Spectra of non-normal random matrices and noise stability

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Regularization by noise

Consider the nilpotent N-by-N matrix

$$T_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

Eigenvalues $\lambda_i = 0$, empirical measure $n^{-1}\sum \delta_{\lambda_i} = \delta_0$.
Let G_N be a matrix with i.i.d. standard Gaussians. For $\gamma > 1/2$, $\|N^{-\gamma} G_N\| \to 0$, almost surely.

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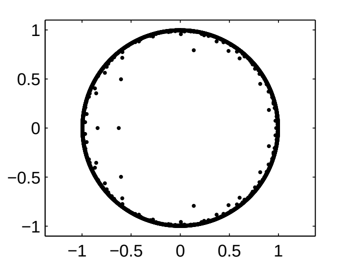
Regularization by noise II

Theorem (Guionnet-Wood-Z. '11)

Set $A_N = T_N + N^{-\gamma} G_N$, eigenvalues η_i, empirical measure $L_N^A = n^{-1} \sum \delta_{\eta_i}$. $\gamma > 1/2$. Then L_N^A converges weakly to the uniform measure on the unit circle in the complex plane.

Thus, $L_N^T = \delta_0$ but for a vanishing perturbation, L_N^A has different limit.
(Generalization to i.i.d. G_N: Wood '13.)



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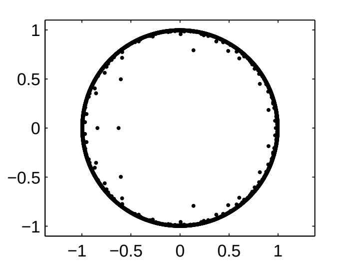

Background

A_N - $N \times N$ matrix, uniformly bounded in operator norm.

Definitions: A_N converges in $*$-moments toward an element a in a W^* probability space $(\mathcal{A}, \|\cdot\|, *, \phi)$ (faithful trace ϕ) if for any non-commutative polynomial P,

$$\frac{1}{N}\mathrm{tr} P(A_N, A_N^*) \xrightarrow{N\to\infty} \phi(P(a, a^*)) .$$

Fuglede–Kadison Determinant $\det(a) = \exp(\phi(\log|a|))$.

Brown measure ν_a of $a \in \mathcal{A}$:

$$\log \det(z - a) = \int \log|z - z'| d\nu_a(z'), \quad z \in \mathbb{C}.$$

Given by

$$\nu_a(dz) = \frac{1}{2\pi}\Delta_z \log(\det(a - z)) .$$

In particular $\log \det(z - a) = \int \log x \, d\nu_a^z(x) \quad z \in \mathbb{C}$, where ν_a^z denotes the spectral measure of the operator $|z - a|$.

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Śniady's theorem

Assume $A_N \to^* a$. Define $A_N(t) = A_N + tN^{-1/2}G_N$.

Theorem (Śniady '02)

$\lim_{t\to 0}\lim_{N\to\infty} L_N^{A_N(t)} = \nu_a$.
In particular, some sequence of noise regularizes empirical measure to the Brown measure.

Main ingredient of proof compares the singular values $\Sigma_A(t) = (\sigma_1^A, \ldots, \sigma_N^A)$ of $A_N + tN^{-1/2}G_N$ to the singular values $\Sigma_0(t) = (\sigma_1, \ldots, \sigma_N)$ of $tN^{-1/2}G_N$; by coupling the SDEs for the evolution of Σ, Σ_A, for f coordinate-wise increasing,

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$a \in \mathcal{A}$ is regular if for f smooth, compactly supported,

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Theorem (Guionnet-Wood-Z. '11)

Assume: $A_N \to^* a$, regular. $L_N^A \to \nu_a$ weakly. $\gamma > 1/2$. Then, $L_N^{A_N + N^{-\gamma} G_N} \to \nu_a$ weakly, in probability.

