ON THE MATHEMATICAL THEORY OF BLACK HOLES

Sergiu Klainerman

Princeton University

October 16, 2017

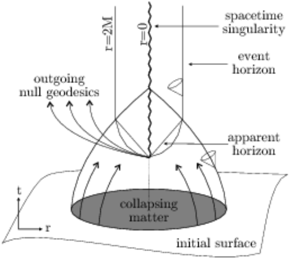
FINAL STATE CONJECTURE

The long time behavior of generic, asymptotically flat, solutions to the Einstein vacuum equations is given by the superposition of a finite number of diverging Kerr black holes plus a radiative decaying term.

FINAL STATE CONJECTURE.

1. Small solutions don't concentrate !
   Stability of Minkowski space.
2. Large data may concentrate into stationary states-BHs.
   Collapse.
3. All stationary states are Kerr solutions.
   Rigidity.
4. Kerr solutions are stable.
   Stability.
5. There can be no singularities outside BHs.
   Cosmic Censorship Conjecture.
6. Two (and more) body problem.
   Collision and merging of Black Holes.

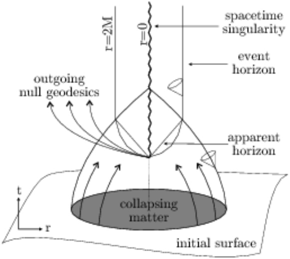
GRAVITATIONAL COLLAPSE



Large energy concentrations may lead to the formation of a *dynamical* black hole settling down, by gravitational radiation, to a Kerr black hole.

- BHs can form dynamically from regular configurations.
  Collapse
- All stationary states are Kerr black holes.
  Rigidity
- Stable under general perturbations.
  Stability

GRAVITATIONAL COLLAPSE



Large energy concentrations may lead to the formation of a *dynamical* black hole settling down, by gravitational radiation, to a Kerr black hole.

- BHs can form dynamically from regular configurations.
  Collapse
- All stationary states are Kerr black holes.
  Rigidity
- Stable under general perturbations.
  Stability

II. WHAT IS A BLACK HOLE ?

Stationary, asymptotically flat, solutions of EVE,

$$\mathrm{Ric}(\mathbf{g}) = \mathbf{0}.$$

EXTERNAL BLACK HOLE

- (M, g) Asymptoticaly flat, globally hyperbolic, diffeomorphic to the complement of a cylinder $\subset \mathbb{R}^{1+3}$.
- Has an **asymptotically** time-like, Killing vectorfield T

$$\mathcal{L}_T g = 0.$$

- Completeness (of Null Infinity)

EVENT HORIZON. {Past of future null infinity} $\cup$ { Future of past null infinity.}

KERR FAMILY $\mathcal{K}(a, m)$

COORDINATES (t, r, θ, φ).

$$-\frac{\rho^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{\rho^2}\left(d\varphi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2(d\theta)^2,$$

$$\begin{cases} \Delta = r^2 + a^2 - 2mr; \\ \rho^2 = r^2 + a^2(\cos\theta)^2; \\ \Sigma^2 = (r^2 + a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases}$$

STATIONARY $\mathbf{T} = \partial_t$

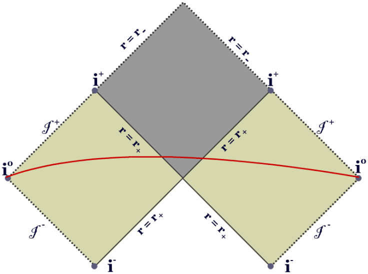
AXISYMMETRIC $\mathbf{Z} = \partial_\varphi$

SCHWARZSCHILD $a = 0$, $m > 0$, static, sph. symmetric.

$$-\frac{\Delta}{r^2}(dt)^2 + \frac{r^2}{\Delta}(dr)^2 + r^2 d\sigma_{\mathbb{S}^2}, \qquad \frac{\Delta}{r^2} = 1 - \frac{2m}{r}$$

