Shadowing lemmas for NHIM's and application to Arnold diffusion

Emerging interactions of geometric and variational methods

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Outline

1. Background
2. Shadowing lemmas for NHIM's
3. Perturbative results
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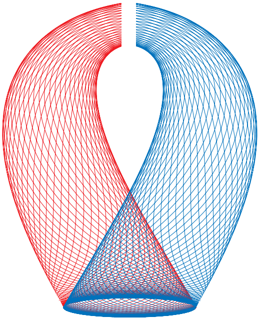
Normal hyperbolicity

- **Normally hyperbolic invariant manifold (NHIM)**:
  - $F : M \to M$, C^r-smooth, $r \geq r_0$, $m = \mathrm{dim} M$.
  - $F(\Lambda) \subset \Lambda$, $n_c = \mathrm{dim}\Lambda$.
  - $TM = T\Lambda \oplus E^u \oplus E^s$
  - $n_s = \mathrm{dim} E^s$, $n_u = \mathrm{dim} E^u$.
  - $m = n_c + n_s + n_u$
  - $\exists\ C > 0$, $0 < \lambda < \mu^{-1} < 1$, s.t. $\forall\ x \in \Lambda$
    $v \in E^s_x \Leftrightarrow \|DF^k_x(v)\| \leq C\lambda^k \|v\|, \forall k \geq 0$
    $v \in E^u_x \Leftrightarrow \|DF^k_x(v)\| \leq C\lambda^{-k} \|v\|, \forall k \leq 0$
    $v \in T_x\Lambda \Leftrightarrow \|DF^k_x(v)\| \leq C\mu^{|k|} \|v\|, \forall k \in \mathbb{Z}$

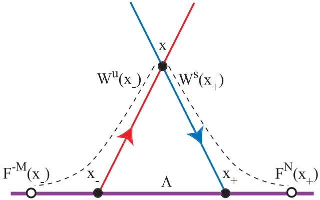


In this case $W^{u,s}(\Lambda) = \bigcup_{x \in \Lambda} W^{u,s}(x)$

Scattering map

Definition

- Assume $W^u(\Lambda)$ intersects transversally $W^s(\Lambda)$ along a homoclinic manifold Γ satisfying certain conditions
- Wave maps: $\Omega^{\pm}:\Gamma\to\Lambda$, $\Omega^{\pm}(x)=x^{\pm}\Leftrightarrow x\in W^{s,u}(x^{\pm})\cap\Gamma$
- Restrict Γ so that $\Omega^{\pm}$ diffeomorphisms
- Scattering map: $\sigma:\Omega^{-}(\Gamma)\to\Omega^{+}(\Gamma)$ given by $\sigma=\Omega^{+}\circ(\Omega^{-})^{-1}$



$\sigma(x^-)=x^+ \rightsquigarrow d(F^{-m}(x),F^{-m}(x^-))\to 0,\ d(F^n(x),F^n(x^+))\to 0$, as $m,n\to\infty$

A general Shadowing Lemma for NHIM's

Theorem 1 [Gidea, de la Llave, S.]

Given $f: M \to M$, is a C^r-map, $r \geq r_0$, $\Lambda \subseteq M$ NHIM, $\Gamma \subseteq M$ homoclinic channel. $\sigma = \sigma^\Gamma : \Omega^-(\Gamma) \to \Omega^+(\Gamma)$ is the scattering map associated to Γ. Assume that Λ and Γ are compact.

Then, for every $\delta > 0$ there exists $n^* \in \mathbb{N}$ and a family of functions $m_i^* : \mathbb{N}^{2i+1} \to \mathbb{N}$, $i \geq 0$, such that, for every pseudo-orbit $\{y_i\}_{i\geq 0}$ in Λ of the form

$$y_{i+1} = f^{m_i} \circ \sigma \circ f^{n_i}(y_i),$$

for all $i \geq 0$, with $n_i \geq n^*$ and $m_i \geq m_i^*(n_0, \ldots, n_{i-1}, n_i, m_0, \ldots, m_{i-1})$, there exists an orbit $\{z_i\}_{i\geq 0}$ of f in M such that, for all $i \geq 0$,

$$z_{i+1} = f^{m_i+n_i}(z_i), \quad \text{and} \quad d(z_i, y_i) < \delta.$$

n^* and m_i^* also depend on the angle between (W^u, W^s) along Γ

Related result: Gelfreich, Turaev Arnold Diffusion in a priori chaotic symplectic maps, Commun. Math. Phys., 2017

