ON THE MATHEMATICAL THEORY OF BLACK HOLES

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TESTS OF REALITY $\quad \mathbf{Ric}(g) = 0$

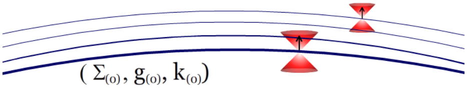
1. **RIGIDITY.** Does the Kerr family $\mathcal{K}(a, m)$, $0 \le a \le m$, **exhaust** all possible vacuum black holes ?

2. **STABILITY.** Is the Kerr family **stable** under arbitrary small perturbations ?

3. **COLLAPSE.** Can black holes form starting from **reasonable** initial data configurations ? Formation of **trapped** surfaces.

INITIAL VALUE PROBLEM: J. Leray, Y. C. Bruhat(1952)

Ric(g)=0



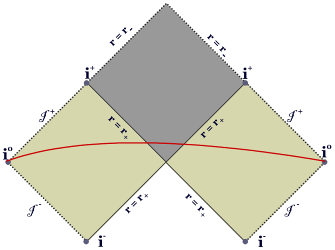
LECTURE II

1. GENERAL STABILITY PROBLEM FOR STATIONARY SOLUTIONS
2. QUANTITATIVE DEFINITION OF LINEAR STABILITY
3. VECTORFIELD METHOD.
4. STABILITY OF MINKOWSKI SPACE
5. GEOMETRIC FRAMEWORK
6. PRESENT STATE OF UNDERSTANDING

II. STABILITY

CONJECTURE[Stability of (external) Kerr].

Small perturbations of a given exterior Kerr ($\mathcal{K}(a, m)$, $|a| < m$) initial conditions have max. future developments converging to **another** Kerr solution $\mathcal{K}(a_f, m_f)$.



GENERAL STABILITY PROBLEM $\mathcal{N}[\phi] = 0.$

NONLINEAR EQUATIONS. $\mathcal{N}[\phi_0 + \psi] = 0, \ \mathcal{N}[\Phi_0] = 0.$

1. ORBITAL STABILITY(OS). ψ bounded for all time.
2. ASYMPT STABILITY(AS). $\psi \longrightarrow 0$ as $t \to \infty$.

LINEARIZED EQUATIONS. $\mathcal{N}'[\phi_0]\psi = 0.$

1. MODE STABILITY (MS). No growing modes.
2. BOUNDEDNESS.
3. QUANTITATIVE DECAY.

GENERAL STABILITY PROBLEM $\mathcal{N}[\phi_0] = 0$

STATIONARY CASE. Possible instabilities for $\mathcal{N}'[\phi_0]\psi = 0$:

- Family of stationary solutions ϕ_λ, $\lambda \in (-\epsilon, \epsilon)$,

$$\mathcal{N}[\phi_\lambda] = 0 \Longrightarrow \mathcal{N}'[\phi_0](\frac{d}{d\lambda}\Phi_\lambda)|_{\lambda=0} = 0.$$

- Mappings $\Psi_\lambda : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$, $\Psi_0 = I$ taking solutions to solutions.

$$\mathcal{N}[\phi_0 \circ \Psi_\lambda] = 0 \Longrightarrow \mathcal{N}'[\phi_0]\frac{d}{d\lambda}(\phi_0 \circ \Psi_\lambda)|_{\lambda=0} = 0.$$

- Intrinsic instability of ϕ_0. Negative eigenvalues of $\mathcal{N}'(\phi_0)$.

GENERAL STABILITY PROBLEM $\mathcal{N}[\phi_0] = 0$

STATIONARY CASE. Expected linear instabilities due to non-decaying states in the kernel of $\mathcal{N}'[\phi_0]$:

1. Presence of continuous family of stationary solutions ϕ_λ implies that the final state ϕ_f may differ from initial state ϕ_0
2. Presence of a continuous family of invariant diffeomorphism requires us to track dynamically the gauge condition to insure decay of solutions towards the final state.

QUANTITATIVE LINEAR STABILITY. After accounting for (1) and (2), all solutions of $\mathcal{N}'[\phi_0]\psi = 0$ decay sufficiently fast.

MODULATION. Method of constructing solutions to the nonlinear problem by tracking (1) and (2).