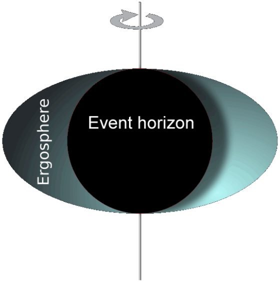
KERR SPACETIME $\mathcal{K}(a,m)$, $|a| \leq m$

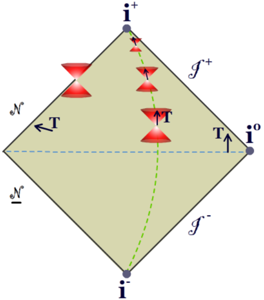
MAXIMAL G.H. EXTENSION. $\Delta(r_-) = \Delta(r_+) = 0$



- **EXTERNAL REGION** $r > r_+$
- **EVENT HORIZON** $r = r_+$
- **BLACK HOLE** $r < r_+$
- **NULL INFINITY** $r = \infty$

EXTERNAL KERR





- **HORIZON** $\mathcal{N} \cup \underline{\mathcal{N}}$
- **ERGOREGION** $\mathbf{g}(\mathbf{T},\mathbf{T}) < 0$
- **TRAPPED NULL GEODESICS**
- **ASYMPTOTIC REGION**

OTHER PROPERTIES OF KERR

- Besides $\mathbf{T}, \mathbf{Z}$ there exists a non-trivial Killing tensor $\mathbf{C}$(*Carter*).
- Possesses two distinct **principal null directions** which diagonalize the curvature tensor.
- Kerr is distinguished, among all stationary solutions of EVE by the **vanishing** of a complex 4-covariant tensorfield called the Mars-Simon tensor $\mathcal{S}$.

TESTS OF REALITY $\quad \mathrm{Ric}(g) = 0$

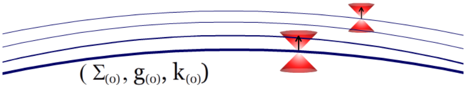
1. **RIGIDITY.** Does the Kerr family $\mathcal{K}(a, m)$, $0 \le a \le m$, **exhaus**t all possible vacuum black holes ?

2. **STABILITY.** Is the Kerr family **stable** under arbitrary small perturbations ?

3. **COLLAPSE.** Can black holes form starting from **reasonable** initial data configurations ? Formation of **trapped** surfaces.

INITIAL VALUE PROBLEM: J. Leray, Y. C. Bruhat(1952)

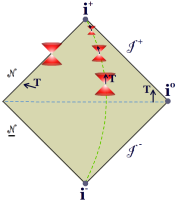
Ric(g)=0



RIGIDITY

RIGIDITY CONJECTURE. Kerr family $\mathcal{K}(a, m)$, $0 \leq a \leq m$, exhaust all **stationary**, asymptotically flat, **regular** vacuum black holes.

- True in the axially symmetric case [Carter-Robinson]
- True in general, under an **analyticity** assumption [Hawking]

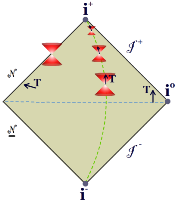


ANALYTICITY - NOT A REASONABLE ASSUMPTION !

RIGIDITY

RIGIDITY CONJECTURE. Kerr family $\mathcal{K}(a, m)$, $0 \leq a \leq m$, exhaust all **stationary**, asymptotically flat, **regular** vacuum black holes.

- True in the axially symmetric case [Carter-Robinson]
- True in general, under an **analyticity** assumption [Hawking]

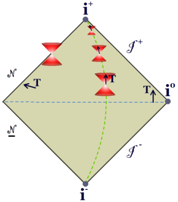


ANALYTICITY - NOT A REASONABLE ASSUMPTION !

RIGIDITY

RIGIDITY CONJECTURE. Kerr family $\mathcal{K}(a, m)$, $0 \le a \le m$, exhaust all **stationary**, asymptotically flat, **regular** vacuum black holes.