The proof uses the regularity (of the limit) to truncate the singularity of the log... and depends crucially on convergence to ν_a. But it is not useful in maximally nilpotent example, since $L_N^A = \delta_0 \not\to \nu_a = \delta_{S^1}$!.

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Theorem (Guionnet-Wood-Z. '11)

Assume: $A_N \to^* a$, regular, $\|E_N\| \to 0$ polynomially. $L_N^{A_N+E_N} \to \nu_a$ weakly. Then $L_N^{A_N+N^{-\gamma}G_N} \to \nu_a$ weakly, in probability.

So it is enough to find a perturbation with correct limiting behavior! Nilpotent example uses a- unitary element (which is regular), E_N is $(N, 1)$ element.

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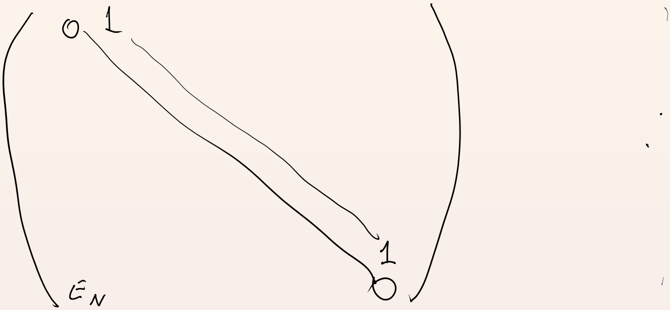

eigenvalues $\varepsilon_N^{1/N}$

$$L_N^{A_N+E_N} \longrightarrow \delta_{S^1}$$

Maybe this always works?

$$T_{b,N} = \begin{bmatrix} T_b & & & \\ & T_b & & \\ & & \ddots & \\ & & & T_b \end{bmatrix}$$

where T_b is maximally nilpotent of dimension b.

Theorem (Guionnet-Wood-Z '11)

If $b = a \log N$ and γ is large enough, then the spectral radius of $T_{b,N} + N^{-\gamma} G_N$ is uniformly strictly smaller than 1. In particular,

$$L_N^{T_{a \log N,N} + N^{-\gamma} G_N} \not\to \delta_{S^1}$$

even though $T_{a \log N,N}$ converges in $*$ moments to random unitary!

What is going on?

Noise Stability-Nilpotent matrices

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If $b = a \log N$ and γ is large enough, then the spectral radius of $T_{b,N} + N^{-\gamma} G_N$ is uniformly strictly smaller than 1. In particular,

$$L_N^{T_{a \log N,N} + N^{-\gamma} G_N} \not\to \delta_{S^1}$$

even though $T_{a \log N,N}$ converges in $*$ moments to random unitary!

What is going on?

Noise Stability-Nilpotent matrices

Maybe this always works?

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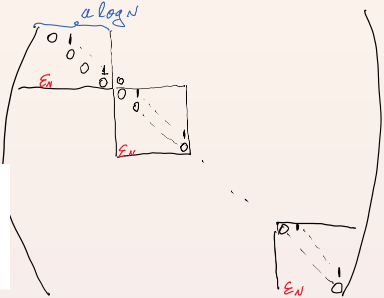

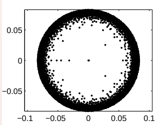

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In block: $(N^{-\gamma})^{\frac{1}{a\log N}} \simeq e^{-\gamma/a}$

$\delta_{e^{-\gamma/a}}$?

Noise Stability-Block Nilpotent

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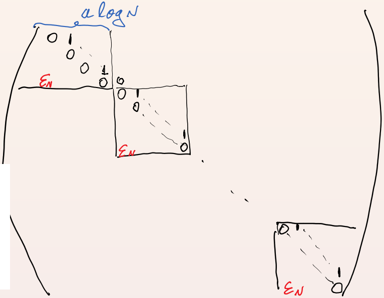

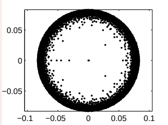

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$Q \quad \delta_{S e^{-\gamma/a}} \quad ?$

Framework: $B^i = B^i(N)$ - Jordan blocks, dimension $a_i(N)\log N$, eigenvalue $c_i(N)$.