STABILITY OF SIMPLEST SOLUTIONS- NWE

$$\Box\Phi = (\partial_t\Phi)^2, \qquad \Phi|_{t=0} = \phi_0, \quad \partial_t\Phi|_{t=0} = \phi_1$$

ENERGY NORM. $E_0[\Phi](t) := \int_{\Sigma_t} |\partial\Phi|^2,$

HIGHER ENERGY. $E_s[\Phi](t).$

To bound $E_s[\Phi]$ we need to control $\int_0^t \|\partial_t\Phi\|_{L^\infty(\Sigma(\tau))} d\tau.$

DECAY. Need integrable decay rates for $\|\partial_t\Phi\|_{L^\infty}$

FACT $\Box\Phi = 0$ we have $\|\partial_t\Phi\|_{L^\infty(\Sigma(t))} \lesssim t^{-\frac{n-1}{2}}$

STABILITY OF SIMPLEST SOLUTIONS- NWE

GENERALIZED ENERGY. **REDEFINE**

$$E_s[\Phi](t) := \sum_{0 \le i \le s} \sum_{X_1, \ldots X_i} E_0[X_1 \ldots X_i \phi](t)$$

vectorfields $X_1, \ldots X_s$ are Killing or conformal Killing.

GLOBAL SOBOLEV. If $s > \frac{n}{2}$ and $E_s[\Phi](t)$ is bounded for $t \ge 0$

$$|\partial \phi(t, x)| \lesssim (1 + t + |x|)^{-\frac{n-1}{2}} (1 + \big||t| - |x|\big|)^{-\frac{1}{2}}$$

PEELING. Relative to the null frame

$$L = \partial_t + \partial_r, \quad \underline{L} = \partial_t - \partial_r, \quad e_A$$

every successive derivative of Φ in L, e_A improves the rate of decay in the wave zone $t \sim |x|$.

STABILITY OF SIMPLEST SOLUTIONS- NWE

$$\Box\phi = F(\phi, \partial\phi, \partial^2\phi) \qquad \text{in} \quad \mathbb{R}^{1+3}.$$

FACT. The trivial solution $\Phi = 0$,

- is unstable for most equations $\qquad \Box\Phi = (\partial_t\Phi)^2$
- is stable if a structural condition on F, **null condition**, is verified.

NULL CONDITION. Typically, nonlinear wave equations derived from a geometric Lagrangian, verify some version (**gauge dependent**) of the null condition.

VECTORFIELD METHOD

Use of well adapted vectorfields, related to

1. Symmetries,
2. Approximate, symmetries,
3. Other geometric features

of specific linear and nonlinear wave equations to derive generalized energy bounds (L^2) and quantitative decay (L^∞) for its solutions.

It applies to tensorfield equations such as Maxwell and Bianchi type equations,

$$dF = 0, \qquad \delta F = 0.$$

and nonlinear versions such as Yang-Mills, EVE etc..

STABILITY OF MINKOWSKI SPACE

THEOREM[Global Stability of Minkowski] Any asymptotically flat initial data set which is sufficiently close to the trivial one has a regular, complete, maximal development. Christodoulou-K

(I) BIANCHI IDENTITIES.
Effective, *invariant*, way to treat the hyperbolic character of the equations.

(II) DECAY OF PERTURBATIONS.
Perturbations radiate and decay *sufficiently fast* (just fast enough !) to insure convergence.

(III) VECTORFIELD METHOD. Construct approximate Killing and conformal Killing fields based on two foliations induced by

- optical function u
- time function t.

GEOMETRIC FRAMEWORK

1. Principal Null Directions e_3, e_4.
2. Horizontal Structure. Null Frames.
3. Null decompositions
   - Connection $\Gamma = \{\chi, \xi, \eta, \ \zeta, \underline{\eta}, \omega, \underline{\xi}, \underline{\omega}\}$
   - Curvature $R = \{\alpha, \beta, \rho, {}^\star\rho, \underline{\beta}, \underline{\alpha}\}$
4. $O(\epsilon)$ -Perturbations
5. $O(\epsilon)$ -Frame Transformations. Invariant quantities.
6. Main Equations

KERR FAMILY $\mathcal{K}(a, m)$

BOYER-LINDQUIST (t, r, θ, φ).

$$-\frac{\rho^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{\rho^2}\left(d\varphi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2(d\theta)^2,$$

$$\begin{cases} \Delta = r^2 + a^2 - 2mr; \\ \rho^2 = r^2 + a^2(\cos\theta)^2; \\ \Sigma^2 = (r^2 + a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases}$$

STATIONARY, AXISYMMETRIC. $\mathbf{T} = \partial_t$, $\mathbf{Z} = \partial_\varphi$

PRINCIPAL NULL DIRECTIONS.

$$\begin{aligned} e_3 &= \frac{r^2 + a^2}{q\sqrt{\Delta}}\partial_t - \frac{\sqrt{\Delta}}{q}\partial_r + \frac{a}{q\sqrt{\Delta}}\partial_\varphi \\ e_4 &= \frac{r^2 + a^2}{q\sqrt{\Delta}}\partial_t + \frac{\sqrt{\Delta}}{q}\partial_r + \frac{a}{q\sqrt{\Delta}}\partial_\varphi. \end{aligned}$$

