Hodge theory aspects of homological mirror symmetry

Jingyu Zhao

Institute of Advanced Study

jzhao@ias.edu

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Hodge decomposition

- Given a complex manifold, one can decompose the de Rham complex $A_X^* := \Omega_{dR}^*(X) \otimes_{\mathbb{R}} \mathbb{C}$ as $A_X^k \cong \oplus_{p+q=k} A^{p,q}(X)$, where $\alpha \in A^{p,q}(X)$ is locally of the form

$$\sum f_{i_1 \cdots \bar{i}_q} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{\bar{i}_1} \wedge \cdots \wedge d\bar{z}_{\bar{i}_q}.$$

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- For Kähler manifolds, Hodge theory gives the Hodge decomposition

$$H^k(X, \mathbb{C}) \cong \oplus_{p+q=k} H^{p,q}(X) \cong \oplus_{p+q=k} H^p(X, \Omega_X^q),$$

where Ω_X^* is the sheaf of holomorphic differential forms. This decomposition depends on the complex structure.

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- On E_1-page, $E_1^{p,q} \cong H^p(X, \Omega_X^q)$ and E_1-degeneration implies the Hodge decomposition for $H^k(X, \mathbb{C})$. In fact, there is a (pure) Hodge structure of weight k on $H^k(X, \mathbb{C})$.

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- In 1990, Greene and Plesser constructed the mirror for the quintic 3-fold in $\mathbb{P}^4$.
- Candelas, de la Ossa, Green and Parkes predicted the genus zero Gromov-Witten invariants (**symplectic**) of X using period integrals (**complex**) on the mirror $X^\vee$ (Ref.Givental, Lian-Liu-Yau).

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1		101		101		1
	0		1		0	
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			1			

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- Mirror symmetry is manifested as a 90 degree rotation of Hodge diamonds.

Homological Mirror Symmetry (HMS)

- Given a symplectic manifold X and a complex manifold $X^\vee$

Categories	Objects	Morphisms
$Fuk(X)$	Lagrangian submanifolds	$CF^*(L_0, L_1) = \mathbb{K}\langle L_0 \pitchfork L_1 \rangle$
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  To do this, need a Hodge theory for categories.

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- If $\mathcal{A}$ is the coordinate ring of a smooth affine variety X, the Hochschild-Konstant-Rosenberg says $HH_*(\mathcal{A}) \cong \Omega^*_X$, and moreover $HKR\colon (HH_*(\mathcal{A}), B) \to (\Omega^*_X, d_{dR})$ is a chain map.

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  Hodge-to-de Rham spectral sequence $H^p(X, \Omega_X^q) \Longrightarrow H^{p+q}(X, \mathbb{C})$ by
  Hochschild-to-cyclic spectral sequence $\quad HH_p(\mathcal{A})u^q \Longrightarrow HC_{p+q}(\mathcal{A})$.

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- Kaledin in 2016 proved the degeneration of noncommutative Hodge-to-de Rham spectral sequence for smooth and proper DG categories.
- Ganatra, Perutz and Sheridan in 2015 used noncommutative Hodge theory to show that HMS for Calabi-Yau manifolds implies enumerative mirror symmetry for the quintic.

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- For open manifolds $U = X \backslash D$ where X is a compact Kähler manifold and D is a normal crossing divisor, Deligne in 1971 constructed a mixed Hodge structure on $H^*(U, \mathbb{C}) = \mathbb{H}^*(X, \Omega_X^\bullet(\log D))$.

Questions and future directions

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- Q2: If it does not degenerate at E_1-page, does it degenerate at E_2-page or so on?
- Q3: Given a (nondegenerate) Liouville manifold, by Ganatra $HH_*(\mathcal{W}(M)) \cong SH^{*+n}(M)$ and $HC_*(\mathcal{W}(M)) \cong SH^{*+n}_{S^1}(M)$.
When the spectral sequence degenerate at E_1, it induces a "Hodge" filtration. It is a symplectic invariant, does it respect symplectomorphisms?

Thank you!