A general Shadowing Lemma for NHIM's: Proof

We have two proofs, one uses the topological method of correctly aligned windows.
The one we present here uses the obstruction argument.
The proof is based on the construction of a nested sequence of closed balls $B_{i+1} \subset B_i$ in a neighborhood of the first point of the pseudo-orbit y_0, such that taking $z_0 \in B_k = \bigcap_{0\leq i\leq k} B_i$ one has that

- $z_0 \in B_\delta(y_0)$
- $z_{i+1} = f^{m_i+n_i}(z_i) \in B_\delta(y_{i+1})$ for $i = 0, 1 \ldots, k$, for any $k \in \mathbb{N}$.

Moreover, taking $z_0 \in B_\infty = \bigcap_{i\geq 0} B_i \neq \emptyset$, one has that:
$z_{i+1} \in B_\delta(y_{i+1})$ for any $i \in \mathbb{N}$.

- The argument will be done by induction.
- At every step of the process we will have several choices which give us different orbits

Choice of n^*

- We will take $\delta > 0$ and consider V_Λ and V_Γ contained in neighborhoods of size δ of the compact manifolds Λ and Γ.
- We define $n^* = n^*(\delta)$ such that: given any point $p \in \Gamma$, for any $n \geq n^*$, one has that $f^{\pm n}(p) \in V_\Lambda$.
- We will give an extra condition to n^*.
- m^* will depend on the previous choices

Choice of n^*

- Assume we have $p \in \Gamma$ and let $p^-, p^+ \in \Lambda$ be the unique points for which $W^u(p^-) \cap W^s(p^+) \cap \Gamma = \{p\}$ $(\sigma(p^-) = p^+)$.
  - Let $x \in W^s(f^{-k}(p^-))$ and $B \subset B_\delta(f^{-k}(p^-))$ be any ball centered at x of fixed radius $\rho > 0$ small enough

$$B \subset V_\Lambda, \quad x \in B \cap W^s(f^{-k}(p^-)) \neq \emptyset.$$

    As $W^s(p^+)$ intersects transversally $W^u(\Lambda)$ at the homoclinic point p, by the Lambda Lemma (L. Sabbah) there exists $n^* > 0$ such that:
    if $k > n^*$, there exists a point $\bar{x} \in W^s(p^+) \cap V_\Gamma$ such that $f^{-k}(\bar{x}) \in B$.
  - The value of n_* depends on ρ (and δ), which is fixed once for all, and also on the angle of intersection of the stable and unstable manifolds of Λ along Γ.
  - By continuity, there exists a ball $V \subset V_\Gamma$ centered at $\bar{x}$ such that $f^{-k}(\bar{x}) \in f^{-k}(V) \subset B$.
  - **Remark:** The point $\bar{x}$ and its neighborhood V depend on the k we choose.

The value of n^* will be fixed from now on.

Choice of m^*

Assume that we also have $p' \in \Gamma$ and p'^-, p'^+ with the same properties as p and p^-, p^+:
$W^u(p'^-) \cap W^s(p'^+) \cap \Gamma = \{p'\}$ $(\sigma(p'^-) = p'^+)$.
and such that $f^{m+k'}(p^+) = p'^-$. Equivalently

$$p^+ = f^{-(k'+m)}(p'^-) \tag{1}$$

Take the point $\bar{x} \in W^s(p^+)$ and the ball $\bar{x} \in V \subset V_\Gamma$ centered at $\bar{x}$ previously chosen.