BASIC QUANTITIES

NULL FRAME $e_3, e_4, (e_a)_{a=1,2},$ $S = \text{span}\{e_1, e_2\}$

CONNECTION COEFFICIENTS. $\chi,\ \xi,\ \eta,\ \zeta,\ \underline{\eta},\ \omega,\ \underline{\xi},\ \underline{\omega}$

$$
\begin{aligned}
\chi_{ab} &= \mathbf{g}(D_a e_4, e_b), \quad \xi_a = \frac{1}{2} g(D_4 e_4, e_a), \quad \eta_a = \frac{1}{2}\mathbf{g}(e_a, D_3 e_4),\\
\zeta_a &= \frac{1}{2}\mathbf{g}(D_a e_4, e_3), \quad \omega = \frac{1}{4}\mathbf{g}(\mathbf{D}_4 e_4, e_3) \ldots
\end{aligned}
$$

CURVATURE COMPONENTS. $\alpha,\ \beta,\ \rho,\ {}^\star\rho,\ \underline{\beta},\ \underline{\alpha}$

$$
\begin{aligned}
\alpha_{ab} &= \mathbf{R}(e_a, e_4, e_b, e_4), \quad \beta_a = \frac{1}{2}\mathbf{R}(e_a, e_4, e_3, e_4),\\
\rho &= \frac{1}{4}\mathbf{R}(e_4, e_3, e_4, e_3), \ldots
\end{aligned}
$$

CRUCIAL FACT.

- In Kerr relative to a principal null frame we have

$$\begin{aligned} \alpha,\, \beta,\, \underline{\beta},\, \underline{\alpha} &= 0, \quad \rho + i\,{}^\star\rho = -\frac{2m}{(r + ia\cos\theta)^3} \\ \xi,\ \underline{\xi},\ \widehat{\chi},\ \underline{\widehat{\chi}} &= 0. \end{aligned}$$

- In Schwarzschild we have in addition

$$^\star\rho = 0, \qquad \eta,\ \underline{\eta},\ \zeta = 0$$

The only nonvanishing components of Γ are

$$\mathrm{tr}\chi,\, \mathrm{tr}\underline{\chi},\, \omega,\, \underline{\omega}$$

- In Minkowski we also have $\quad \omega,\, \underline{\omega},\, \rho = 0.$

$O(\epsilon)$ - PERTURBATIONS

ASSUME. There exists a null frame e_3, e_4, e_1, e_2 such that

$$\xi,\ \underline{\xi},\ \widehat{\chi},\ \underline{\widehat{\chi}},\ \alpha,\ \underline{\alpha},\ \beta,\ \underline{\beta} = O(\epsilon)$$

FRAME TRANSFORMATIONS, $(f_a)_{a=1,2}$, $(\underline{f}_a)_{a=1,2} = O(\epsilon)$

$$\begin{aligned} e_4' &= e_4 + f_a e_a + O(\epsilon^2) \\ e_3' &= e_3 + \underline{f}_a e_a + O(\epsilon^2) \\ e_a' &= e_a + \frac{1}{2}\underline{f}_a e_4 + \frac{1}{2} f_a e_3 + O(\epsilon^2) \end{aligned}$$

FACT.

- The curvature components $\alpha, \underline{\alpha}$ are $O(\epsilon^2)$ invariant with respect to $O(\epsilon)-$ gauge transformations
- For $O(\epsilon)$-perturbations of Minkowski all null components of R are $O(\epsilon^2)$-invariant.

BASIC EQUATIONS

NULL STRUCTURE EQTS. (Transport)

$$\nabla_4 \Gamma = \mathbf{R} + \Gamma \cdot \Gamma, \qquad \nabla_3 \Gamma = \mathbf{R} + \Gamma \cdot \Gamma$$

NULL STRUCTURE EQTS. (Codazzi)

$$\nabla \Gamma = \mathbf{R} + \Gamma \cdot \Gamma,$$

NULL BIANCHI.

$$\nabla_4 \mathbf{R} = \nabla \mathbf{R} + \Gamma \cdot \mathbf{R}, \quad \nabla_3 \mathbf{R} = \nabla \mathbf{R} + \Gamma \cdot \mathbf{R}$$

BASIC EQUATIONS

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KERR STABILITY-MAIN DIFFICULTIES

