# Some Boundary States For Bosons

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According to inSpire, I've written 18 papers with Nati – possibly his second most frequent co-author, after Michael Dine.

Three particularly memorable topics might be

(*) Worldsheet instantons, with Dine and X.-G. Wen (1986-7)

(**) Effective action of $N = 2$ Super Yang-Mills Theory (1994)

(***) String Theory And Noncommutative Geometry (1999)

Rather than reminiscences, it seemed more appropriate today to talk about something current. Nat and I have written two papers this year:

(*) Gapped Boundary States Of Topological Insulators via Weak Coupling, arXiv:1602.04251

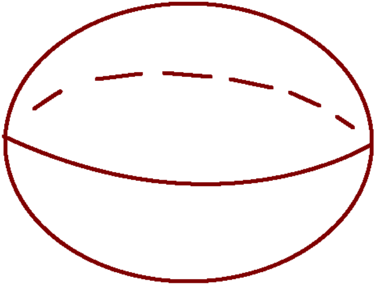
(**) A Duality Web In $2+1$ Dimensions And Condensed Matter Physics, with T. Senthil and C. Wang, arXiv:1606.01989.

Nati has given many talks on the second paper and does it much better than I could, so I wouldn't seriously consider talking about that one. I considered talking about the first paper. It would make a nice talk, which I haven't really given so it would be new. But there was one problem – I couldn't see how to make such a talk interesting for Nati.

So instead I will describe more recent work that in a sense deals with the same problem that we studied in that paper, but for bosons rather than fermions. The idea was described in an abstract way in my paper arXiv:1605.02391, and I am currently working with Juven Wang and Xiao-Gang Wen to understand it better and make it more concrete.

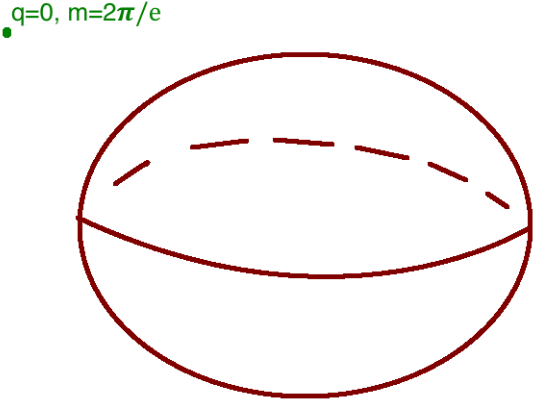
First of all, in very general terms, the problem is as follows. One has a material of some kind – possibly a topological insulator – which has a certain global symmetry group. In the case of the topological insulator, the relevant symmetries are time-reversal symmetry $T$ and conservation of electric charge.

The bulk of the system is gapped

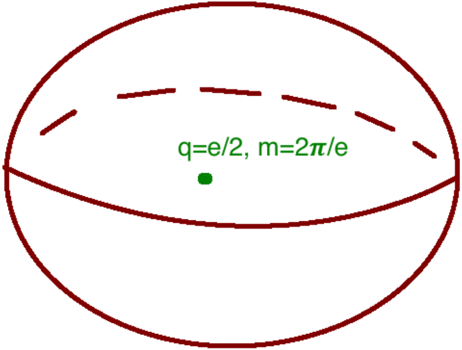


but it is not a “trivial” gapped phase. What is nontrivial about it is that it responds in an unusual way to some external perturbation.

In the case of a topological insulator, what is unusual is that an external magnetic monopole, if it is moved from vacuum into the material, has its electric charge shifted by 1/2. Here the monopole is outside the topological insulator:



In the case of a topological insulator, what is unusual is that an external magnetic monopole, if it is moved from vacuum into the material, has its electric charge shifted by 1/2. And here it is inside:



Obviously, when the magnetic monopole is moved inside the material, an electric charge $\pm e/2$ is deposited on the surface. But what kind of material can support boundary excitations of electric charge $\pm e/2$?