Choice of m^*

- We know that $f^{n^*}(\bar{x}) \in V_\Lambda \cap W^s(f^{n^*}(p^+))$, and there exists a ball U centered at $f^{n^*}(\bar{x})$ such that:

$$U \subset V_\Lambda, \; f^{-n^*}(U) \subset V \subset V_\Gamma$$

$$f^{n^*}(\bar{x}) \in U \cap W^s(f^{n^*}(p^+)) \neq \emptyset.$$

- As, $p^+ = f^{-(k'+m)}(p'^-)$, we have $f^{n^*}(p^+) = f^{-(k'+m-n^*)}(p'^-))$: $f^{n^*}(\bar{x}) \in U \cap W^s(f^{-(k'+m-n^*)}(p'^-))) \neq \emptyset.$
- Now we apply the Lambda Lemma to U; as $W^s(p'^+)$ intersects transversally $W^u(\Lambda)$ at p', if $k' + m - n^* > m^*$ big enough (depending of the size of U), there exists $\bar{x}' \in W^s(p'^+)$ such that:
  - $f^{-(k'+m-n^*)}(\bar{x}') \in U.$
- By continuity, there exists a ball centered at $\bar{x}' \in V' \subset V_\Gamma$, such that
  - $f^{-(k'+m-n^*)}(V') \subset U.$
- As $k' \geq n^*$, we can choose x' and V' satisfying: $f^{-k'}(\bar{x}) \in f^{-k'}(V') \subset B_\delta(f^{-k'}(p'^-))$,

Summarizing the construction

Given a point $x \in W^s(f^{-k}(p^-))$ and a ball B centered at x of fixed radius $\rho > 0$ small enough with the property that

$$B \subset B_\delta(f^{-k}(p^-)) \subset V_\Lambda, \quad x \in B \cap W^s(f^{-k}(p^-)) \neq \emptyset,$$

we have produced, for $k, k' \geq n^*$ and $k' + m - n^* \geq m^*$:

1. a ball $V \subset V_\Gamma$, centered at a point $\bar{x} \in W^s(p^+) \cap V_\Gamma$ such that $f^{-k}(V) \subset B$.
2. A ball $U \subset B_\delta(f^{n^*}(p^+)) \subset V_\Lambda$ centered at the point $f^{n^*}(\bar{x}) \in W^s(f^{n^*}(p^+)) \cap U$ such that $f^{-n^*}(U) \subset V$.
3. a ball $V' \subset V_\Gamma$, centered at a point $\bar{x}' \in W^s(p'^+) \cap V_\Gamma$ such that:
    - $f^{-(k'+m-n^*)}(V') \subset U$.
    - $f^{-k'}(V') \subset B_\delta(f^{k'}(p'^-))$.

$k, k' \geq n^*$, but the value of m^* depends on the size of U and $m^* > n^*$, but it is independent of the points $p, p', p^\pm, (p')^\pm$. As the balls U, V will decrease in size during the induction process, the value of m^* will increase depending of the previous iterates.

Inductive construction. First step

We construct the shadowing orbit $\{z_i\}$ once the pseudo-orbit $\{y_i\}$ is given.
Remember $y_{i+1} = f^{m_i}(\sigma(f^{n_i}(y_i)))$.
The required values of n^*, and m_i^* does not depend of the given pseudo-orbit, but only on the numbers n_i, m_j.
Fisrt step:
$p^- = f^{n_0}(y_0)$, $p^+ = \sigma(f^{n_0}(y_0))$, and $k = n_0$.

- Choose $x_0 \in W^s(y_0)$ and B_0 be any ball centered at x_0 of fixed radius $\rho > 0$ such that

$$B_0 \subset B_\delta(y_0) \subset V_\Lambda, \quad x_0 \in B_0 \cap W^s(y_0) \neq \emptyset.$$

- The previous construction gives:
  a point $\bar{x}_0 \in W^s(\sigma(f^{n_0}(y_0))) \cap V_\Gamma$ and a a ball $V_0 \subset V_\Gamma$ centered at $\bar{x}_0$ such that

$$f^{-n_0}(\bar{x}_0) \in f^{-n_0}(V_0) \subset B_0 \subset B_\delta(y_0) \subset V_\Lambda. \tag{2}$$