- True in the axially symmetric case [Carter-Robinson]
- True in general, under an **analyticity** assumption [Hawking]

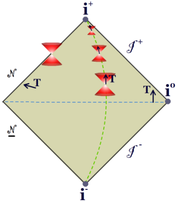


ANALYTICITY - NOT A REASONABLE ASSUMPTION !

RIGIDITY

RIGIDITY CONJECTURE. Kerr family $\mathcal{K}(a, m)$, $0 \leq a \leq m$, exhaust all **stationary**, asymptotically flat, **regular** vacuum black holes.

- True in the axially symmetric case [Carter-Robinson]
- True in general, under an **analyticity** assumption [Hawking]

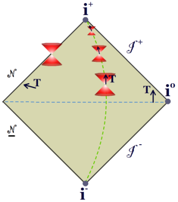


ANALYTICITY - NOT A REASONABLE ASSUMPTION !

RIGIDITY. MAIN NEW RESULTS

RIGIDITY CONJECTURE. Kerr family $\mathcal{K}(a, m)$, $0 \le a \le m$, exhaust all **stationary**, asymptotically flat, **regular** vacuum black holes.

- True if coincides with Kerr on $\mathcal{N} \cap \underline{\mathcal{N}}$ [Ionescu-Kl]
- True if close to a Kerr space-time [Alexakis-Ionescu-Kl]

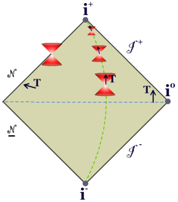


CONJECTURE. [Alexakis-Ionescu-Kl]. Rigidity conjecture holds true provided that there are **no T-trapped** null geodesics.

RIGIDITY. MAIN NEW RESULTS

RIGIDITY CONJECTURE. Kerr family $\mathcal{K}(a, m)$, $0 \leq a \leq m$, exhaust all **stationary**, asymptotically flat, **regular** vacuum black holes.

- True if coincides with Kerr on $\mathcal{N} \cap \underline{\mathcal{N}}$ [Ionescu-Kl]
- True if close to a Kerr space-time [Alexakis-Ionescu-Kl]

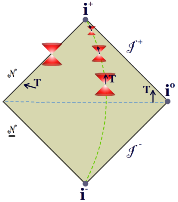


CONJECTURE. [Alexakis-Ionescu-Kl]. Rigidity conjecture holds true provided that there are **no T-trapped** null geodesics.

RIGIDITY. MAIN NEW RESULTS

RIGIDITY CONJECTURE. Kerr family $\mathcal{K}(a, m)$, $0 \le a \le m$, exhaust all **stationary**, asymptotically flat, **regular** vacuum black holes.

- True if coincides with Kerr on $\mathcal{N} \cap \underline{\mathcal{N}}$ [Ionescu-Kl]
- True if close to a Kerr space-time [Alexakis-Ionescu-Kl]



CONJECTURE. [Alexakis-Ionescu-Kl]. Rigidity conjecture holds true provided that there are **no T-trapped** null geodesics.

LOCAL EXTENSION

DEFINITION. $O \subset \mathbf{M}$ is (**T**) null-convex at $p \in \partial O$ if, for any defining function f,
$\mathbf{D}^2 f(X,X)(p) < 0, \quad \forall X \in T_p(O), \quad g(X,X) = 0, \ (\mathbf{g}(X,T) = 0)$

THEOREM (Ionescu-Kl) $(\mathbf{M}, \mathbf{g})$ Ricci flat, pdo-riemannian manifold. $(O \subset \mathbf{M},\ Z)$ verify:

- Z Killing v-field in O,
- ∂O is strongly null-convex at $p \in \partial O$

$\Rightarrow$ Z extends as a Killing vector-field to a neighborhood of p.