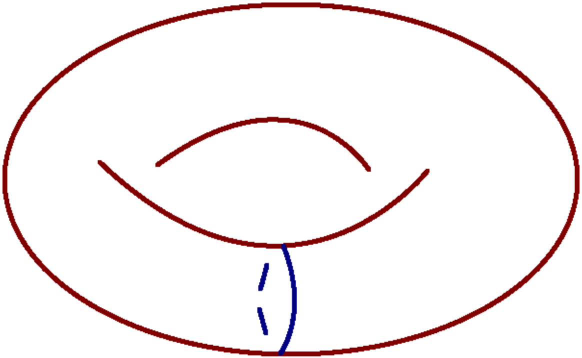
One option – realized in the lab – is that the boundary supports gapless fermions. When there is a net magnetic flux passing through the surface, those gapless fermions have zero-modes. Quantization of the zero-modes leads to states of electric charge $\pm e/2$, as shown long ago by Jackiw and Rebbi in another context.

It is also possible for the symmetry to be explicitly or spontaneously broken on the boundary. For example, in the case of the topological insulator, if we did not have electric charge conservation, there would be no issue. (It is also true that if we did not have time-reversal symmetry, there would be no issue, but this takes longer to explain.)

These phases are said to be "symmetry-protected" – they have subtle properties but the subtlety disappears if one is allowed to violate the symmetries.

Keeping the symmetries, it is also possible for the boundary to be gapped. But in this case, when the magnetic monopole passes through the surface, a quasiparticle of electric charge $\pm e/2$ is created. It turns out that in a theory that microscopically has integer charges only, to have a gapped phase with fractionally charged quasiparticles, that phase must actually be described by a nontrivial topological field theory.

Very roughly, that is because otherwise, one cannot make sense of an amplitude in which a charge $e/2$ quasiparticle goes around a noncontractible loop:

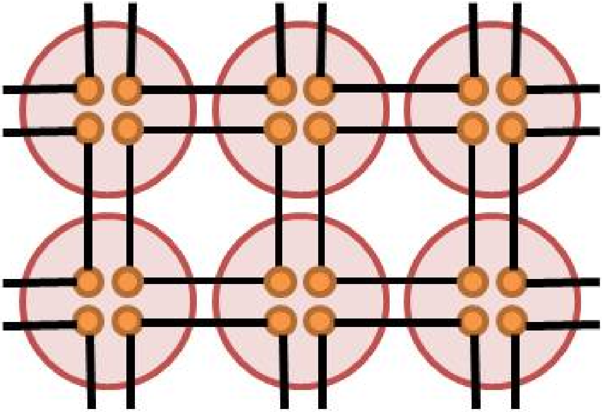


The problem that Nati and I treated in the first of our recent papers was to describe gapped and symmetry-preserving but topologically nontrivial boundary states of a topological insulator or superconductor. (There were a number of previous papers, e.g. Metlitski, Kane, and Fisher (2013), Chen, Fidkowski, and Vishwanath (2013), Wang, Potter, and Senthil (2013), Mross, Essin, and Alicea (2014).)

Instead, today I will talk about analogous boundary states for gapped phases of bosons that are "symmetry-protected topological (SPT)," meaning that the bulk theory realizes some global symmetry group $G$ in an unusual fashion that forces unusual boundary behavior. There is a systematic approach to such phases (Chen, Gu, Liu and Wen (2013)) using group cohomology as in the study of Chern-Simons theory of a finite group (Dijkgraaf and EW (1990)). However, the formulas are a little abstract and it is difficult to make them understandable on the fly.

In some simple cases there are very concrete constructions of these phases, and in particular today I will follow the approach of Chen, Liu, and Wen arXiv:1106.4752 of a very simple model (the "CZX model") with global symmetry $\mathbb{Z}_2$. This actually comes at a certain cost, because we will be in dimension $2+1$ and some things of importance in condensed matter really only behave generically in dimension $\geq 3+1$. In the long run, the general approach is more powerful, but it is nice to have an expiicit intuitive example. (In my work with J. Wang and X.-G. Wen, we use the abstract general approach as well as the concrete one that I follow today. An example of a gapped boundary state for bosons was described by Chen, Burnell, Vishwanath, and Fidkowski (2014).)