Inductive construction. Second step

1. We know that

$$f^{n^*}(\bar{x}_0) \in W^s(f^{n^*}(\sigma(f^{n_0}(y_0)))) \in V_\Lambda.$$

2. The previous construction gives a ball U_1 centered at $f^{n^*}(\bar{x}_0)$ such that:

$$\begin{aligned} U_1 &\subset V_\Lambda, \\ f^{n^*}(\bar{x}_0) &\in U_1 \cap W^s(f^{n^*}(\sigma(f^{n_0}(y_0)))), \\ f^{-n^*}(U_1) &\subset V_0 \subset V_\Gamma. \end{aligned} \tag{3}$$

Inductive construction. Second step

1. Recall that $y_1 = f^{m_0}(\sigma(f^{n_0}(y_0)))$, and therefore $f^{n^*}(\sigma(f^{n_0}(y_0))) = f^{-(m_0+n_1-n^*)}(f^{n_1}(y_1))$.
2. $f^{n^*}(\bar{x}_0) \in U_1 \cap W^s(f^{-(m_0+n_1-n^*)}(f^{n_1}(y_1)))$,
3. The next step is the second application of the Lambda Lemma. Now $p'^- = f^{n_1}(y_1)$, $p'^+ = \sigma(f^{n_1}(y_1))$ and $k' = n_1$. As $W^u(\Lambda)$ intersects transversally $W^s(\sigma(f^{n_1}(y_1)))$ at an homoclinic point that we will call p_1, if we take $n_1 \geq n^*$ and $m_0 > m_0^*$, where m_0^* is the value m^* given in the general step and depends on the size of U_1 and therefore on n_0, one has that:
   $m_0 + n_1 - n^* > m_0 + n^* - n^* = m_0 > m_0^* = m_0^*(n_0)$
   and there exists
   $x_1 \in W^s(\sigma(f^{n_1}(y_1)))$ and a ball V_1 centered at x_1 such that:

$$f^{-n_1}(x_1) \in f^{-n_1}(V_1) \subset B_\delta(y_1), \tag{4}$$
$$f^{-(m_0+n_1-n^*)}(x_1) \in f^{-(m_0+n_1-n^*)}(V_1) \subset U_1. \tag{5}$$

Conclusions of the first two steps of the induction process

1. If we now take $B_1 = f^{-(n_0+n_1+m_0)}(V_1)$, we have:

$$\begin{aligned} B_1 = f^{-(n_0+n_1+m_0)}(V_1) &= f^{-(n_0+n^*)} \circ f^{-(m_0+n_1-n^*)}(V_1) \\ &\subset f^{-(n_0+n^*)}(U_1) \subset f^{-n_0}(V_0) \subset B_0. \end{aligned} \tag{6}$$

Moreover, if we take $z_0 \in B_1$ it satisfies:

$$\begin{aligned} z_0 &\in B_0 \subset B_\delta(y_0), \\ f^{n_0+m_0}(z_0) &\in f^{-n_1}(V_1) \subset B_\delta(y_1). \end{aligned}$$

And we proceed by induction

Shadowing Lemma for pseudo-orbits of the scattering map

Theorem 2 [Gidea, de la Llave, S.]

$f : M \to M$ smooth map, $\Lambda \subseteq M$ is a NHIM, $\Gamma \subseteq M$ homoclinic channel and σ is the scattering map associated to Γ.
f preserves a measure μ absolutely continuous with respect to the Lebesgue measure on Λ,
σ sends positive measure sets to positive measure sets.
Let $\{x_i\}_{i=0,\ldots,n}$ be a finite pseudo-orbit of the scattering map in Λ, i.e., $x_{i+1} = \sigma(x_i)$, $i = 0, \ldots, n-1$, $n \geq 1$, that is contained in some open set $\mathcal{U} \subseteq \Lambda$ with almost every point of $\mathcal{U}$ recurrent for $f_{|\Lambda}$. (The points $\{x_i\}_{i=0,\ldots,n}$ do not have to be themselves recurrent.)
Then, for every $\delta > 0$ there exists an orbit $\{z_i\}_{i=0,\ldots,n}$ of f in M, with $z_{i+1} = f^{k_i}(z_i)$ for some $k_i > 0$, such that $d(z_i, x_i) < \delta$ for all $i = 0, \ldots, n$.