LOCAL EXTENSION

DEFINITION. $O \subset \mathbf{M}$ is **(T)** null-convex at $p \in \partial O$ if, for any defining function f,
$\mathbf{D}^2 f(X,X)(p) < 0, \quad \forall X \in T_p(O), \quad g(X,X) = 0, \ (\mathbf{g}(X, T) = 0)$

THEOREM (Ionescu-Kl) $(\mathbf{M}, \mathbf{g})$ Ricci flat, pdo-riemannian manifold. $(O \subset \mathbf{M}, \ Z)$ verify:

- Z Killing v-field in O,
- ∂O is *s*trongly null-convex at $p \in \partial O$

$\Rightarrow$ Z extends as a Killing vector-field to a neighborhood of p.

LOCAL EXTENSION

DEFINITION. $O \subset \mathbf{M}$ is **(T)** null-convex at $p \in \partial O$ if, for any defining function f,
$\mathbf{D}^2 f(X,X)(p) < 0, \quad \forall X \in T_p(O), \quad g(X,X) = 0, \; (\mathbf{g}(X,T) = 0)$

THEOREM (Ionescu-Kl) $(\mathbf{M}, \mathbf{g})$ Ricci flat, pdo-riemannian manifold. $(O \subset \mathbf{M}, \; Z)$ verify:

- Z Killing v-field in O,
- ∂O is *s*trongly null-convex at $p \in \partial O$

$\Rightarrow$ Z extends as a Killing vector-field to a neighborhood of p.

LOCAL EXTENSION

DEFINITION. $O \subset \mathbf{M}$ is (**T**) null-convex at $p \in \partial O$ if, for any defining function f,
$\mathbf{D}^2 f(X,X)(p) < 0, \quad \forall X \in T_p(O), \quad g(X,X) = 0, \ (\mathbf{g}(X,T) = 0)$

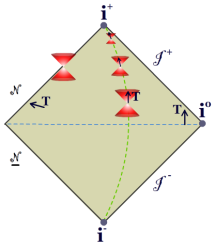
THEOREM (Ionescu-Kl) $(\mathbf{M}, \mathbf{g})$ Ricci flat, pdo-riemannian manifold. $(O \subset \mathbf{M}, \ Z)$ verify:

- Z Killing v-field in O,
- ∂O is strongly null-convex at $p \in \partial O$

$\Rightarrow$ Z extends as a Killing vector-field to a neighborhood of p.

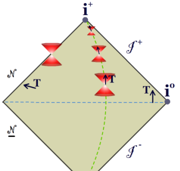
LOCAL EXTENSION

THEOREM (Ionescu-Kl) $\exists$ smooth, **stationary**, extensions of $\mathcal{K}(a, m)$, $0 < a < m$, locally defined in a neighb. of a point on the horizon $p \in \mathcal{N} \cup \underline{\mathcal{N}} \setminus \mathcal{N} \cap \underline{\mathcal{N}}$, which posesses **no additional** Killing v-fields.



RIGIDITY - METHODOLOGY

1. Measure closeness to Kerr using the Mars-Simon tensor
2. Turn rigidity, i.e. **uniqueness**, into an **extension** problem using **unique continuation** arguments.
   - Killing vectorfields can be extended past **null convex** boundaries.
   - Bifurcate horizons are null convex.
   - Carleman Estimates
3. Obstruction to extension. Presence of **T-trapped** null geodesics.
   - No such obstruction in Kerr.



TESTS OF REALITY $\mathbf{Ric(g) = 0}$

1.

2.

3. **COLLAPSE.** Can black holes **form** starting from **reasonable** initial data configurations ? Formation of trapped surfaces.

COLLAPSE

GOAL. Investigate the mechanism of formation of **black holes** starting with reasonable initial data configurations.

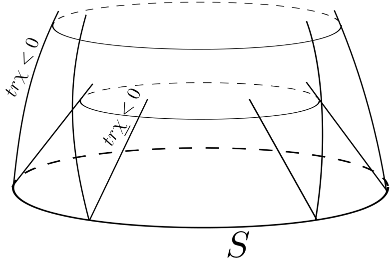
TRAPPED SURFACE. Concept introduced by Penrose in connection to his incompleteness theorem.

