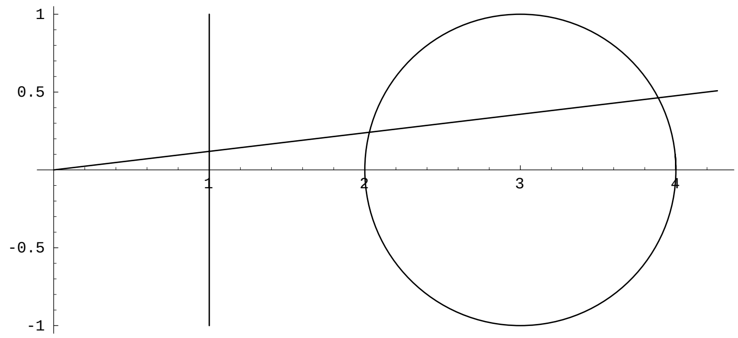
D at $(0,0)$; center at $(a,0)$; radius is r; $C=(x_1,y_1)$, $A=(x_2,y_2)$

$$DC=\sqrt{x_1^2+y_1^2}\quad DA=\sqrt{x_2^2+y_2^2}$$

The proposition implicitly affirms that

$$DC\cdot DA=\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$$

is independent of line, provided that it falls on the circle, so that this number equals its value in the extreme case that $A=C$, thus is equal to DB^2



Suppose $(1, \alpha)$ is the point where the line $x = 1$ crosses the line DA. By the theory of similar triangles, a point (x, y) lies on the line exactly when it is of the form $(x, y) = (x, \alpha x)$, thus $y = \alpha x$. Both (x_1, y_1) and (x_2, y_2) are solutions of the equation

$$(x-a)^2 + y^2 = r^2 \text{ or } (x-a)^2 + \alpha^2 x^2 = r^2$$

or

$$x^2 - 2ax + a^2 + \alpha^2 x^2 = a^2 \text{ or } (1-\alpha^2)x^2 - 2ax + a^2 - r^2 = 0$$

Recall

$$Ax^2 + Bx + C = 0$$

has solutions

$$x_2 = \frac{-B + \sqrt{B^2 - AC}}{2A}, \quad x_1 = \frac{-B - \sqrt{B^2 - AC}}{2A}$$

Thus

$$x_1 x_2 = \frac{B^2 - B^2 + 4AC}{4A^2} = \frac{C}{A}$$

Since $y_1 = \alpha x_1$ and $y_2 = \alpha x_2$,

$$\sqrt{x_1^2 + y_1^2} = \sqrt{1+\alpha^2}x_1, \quad \sqrt{x_2^2 + y_2^2} = \sqrt{1+\alpha^2}x_2$$

$$\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} = (1+\alpha^2)x_1 x_2 = (1+\alpha^2)\frac{a^2 - r^2}{1+\alpha^2} = a^2 - r^2$$

Solve:

$$ax^2 + bx + c = 0$$

Complete the square.

$$a(x^2 + 2\frac{b}{2a} + \frac{b^2}{4a^2}) + (c - \frac{b^2}{4a}) = a(x^2 + 2\frac{b}{2a} + \frac{b^2}{4a^2}) + \frac{(4ac - b^2)}{4a} = 0$$

We divide by a.

$$(x^2 + 2\frac{b}{2a} + \frac{b^2}{4a^2}) + \frac{(4ac - b^2)}{4a^2} = (x + \frac{b}{2a}{}^{2} + \frac{(4ac - b^2)}{4a^2} = 0$$

Thus

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$