UNLIKE STABILITY OF THE MINKOWSKI SPACE

1. Some null curvature components (middle components) are nontrivial. Bianchi system admits non-decaying states.

2. All other null components of of the curvature tensor are **sensitive** to frame transformations.

3. Principal null directions are not integrable.

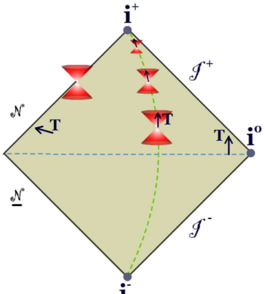
4. Have to track the parameters (a_f, m_f) of the final Ker and the correct gauge condition. **They have to emerge dynamically !.**

5. Obstacles to prove decay for the simplest linear equations $\square_{\mathbf{g}} \Phi = 0$ on a fixed Kerr.

KERR STABILITY-MAIN DIFFICULTIES

OBSTACLES TO PROVE DECAY FOR $\Box_{Kerr}\Phi = 0$.

- Degeneracy of the horizon.
- Trapped null geodesics.
- Superradiance - absent in Schw. and (in general) for axially symmetric solutions.
- Superposition problem.



1'ST BREAKTHROUGH. TEUKOLSKI EQTS.

FACT[Teukolski 1973).] Extreme curvature components α, $\underline{\alpha}$ verify, up to $O(\epsilon^2)$-errors, **decoupled**, albeit **non-conservative**, linear wave equations.

- Whiting(1989). Mode Stability. Teukolski linearized gravity equations, have **no exponentially growing** modes.
- Y. Schlapentokh Rothman (2014). Quantitative mode stability for $\square_{a,m}\Phi = 0$, $|a| < m$.
- Dafermos-Rondianski-Rothman(2015) Make use of the New Vectorfield Method and Yacov's result to deduce quantitative decay estimates for $\square_{a,m}\Phi = 0$, $|a| < m$.

2'ND BREAKTHROUGH. CHANDRASKHAR TR.

FACT[Chandrasekhar(1975).] There exist a transformation $\alpha \longrightarrow P$ which takes solutions of the Teukolski equation on a Schwarzschild background to solutions of the Reggee-Wheeler equation,

$$\Box_{Schw} P + VP = 0.$$

- Dafermos-Holzegel-Rodnianski(DHR 2016). Prove quantitative decay[1] for P and therefore also for $\alpha, \underline{\alpha}$. They use this as a first step to prove linear stability of Schwarzschild.

- S. Ma(2017.) Can extend the analysis to control the Teukolski equation for Kerr(a, m), $|a| \ll m$.

[1] Based on the technology developed in the last 15 years., See next slides.

3'RD BREAKTHROUGH. VF. METHOD

CLASSICAL VF. METHOD.

- Generalized energy estimates, based on the symmetries of Minkowski, to derive robust uniform decay. Global existence results for nonlinear wave equations. Null Condition.
- Nonlinear Stability of Minkowski. Uses generalized energy estimates, based on constructed approximate symmetries, to get uniform decay estimates for the curvature tensor.

NEW VECTORFIELD METHOD $\square_{a,m}\Phi = 0$

Compensates for the lack of enough symmetries of $Kerr(a, m)$ by introducing new geometric quantities to deal with:

- Degeneracy of the horizon.
- Trapped null geodesics.
- Superradiance
- Low decay at null infinity.

The new method has emerged in the last 15 years in connection to the study of boundedness and decay for the scalar wave equation,

$$\square_{\mathbf{g}_{a,m}}\phi = 0$$

Dafermos-Rodnianski-Shlapentokh-Rothman (2014)

Previous Results. Soffer-Blue(2003), Blue-Sterbenz, Daf-Rod, MMTT, Blue-Anderson, Tataru-Tohaneanu, etc.

LINEAR STABILITY OF SCHWARZSCHILD

Dafermos-Holzegel-Rodnianski(2016). Schwarzschild Space $Kerr(0, m)$ is linearly stable, once we mod out the unstable modes related to:

- Continuous two parameter family of nearby stationary solutions.
- Linearized gauge transformations

- **CHANDRASEKHAR TRANSF.** Derive sharp decay bounds for $\alpha, \underline{\alpha}$.
- **RECONSTRUCTION.** Find appropriate gauge conditions, to derive bounds and decay for all other quantities of of the linearized Einstein equations on Schwarzschild.

Hung-Keller- Wang (2017). Alternative approach based on Regge-Weeler, Zerilli coordinate approach.

CONCLUSIONS

WHAT WE UNDERSTAND. In light of the recent advances we now have tools to control, in principle, $\alpha, \underline{\alpha}$. This replaces the methods used in the stability of Minkowski based on the analysis of the Bianchi system.

WHAT REMAINS TO DO.

- Find quantities that track the mass and angular momentum.
- Find an effective, dynamical, solution to fix the gauge problem.
- Determine the decay properties of all important quantities and close the estimates for the full nonlinear problem.