The model is on a 2d square lattice. Each lattice site (represented below by a large disc) contains 4 qubits (corresponding to the small darker discs)



We denote the states of a qubit as $|0\rangle$ and $|1\rangle$. There is a $\mathbb{Z}_2$ symmetry that acts on the four qubits on a site, but the way that it acts is a little subtle. Let $X_i$ be the operator that flips the $i^{th}$ qubit:

$$X_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The obvious $\mathbb{Z}_2$ symmetry for four qubits on a site is

$$X^* = X_1 X_2 X_3 X_4.$$

However in the CZX model, this is combined with signs.

We define the operator

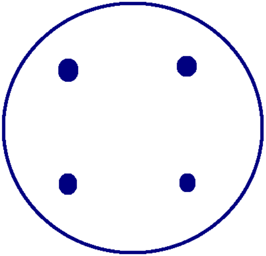
$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for any qubit (in the basis $|1\rangle,\ |0\rangle$). If $i$ and $j$ are any two qubits, we define the "controlled $Z$" operator as

$$CZ_{ij}|1\,1\rangle = -|1\,1\rangle, \quad CZ_{ij}|a\,b\rangle = +|a, b\rangle \text{ if } |a, b\rangle \neq |1, 1\rangle.$$

In other words if the first spin is $|1\rangle$, we measure $-Z$ of the second spin, and otherwise we do nothing.

For four spins in cyclic order



we define the "total CZ" as

$$CZ^* = CZ_{12} CZ_{23} CZ_{34} CZ_{41}.$$

Thus $CZ^*$ gives a minus sign for every pair of adjacent spins that are both in state $|11\rangle$.

The $\mathbb{Z}_2$ generator for four qubits on a site is now defined as the "total CZX":

$$CZX^* = X^* CZ^*.$$

To check that $(CZX^*)^2 = 1$ on all states, one has to look at a few special cases, for example acting with $X^*$ exchanges

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and these states both have $CZ^* = +1$. Another example is

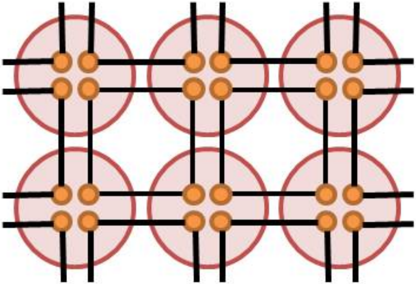
$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

where both have $CZ^* = -1$.

In all cases $(CZX^*)^2 = (X^* CZ^*)^2 = +1$ because $X^*$ exchanges two states with the same eigenvalues of $CZ^*$.

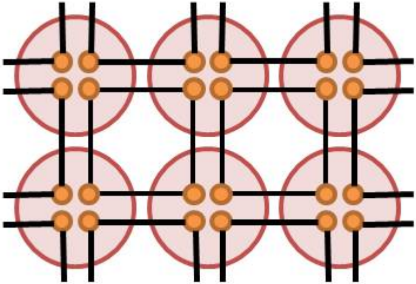
What we have done so far is not useful by itself. By making a unitary transformation of the 4 qubits, we could put the $\mathbb{Z}_2$ action in a standard form. But this would make it hard to describe the entanglement pattern of the ground state, which we will come to shortly.

Before going on, I want to mention the following. Although the $\mathbb{Z}_2$ action was described in a slightly nonstandard fashion, it was "on-site," meaning that the qubits at any one lattice site



transform separately under their own $\mathbb{Z}_2$ symmetry. Any on-site global symmetry of a lattice system is anomaly-free, in the sense that it can be gauged – that is, any lattice system with the symmetry that I have described can be coupled to $\mathbb{Z}_2$ lattice gauge fields. Non-on-site symmetries can be anomalous and ungaugeable.