Shadowing Lemma for pseudo-orbits of the scattering map: Proof

- Choose a small open disk B_0 of x_0 in Λ, with $B_0 \subseteq \mathcal{U}$ such that $B_i := \sigma^i(B_0) \subseteq \mathcal{U}$, and $\mathrm{diam}(B_i) \leq \delta/2$, for all $i = 0, \ldots, n$.
- For the given pseudo-orbit $\{x_i\}$ of σ, with $x_{i+1} = \sigma(x_i)$, we have that $x_i \in B_i$ for all i.
- We will use Poincaré recurrence to produce a new pseudo-orbit $\{y_i\}$, with $y_{i+1} = f^{m_i} \circ \sigma \circ f^{n_i}(y_i)$, where m_i, n_i are as in previous theorem, such that $y_i \in B_i$ for all i, and hence $d(y_i, x_i) \leq \delta/2$.
- The shadowing theorem will provide us with a true orbit $\{z_i\}$ with $z_{i+1} = f^{m_i+n_i}(z_i)$, such that $d(z_i, y_i) \leq \delta/2$, hence $d(z_i, x_i) < \delta$.

First recurrence property.

- Given an open set $B \subseteq \mathcal{U} \subseteq \Lambda$, a subset $A \subseteq B$ of positive measure in B, and $k^* > 0$, consider the set $P_\tau^{k^*}(A, B)$ of points which return to B at time $k^*\tau$.
- Since μ-a.e. point in $\mathcal{U}$ is recurrent, there exists $\tau^* \geq 1$ such that $\mu(P_{\tau^*}^{k^*}(A, B)) > 0$
- Since f^{k^*} is area preserving, the set $Q_{\tau^*}^{k^*}(B, A) := f^{k^*\tau^*}(P_{\tau^*}^{k^*}(A, B)) \subseteq B$ has positive measure in B (in fact $\mu(Q_{\tau^*}^{k^*}(B, A) = \mu(P_{\tau^*}^{k^*}(A, B)) > 0$.)

In terms of f, every point in $P_{\tau^*}^{k^*}(A, B) \subseteq A \subseteq B$ will return to a point in $Q_{\tau^*}^{k^*}(A, B) \subseteq B$ in exactly $k^*\tau^* \geq k^*$ iterates.

Second recurrence property.

Consider now two open sets $B \subseteq \mathcal{U}$ and $B' = \sigma(B) \subseteq \mathcal{U}$.
Let A be a subset of B of positive measure.
By the above, $P_{T^*}^{k^*}(A, B)$ and $Q_{T^*}^{k^*}(A, B)$ are positive measure subsets of B.
Since the scattering map σ sends positive measure sets onto positive measure sets, it follows that

$$A' := \sigma(Q_{T^*}^{k^*}(A, B)) \subset B' \tag{7}$$

is a positive measure subset of B'.

Inductive construction of pseudo-orbits.

Starting with B_0, we construct inductively a nested sequence of subsets $\Sigma_i \subset B_0$ of positive measure of B_0, such that each set is carried onto a positive measure subset of B_i, $i = 1, \ldots, n$, via successive applications of some large powers of f interspersed with applications of σ.

Consder the value n^* provided by the previous theortem for $\delta/2$.

- Let $A_0 := B_0$, let $\tau_0 \geq 1$ such that $P^{n^*}_{\tau_0}(A_0, B_0) \subset A_0$ has positive measure, and

  $$\Sigma_0 := P^{n^*}_{\tau_0}(A_0, B_0) \subseteq A_0.$$

  Consider the set $Q^{n^*}_{\tau_0}(A_0, B_0) \subseteq B_0$, which has positive measure.
- Then consider the set $A'_1 := \sigma(Q^{n^*}_{\tau_0}(A_0, B_0)) \subseteq B_1$, which has positive measure in B_1.
- Let $n_0 := n^*\tau_0$ and consider the value $m^*_0 = m^*_0(n_0)$ given by previous Theorem for $\delta/2$.