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Simulations...

Noise Stability-Block Nilpotent III

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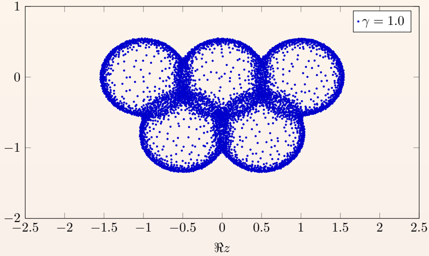
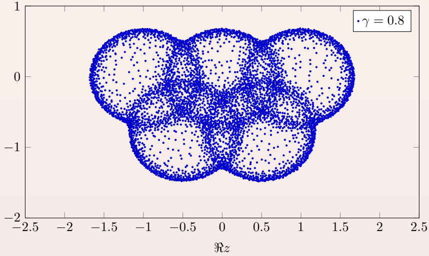
Simulations...

Noise Stability-Block Nilpotent IV

Simulations inconclusive!





Noise Stability-Block Nilpotent IV

A_N block matrix, each block of size $a_i \log N$. c_i on diagonal.
$B_N = A_N + N^{-\gamma} G_N$.
Define $r_i(N) = e^{(-\gamma+1/2)/a_i} \leq 1$. Set $\mu_N = \frac{1}{N} \sum_{i=1}^{\ell(N)} a_i \log N \nu_{c_i,r_i}$ where $\nu_{c,r}$ uniform on circle of radius r centered on c.

Theorem (Feldheim, Paquette, Z. '14)

For $\gamma > 1$ *and* $\ell(N) = o(N/\log\log(N))$,

$$d(L_N^{B_N}, \mu_N) \to_{N\to\infty} 0$$

Analogous result for $\gamma \in (1/2,]$ if collection of circles "does not spread too much" (e.g., olympics rings example OK).

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Noise Stability-Block Nilpotent V

Again, logarithmic potential plays a crucial role in the proof.
By general results, enough to show that for Lebesgue a.e. z,

$$|U_{L_N^B}(z) - U_{\mu_N}(z)| \to 0\,,$$

in probability, where $U_\nu(z) = \int \log|z-x|\nu(dx)$.
For L_N^B, $U_{L_N^B}(z) = \frac{1}{2N}\log\det(z-B_N)(z-B_N)^*$.
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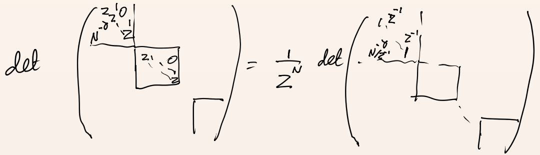
Sketch of UB: all blocks equal, $c=0$

Consider separately $|z|$ small & $|z|$ large.



Expand in minors: main minor $=1$, z^{-1} minors:

$$(z^{-1})^N \cdot (N^{-\gamma})^{\ell} \cdot \det(G)_{\ell\times\ell}$$

$$\simeq (z^{-1})^N \cdot N^{-\gamma\ell} \cdot N^{\ell/2(1+o(1))}$$

For $|z|$ small, competition between main diag & off diag – latter wins.

For $|z|$ large – 1 wins. Cutoff at radius.

LB

As in UB, perform row and column permutations and then write

$$B_N - zI = \begin{bmatrix} T + G_1 & * \\ * & G_2 \end{bmatrix},$$

We need to fight cancelations between possible contributions to the determinant. Using Schur complement,

$$\det(B_N - zI) = \det(T + G_1)\det(G_2 - C)$$

For Gaussian matrices G_2, easy to bound second determinant from below, independently of C, by height $\times$ area formula.
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