Now, the CZX model of Chen, Liu and Wen has short-range entanglement and in particular is gapped. But the entanglement is naturally described in terms of “plaquettes,” drawn in the figure as squares, rather than “sites” (the large shaded discs):



It is easiest to first describe the ground state wavefunction before describing the $\mathbb{Z}_2$-invariant Hamiltonian that has this ground state. The ground state wavefunction is

$$\Psi = \prod_{\text{plaquettes}} \frac{1}{\sqrt{2}} \left( |0000\rangle + |1111\rangle \right).$$

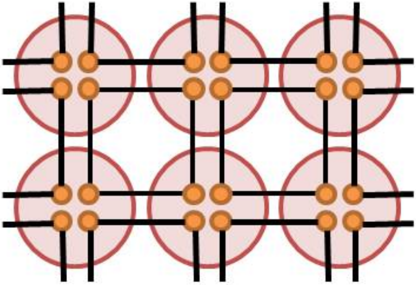
Thus the four qubits in any one plaquette are in a highly entangled state, but there is no entanglement between spins that are not in the same plaquette.

The most obvious Hamiltonian that has $\Psi$ for a ground state is

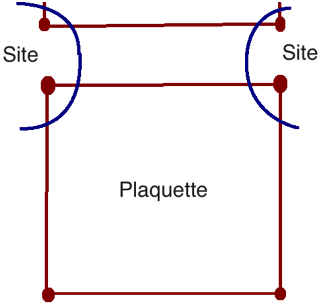
$$H_0 = -\sum_{\text{plaquettes}} \left( |0000\rangle\langle 1111| + |1111\rangle\langle 0000| \right),$$

in other words the sum over plaquettes of the projector onto the state $\frac{1}{\sqrt{2}}\left(|0000\rangle + |1111\rangle\right)$ on each plaquette. The Hamiltonian $H_0$ would have the obvious $\mathbb{Z}_2$ symmetry $X^*$, but it does not have the more subtle $CZX^*$ symmetry.

That is because if we flip the four spins in a plaquette, we can change the sign of an odd number of the $CZ$ operators of adjacent spins in a site:



What will save the day will be if we only flip the spins in a given plaquette if pairs of spins in the next plaquette are equal:




For each plaquette $\mathcal{P}$, let $\Pi_{\mathcal{P}}$ be the projector onto states that obey that condition. Then the Hamiltonian

$$H = -\sum_{\mathcal{P}} \left(|0000\rangle\langle 1111| + |1111\rangle\langle 0000|\right)_{\mathcal{P}} \Pi_{\mathcal{P}}$$

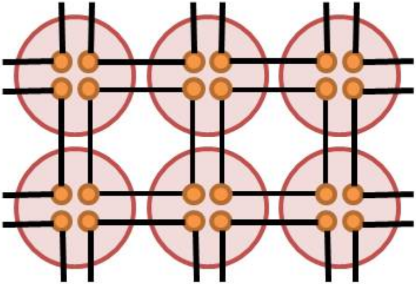
commutes with $CZX^*$ and has $\Psi$ for its ground state.

This is the *CZX* model. The symmetry action is trivial in the sense that it is on-site, and the entanglement pattern of the ground state (or any other eigenstate of this Hamiltonian) is also trivial in the sense that it has a very short range. In fact, if we ignore the $\mathbb{Z}_2$ global symmetry, we could declare the four qubits in a plaquette (rather than in a disc) to represent a single site and then the wavefunction would be a trivial product wavefunction – that is, a product of one-site wavefunctions.

But there is a mismatch between the symmetry pattern and the entanglement pattern. They cannot be both made trivial at once and in fact the model is in a topologically nontrivial phase of gapped systems with $\mathbb{Z}_2$ global symmetry.

We see what is nontrivial most directly if we look at the behavior near the boundary. Here we have to face the fact that there are different possible boundary states. I will first consider the original boundary state of Chen, Liu, and Wen (CLW) and then I will describe an alternative that I've been looking at with Wang and Wen.

If we consider a system with an integer number of sites, all of the type I've described, we run into the fact that near the boundary we have some incomplete plaquettes:

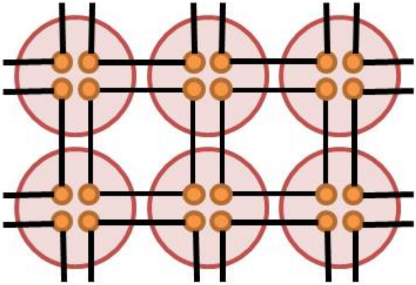


What CLW did was to define the Hamiltonian as a sum over *complete* plaquettes

$$H = -\sum_{\mathcal{P}-\text{complete}} \left( |0000\rangle\langle 1111| + |1111\rangle\langle 0000| \right)_{\mathcal{P}} \Pi_{\mathcal{P}}$$

This makes sense and is $\mathbb{Z}_2$-invariant, but it does not give a gapped system because the spins on the boundary are not fully constrained.