  There exists $\tau'_0 \geq 1$ such that the set $P^{m^*_0}_{\tau'_0}(A'_1, B_1) \subseteq A'_1 \subseteq B_1$ has positive measure.
- Then the set $A_1 = Q^{m^*_0}_{\tau'_0}(A'_1, B_1) \subseteq B_1$ also has positive measure in B_1.

Inductive construction of pseudo-orbits.

Call $m_0 = m_0^* \tau_0'$ and $n_0 = n^* \tau_0$

- Each point $y_1 \in A_1 = Q_{\tau_0'}^{m_0^*}(A_1', B_1)$ is of the form $y_1 = f^{m_0}(x')$, for some $x' \in P_{\tau_0'}^{m^*}(A_1', B_1)$
- Such x' is of the form $x' = \sigma(x)$ for some $x \in Q_{\tau_0}^{n^*}(A_0, B_0)$; and each such x is of the form $x = f^{n_0}(y_0)$ for some $y_0 \in P_{\tau_0}^{n^*}(A_0, B_0) = \Sigma_0$ and $\tau_0 \geq 1$.
- Each $y_1 \in A_1$ can be written as

$$y_1 = f^{m_0} \circ \sigma \circ f^{n_0}(y_0)$$

  for some $y_0 \in \Sigma_0$, $n_0 \geq n^*$ and $m_0 \geq m^*$.
- Denote by Σ_1 the set of points $y_0 \in \Sigma_0$ which correspond, to some point $y_1 \in A_1$.
- We obviously have $\Sigma_1 \subseteq \Sigma_0$ and is a positive measure subset of B_0.

Proceeding by induction we will find subsets $A_j \subseteq B_j$, which have positive measure in B_j, such that each point $y_j \in A_j$ is of the form

$$y_j = f^{m_{j-1}} \circ \sigma \circ f^{n_{j-1}} \circ \ldots \circ f^{m_0} \circ \sigma \circ f^{n_0}(y_0), \tag{8}$$

some $y_0 \in A_0 \subset B_0$,
Σ_j is the set of points y_0 for which the corresponding y_j given by (8) is in A_j.
Then we have that $\Sigma_j \subseteq \Sigma_{j-1} \subseteq \ldots \subseteq \Sigma_0$, and that Σ_j is a positive measure subset of B_0.

Next step in the induction will be given by the positive measure sets:

- $P^{n^*}_{\tau_j}(A_j, B_j) \subseteq B_j$, $n_j = n^*\tau_j \geq n^*$
- $Q^{n^*}_{\tau_j}(A_j, B_j) = f^{n_j}(P^{n^*}_{\tau_j}(A_j, B_j)) \subseteq B_j$
- $A'_{j+1} := \sigma(Q^{n^*}_{\tau_j}(A_j, B_j)) \subset B_{j+1}$.
- $P^{m_j^*}_{\tau'_j}(A'_{j+1}, B_{j+1}) \subseteq A'_{j+1} \subseteq B_{j+1}$, $m_j = m_j^*\tau'_j \geq m_j^*$
- $A_{j+1} = Q^{m_j^*}_{\tau'_j}(A'_{j+1}, B_{j+1}) \subseteq B_{j+1}$

Then each point $y_{j+1} \in A_{j+1}$ is of the form

$$y_{j+1} = f^{m_j} \circ \sigma \circ f^{n_j}(y_j) \tag{9}$$

for some $y_j \in A_j$, where $n_j = n^*\tau_j \geq n^*$ and $m_j = m_j^*\tau'_j \geq m_j^*$, with $\tau_j, \tau'_j \geq 1$.

$$y_{j+1} = f^{m_j} \circ \sigma \circ f^{n_j} \circ \ldots \circ f^{m_0} \circ \sigma \circ f^{n_0}(y_0), \tag{10}$$

for some $y_0 \in \Sigma_0$, with $n_0 \geq n^*, \ldots, n_{j-1} \geq n^*$, and $m_0 \geq m_0^*, \ldots, m_j \geq m_j^*$. Denoting by Σ_{j+1} the set of points $y_0 \in \Sigma_0$ that yield these points y_{j+1}, we obtain that $\Sigma_{j+1} \subseteq \Sigma_j$ is of positive measure. This completes the induction step.

