

# Hodge theory aspects of homological mirror symmetry

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September 28, 2016

# Hodge decomposition

- Given a complex manifold, one can decompose the de Rham complex  $A_X^* := \Omega_{dR}^*(X) \otimes_{\mathbb{R}} \mathbb{C}$  as  $A_X^k \cong \bigoplus_{p+q=k} A^{p,q}(X)$ , where  $\alpha \in A^{p,q}(X)$  is locally of the form

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- For Kähler manifolds, Hodge theory gives the Hodge decomposition

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X) \cong \bigoplus_{p+q=k} H^p(X, \Omega_X^q),$$

where  $\Omega_X^*$  is the sheaf of holomorphic differential forms. This decomposition depends on the complex structure.

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- Mirror symmetry is first discovered for pairs of Calabi-Yau 3-folds, denoted as  $X$  and  $X^\vee$ .
- In 1990, Greene and Plesser constructed the mirror for the quintic 3-fold in  $\mathbb{P}^4$ .
- Candelas, de la Ossa, Green and Parkes predicted the genus zero Gromov-Witten invariants (**symplectic**) of  $X$  using period integrals (**complex**) on the mirror  $X^\vee$  (Ref. Givental, Lian-Liu-Yau).

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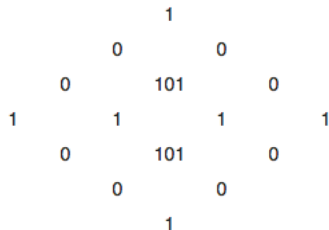
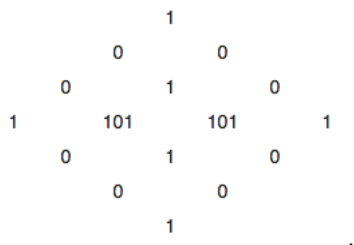
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- Mirror symmetry is manifested as a 90 degree rotation of Hodge diamonds.

# Homological Mirror Symmetry (HMS)

- Given a symplectic manifold  $X$  and a complex manifold  $X^\vee$

Categories	Objects	Morphisms
$Fuk(X)$	Lagrangian submanifolds	$CF^*(L_0, L_1) = \mathbb{K}\langle L_0 \frown L_1 \rangle$
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- Question: Can we use HMS to transfer the well-studied Hodge theory from the complex side to the symplectic side?  
To do this, need a Hodge theory for categories.

# Noncommutative Hodge-to-de Rham spectral sequence

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- Given an associative algebra  $\mathcal{A}$ , on Hochschild chains  $C_*(\mathcal{A})$  one has two differentials, the Hochschild differential  $b$  and Connes differential  $B$ , such that  $bB + Bb = 0$ .

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- For associative algebra, or a differential graded (DG) category  $\mathcal{A}$  (such as  $Coh(X)$ ), one can replace Hodge-to-de Rham spectral sequence  $H^p(X, \Omega_X^q) \implies H^{p+q}(X, \mathbb{C})$  by Hochschild-to-cyclic spectral sequence  $HH_p(\mathcal{A})u^q \implies HC_{p+q}(\mathcal{A})$ .

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Let  $\mathcal{A}$  be a smooth and proper DG category. (e.g.  $\text{Coh}(X)$  of a projective variety, **proper**: morphism space in  $\mathcal{A}$  has finite homological dimension.)

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- Kaledin in 2016 proved the degeneration of noncommutative Hodge-to-de Rham spectral sequence for smooth and proper DG categories.
- Ganatra, Perutz and Sheridan in 2015 used noncommutative Hodge theory to show that HMS for Calabi-Yau manifolds implies enumerative mirror symmetry for the quintic.

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- In order for HMS to hold, one needs a version of Fukaya category which is possibly nonproper. This is the **wrapped Fukaya category**  $\mathcal{W}(X)$  (Abouzaid-Seidel).
- For open manifolds  $U = X \setminus D$  where  $X$  is a compact Kähler manifold and  $D$  is a normal crossing divisor, Deligne in 1971 constructed a mixed Hodge structure on  $H^*(U, \mathbb{C}) = \mathbb{H}^*(X, \Omega_X^\bullet(\log D))$ .

# Questions and future directions

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- Q2: If it does not degenerate at  $E_1$ -page, does it degenerate at  $E_2$ -page or so on?
- Q3: Given a (nondegenerate) Liouville manifold, by Ganatra  $HH_*(\mathcal{W}(M)) \cong SH^{*+n}(M)$  and  $HC_*(\mathcal{W}(M)) \cong SH_{S^1}^{*+n}(M)$ .  
When the spectral sequence degenerate at  $E_1$ , it induces a "Hodge" filtration. It is a symplectic invariant, does it respect symplectomorphisms?

# Thank you!