The low energy physics is interesting. Let us look at the bottom row consisting of qubits that are not in complete plaquettes:



Those spins are not completely free to fluctuate because to minimize the energy, the projectors $\Pi_{\mathcal{P}}$ must all equal 1. Each pair of boundary spins that are in a "partial plaquette" appear in one of those projectors and they must be equal to minimize the energy. Thus for an effective description, we can use "composite qubits" with one qubit for every partial plaquette on the boundary.

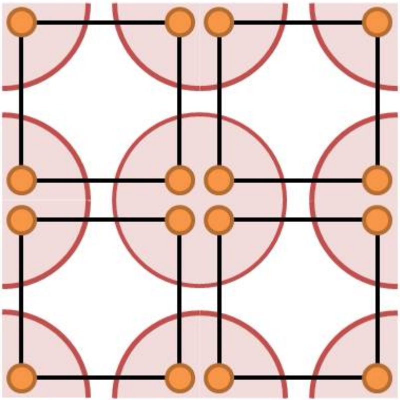
There is an effective $\mathbb{Z}_2$ symmetry for these composite qubits, but it is *not on-site*. The $\mathbb{Z}_2$ generator is

$$CZX' = \prod_i X_i \prod_j CZ_{j\ j+1}$$

with a $CZ$ operator for each adjacent pair of composite qubits. It is not hard to show that $(CZX')^2 = 1$, so $CZX'$ generates a $\mathbb{Z}_2$ symmetry. But it is not onsite because every adjacent pair of composite qubits is linked by one of the CZ factors. No matter how we combine composite qubits into sites, the symmetry is not on-site.

It can be shown that a chain of qubits with this $\mathbb{Z}_2$ symmetry cannot be gapped. However it is possible to remove most of the $2^N$-fold ground state degeneracy that we have so far ($N$ being the number of composite qubits) with a suitable perturbation. One can flow to a $c = 1$ system with the $\mathbb{Z}_2$ acting as a discrete chiral symmetry, preventing the system from being gappable.

I cannot say more about this boundary state as I want to describe the alternative boundary state that I've been looking at lately, with Wang and Wen. To construct it, the first step is to just throw away all of the boundary spins that are not in full plaquettes. Thus on the boundary we have partial sites, i.e. some sites have only 2 qubits, not 4:



With the same Hamiltonian as before, the system is now gapped. Moreover, it has the symmetry $CZX^*$ if we define this symmetry in a fairly obvious way for partial sites. For a boundary site $i$ with two spins $a, b$, we define

$$CZX_i^* = X_a X_b CZ_{ab}.$$

Provided we include the $CZ_{ab}$ factor here, the total $CZX^*$ (defined as a product over all bulk and boundary sites) commutes with the Hamiltonian and generates a global symmetry.

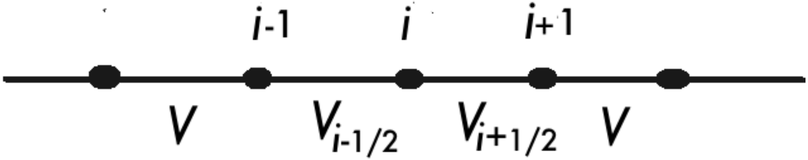
So what goes wrong? The Hamiltonian is gapped and has $CZX^*$ symmetry. But it is no longer true that $(CZX^*)^2 = 1$. For a boundary site $i$ containing two qubits $a, b$, it turns out that $W_i = (CZX_i^*)^2$ is equal to $+1$ if qubits $a, b$ are opposite ($|01\rangle$ or $|10\rangle$) and $-1$ if they are equal. Thus $W_i^2 = 1$. Set $W^* = \prod_i W_i$, where the product is over all boundary sites. The global symmetry of the system is now $\mathbb{Z}_4$, since $(CZX^*)^2 = W^*$, where $W^* \neq 1$ but $(W^*)^2 = 1$. Here $W^*$ is a $\mathbb{Z}_2$ symmetry that only acts on the boundary.