Theorem 3 [Gidea, de la Llave, S.] A Perturbative result

Given H_ε. Assume for all $0 < \varepsilon < \varepsilon_0$ there exist

- NHIM Λ_ε
- Homoclinic channel Γ_ε and corresponding scattering map $s_\varepsilon = \mathrm{Id} + \mu(\varepsilon) J\nabla S + g(\mu(\varepsilon))$, $g(\mu(\varepsilon)) = o(\mu(\varepsilon))$, and $\mu(0) = 0$ ($\mu(\varepsilon) = \varepsilon$, $g(\mu(\varepsilon)) = \varepsilon^2$ classical case)
- Suppose that $J\nabla S(x_0) \neq 0$ at some point $x_0 \in \Lambda_0$. Let $\tilde{\gamma} : [0,1] \to \Lambda_0$ be an integral curve through x_0 for the vector field $\dot{x} = J\nabla S(x)$.
- Suppose that there exists a neighborhood $\mathcal{U}$ of $\tilde{\gamma}([0,1])$ in Λ_ε such that a.e. point in $\mathcal{U}$ is recurrent for $F_{\varepsilon|\Lambda_0}$.

Then for every $\delta > 0$, there exists an orbit $\{z_i\}_{i=0,\ldots,n}$ of F_ε in M, with $n = O(\mu(\varepsilon)^{-1})$, such that for all $i = 0, \ldots, n-1$,

$$z_{i+1} = F_\varepsilon^{k_i}(z_i), \quad \text{for some } k_i > 0, \text{ and}$$

$$d(z_i, \gamma_\varepsilon(t_i)) < \delta + K(\mu(\varepsilon) + |g(\mu(\varepsilon))/\mu(\varepsilon)|), \text{ for } t_i = i \cdot \mu(\varepsilon),$$

where $0 = t_0 < t_1 < \ldots < t_n \leq 1$.

Proof of Theorem 3

Case $\mu = \varepsilon$ and $g = \varepsilon^2$. The main idea is that the scattering map is given by $s_\varepsilon = \mathrm{Id} + \varepsilon J\nabla S + O(\varepsilon^2)$ therefore, its orbits are close to the orbits obtained by applying the Euler method of step ε to the vector field

$$\dot{x} = J\nabla S(x)$$

Therefore, one can find an orbit $x_{i+1} = s_\varepsilon(x_i)$ such that

$$x_0 = \gamma(0), \quad x_{i+1} = s_\varepsilon(x_i) \in \mathcal{U} \subset \Lambda,$$

and

$$d(\gamma(t_i), x_i) < K\varepsilon, \quad i = 0, \dots, n, \ n = O(1/\varepsilon)$$

then we apply Theorem 2 to obtain an orbit $z_{i+1} = F_\varepsilon^{k_i}(z_i)$ in M, for some $k_i > 0$, s.t. $d(z_i, x_i) < \delta$ for all $i = 0, \dots, n$

A general diffusion result

Corollary [Gidea, de la Llave, S.]

Given $H_\varepsilon = H_0 + \varepsilon H_1$. Assume for all $0 < \varepsilon < \varepsilon_0$ there exist

- NHIM $\Lambda_\varepsilon = k_\varepsilon(\Lambda_0)$
- Homoclinic channel Γ_ε and corresponding scattering map σ_ε with $s_\varepsilon = \mathrm{Id} + \varepsilon J\nabla S + O(\varepsilon^2)$, where $s_\varepsilon = k_\varepsilon^{-1} \circ \sigma_\varepsilon \circ k_\varepsilon$
- $\Lambda_0 \subseteq \mathbb{R}^d \times \mathbb{T}^d \ni (I, \phi)$

If $J\nabla S(I, \phi)$ is transverse to some level set $\{I = I_*\}$ of I, then $\exists \varepsilon_1 < \varepsilon_0$, $\exists C > 0$, s.t. $\forall \varepsilon < \varepsilon_1$ $\exists x(t)$ with

$$\|I(x(T)) - I(x(0))\| > C, \text{ for some } T > 0.$$

- **Remark:**
  - There are no requirements on the inner dynamics, except of being conservative

Proof of the Corollary

- Given $J\nabla S(I,\phi)$ transverse to $\{I = I_0\}$
  $\Rightarrow J\nabla S(I,\phi)$ transverse to $\{I = I_*\}$ with $\|I_* - I_0\| < \delta$, for some $\delta > 0$ independent of ε
  $\Rightarrow$ there is a strip $\mathcal{S}$ of ϕ-size $O(1)$ consisting of trajectories of the Hamiltonian system $\dot{x} = J\nabla S(x)$ along which I changes $O(1)$.
  $\Rightarrow$ there are orbits of the map s_ε along which I changes $O(1)$
- We have two possibilities
  - There is a bounded domain through the inner dynamics, then we have Poincaré recurrence and Theorem 3 applies
  - There is diffusion using only the inner dynamics

Application

Diffusion in an a priori unstable system

$$H_\varepsilon(p,q,I,\phi,t) = \underbrace{h_0(I) + \sum_{i=1}^{n} \pm \left(\frac{1}{2}p_i^2 + V_i(q_i) \right)}_{H_0} + \varepsilon H_1(p,q,I,\phi,t;\varepsilon),$$

$$(p,q,I,\phi,t) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^d \times \mathbb{T}^d \times \mathbb{T}^1$$

Theorem 4 [Gidea, de la Llave, S.]

Under the earlier assumptions,
there exists $\varepsilon_0 > 0$, and $C > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, there exists a trajectory $x(t)$ such that

$$\|I(x(T)) - I(x(0))\| > C \text{ for some } T > 0.$$

- We make no asumptions on the dynamics of h_0. No need of KAM tori, Aubry Mather sets etc, do not require any property on $\partial^2 h_0/\partial I^2 \neq 0$
- No convexity of the unperturbed Hamiltonian; the argument works even if $\partial^2 h_0/\partial I^2$ degenerate or non-positive definite (e.g., non-twist maps)
- We allow strong resonances etc.
- Any dimension.
- Works for perturbations in an open and dense set satisfying explicit non-degeneracy conditions

Proofs

- **Proof of Theorem 4:**
- Penduli $\rightsquigarrow$ homoclinic orbit $(p_i^0(\sigma), q_i^0(\sigma))$ to $(0,0)$
- Let
$$L(\tau, I, \phi, s) = -\int_{-\infty}^{\infty} \Big[H_1(p^0(\tau+\sigma), q^0(\tau+\sigma), I, \phi+\omega(I)\sigma, s+\sigma; 0) \\ -H_1(0,0,I,\phi+\omega(I)\sigma, s+\sigma; 0)\Big]\, dt$$
- For generic H_1, the equation $\frac{\partial}{\partial\tau} L(\tau, I, \phi, s) = 0$ has a non degenerate solution $\tau = \tau^*(I, \phi, s)$
- Define $\mathcal{L}(I, \phi, s) = L(\tau^*(I, \phi, s), I, \phi, s)$ and $\mathcal{L}^*(I, \theta) = \mathcal{L}(I, \theta, 0)$
- $s_\varepsilon(I, \phi) = \mathrm{Id}(I, \phi) + \varepsilon J \nabla \mathcal{L}^*(I, \phi - \omega(I)s) + O(\varepsilon^2)$
- For generic H_1, $\nabla \mathcal{L}^*$ is transverse to some level set $\{I = I_0\}$
- Apply Theorem 3 and Corollary.