We have found a system with an emergent $\mathbb{Z}_2$ global symmetry on the boundary. The bulk symmetry is $\mathbb{Z}_2$ generated by $CZX^*$, but along the boundary the symmetry is enhanced to $\mathbb{Z}_4$, with a $\mathbb{Z}_2$ subgroup generated by $W^* = (CZX^*)^2$ that only acts on boundary spins.

Is this physically sensible? It is common in condensed matter physics to find emergent global symmetries in the infrared, but they are always *approximate* symmetries, explicitly broken by irrelevant interactions. In the present situation, the emergent symmetry will have to be *exact*, since it is generated by $W^* = (CZX^*)^2$, where (by hypothesis) $CZX^*$ is an exact, microscopic $\mathbb{Z}_2$ symmetry generator. In other words, we could not explicitly break the symmetry generated by $W^*$ without also breaking the symmetry generated by $CZX^*$.

The only way out is to *gauge* the $\mathbb{Z}_2$ symmetry generated by $W^*$. In other words, the full symmetry group is $\mathbb{Z}_4$ generated by $CZX^*$, and we are going to gauge the $\mathbb{Z}_2$ subgroup generated by $W^* = (CZX^*)^2$. Note that in condensed matter physics there is no problem to have emergent *gauge* symmetries that are exact – the canonical example is the fractional quantum Hall effect.

Gauging the boundary symmetry generated by $W^*$ is actually no problem, because this symmetry is on-site. We introduce a $\mathbb{Z}_2$ gauge field only on the boundary. On each link connecting two boundary sites we have a $\mathbb{Z}_2$-valued field $V = \pm 1$ that represents the parallel transport between two sites:



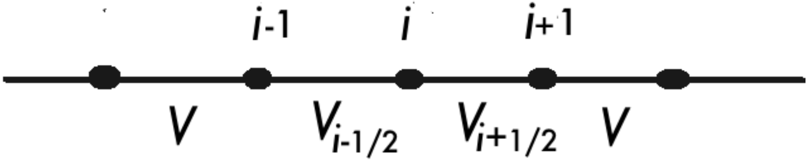
We can think of $V$ as an operator that acts on a new qubit:

$$V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad V^2 = 1.$$

We also need a discrete electric field $E$ that flips $V$:

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E^2 = 1, \quad EV = -VE.$$

Now we need to define the Gauss law constraint. A gauge transformation at boundary site $i$ acts by $W_i$ on the matter fields at that site, but it also flips the gauge fields at links $i - 1/2$ and $i + 1/2$ (i.e. the links that connect to site $i$ on the left or the right):



So the constraint operator at site $i$ is

$$\Lambda_i = E_{i-1/2} W_i E_{i+1/2}.$$

A physical state $|\Psi\rangle$ has to obey

$$\Lambda_i |\Psi\rangle = |\Psi\rangle$$

for all $i$. But the product of all the constraint operators is the generator $W^*$ of the emergent $\mathbb{Z}_2$, since $E^2 = 1$ for all boundary sites:

$$\prod_i \Lambda_i = \dots (E_{i-1/2} W_i E_{i+1/2})(E_{i+1/2} W_{i+1} W_{i+3/2}) = \prod_i W_i = W^*.$$

Thus a physical state is invariant under $W^*$, as a result of which the symmetry that acts on physical states collapses from $\mathbb{Z}_4$ to the original $\mathbb{Z}_2$.

This is not the end, however, because we need to discuss the origin of the $\mathbb{Z}_2$ gauge symmetry that was an important part of the construction. In condensed matter physics, there are no microscopic gauge symmetries other than electromagnetism, and any other gauge symmetry is "emergent." The following precise definition of an emergent gauge symmetry was suggested to me by Wen. In condensed matter physics, not only are the microscopic symmetries on-site, but the Hilbert space is on-site, meaning that the full Hilbert space $\mathcal{H}$ is a tensor product

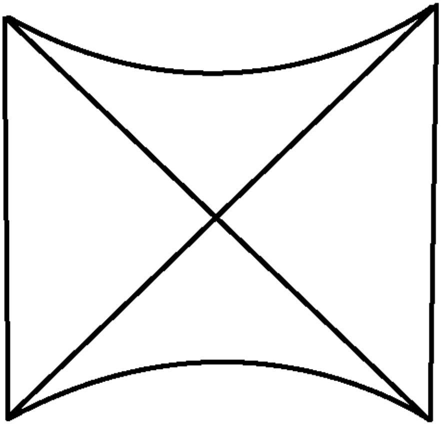
$$\mathcal{H} = \otimes_i \mathcal{H}_i,$$

where $\mathcal{H}_i$ is a Hilbert space associated to the $i^{th}$ site. (By the way, the definition of a bosonic system is that this is an ordinary tensor product rather than a $\mathbb{Z}_2$-graded tensor product.) By contrast, in conventional lattice gauge theory, as I've formulated it, the Hilbert space is not on-site: we introduced a qubit associated to a link rather than to a point.

An emergent gauge theory is a theory that microscopically has an on-site Hilbert space but at long distances can be described by a gauge theory. In $1+1$ dimensions, the $\mathbb{CP}^n$ model (D'Adda, DiVecchia, and Luscher 1978, EW 1978) gives a relatively well-known model of emergent $U(1)$ gauge symmetry. It can be easily modified to the $\mathbb{RP}^n$ model to get an a model of emergent $\mathbb{Z}_2$ gauge symmetry, which one can use in our problem. (Condensed matter physicists have other tricks to describe lattice models with emergent gauge symmetry.)

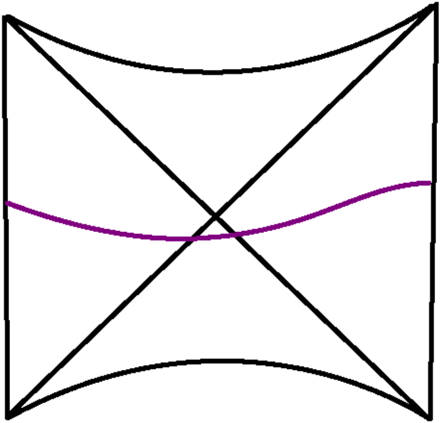
I was originally intending to describe the $\mathbb{RP}^n$ model in this talk, but when I planned the talk in detail, I realized that there would not be time. Instead I want to end by pointing out a recent development in string theory/quantum gravity that has an obvious analogy with what I've just told you.

This involves the eternal, two-sided black hole in Anti de Sitter space:



The spacetime is connected, even though the Hilbert space is supposed to be a simple tensor product $\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_r$ of spaces $\mathcal{H}_\ell$, $\mathcal{H}_r$ that can be defined on the left or right boundary.

This is puzzling, and the puzzle was recently sharpened by Harlow (arXiv:1510.07911, also Harlow and Ooguri to appear) who considered a case in which the AdS radius is large (so effective field theory can be used) and there is a gauge field in AdS. Then one can consider gauge-invariant Wilson operators that link the two sides:



This is a puzzle, because if $\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_r$, then any operator on $\mathcal{H}$ is the sum of products of operators on $\mathcal{H}_\ell$ and operators on $\mathcal{H}_r$. To the naked eye, the Wilson operator that connects the two sides appears not to have this property. To restore factorization, it was necessary to assume that any gauge field in AdS, i.e. in a world of negative cosmological constant, is emergent. By extension, one expects that this is true also if the cosmological constant is zero or positive.

Thus the factorization of the black hole Hilbert space as $\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_r$ leads to the same conclusion in quantum gravity that the on-site nature of the microscopic Hilbert space plays in condensed matter: it forces any gauge fields to be emergent. With the help of an emergent $\mathbb{Z}_2$ gauge field, one can, finally, complete the construction of a gapped symmetry-preserving boundary state for the CZX model.