COROLLARY. If **WCC** holds true the presence of a **trapped surface** detects the presence of a black hole.

PENROSE SINGULARITY THEOREM

THEOREM. Space-time (M, g) cannot be future null geodesicaly complete, if

- $\mathrm{Ric}(g)(L, L) \geq 0, \qquad \forall L$ *null*
- M contains a non-compact Cauchy hypersurface
- M contains a closed **trapped** surface S



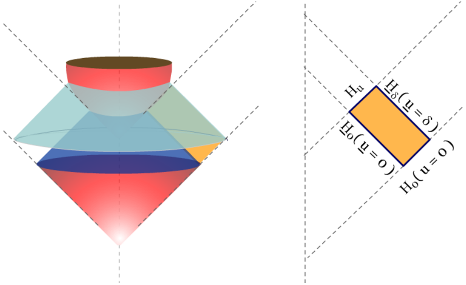
Null expansions $\quad \mathrm{tr}\chi, \ \mathrm{tr}\underline{\chi}$. Raychadhouri equations

QUESTIONS

- Can trapped surfaces form in evolution? In vacuum?
- Can trapped surfaces form starting with **non-isotropic**, initial configurations?
- What is the nature of the singularities predicted by Penrose?

MAIN RESULTS

THEOREM[Christ(2008)]. (∃) open set of regular, vacuum, data whose MGFHD contains a trapped surface.

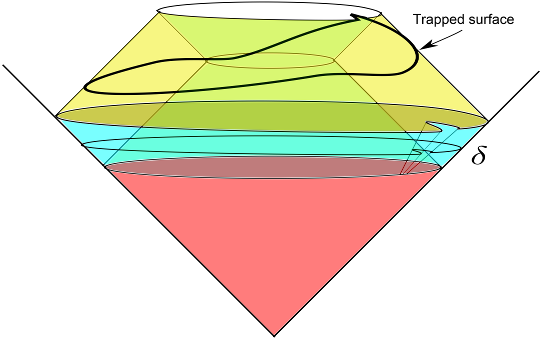


1. Specify **short pulse** characteristic data, for which one can prove a general semi-global result, with **detailed control**.
2. If, **in addition**, the data is sufficiently large, **uniformly** along all its null geodesic generators, a trapped surface must form.
3. Similar result for data given at past null infinity.

FORMATION OF TRAPPED SURFACES

THEOREM[Kl-Luk-Rodnianski(2013)] Result holds true for **non-isotropic** data concentrated near one null geodesic generator.

1. Combines all ingredients in Christodoulou's theorem with a **deformation argument** along incoming null hypersurfaces.
2. Reduces to a simple differential inequality on $S_{0,0} = H_0 \cap \underline{H}_0$.



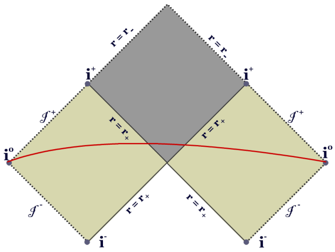
TESTS OF REALITY $\mathbf{Ric(g) = 0}$

1.
2. **STABILITY.** Is the Kerr family **stable** under arbitrary small perturbations ?
3.

II. STABILITY

CONJECTURE[Stability of (external) Kerr].

Small perturbations of a given exterior Kerr ($\mathcal{K}(a, m)$, $|a| < m$) initial conditions have max. future developments converging to **another** Kerr solution $\mathcal{K}(a_f, m_f)$.



GENERAL STABILITY PROBLEM $\mathcal{N}[\phi] = 0.$

NONLINEAR EQUATIONS. $\mathcal{N}[\phi_0 + \psi] = 0, \ \mathcal{N}[\Phi_0] = 0.$

1. ORBITAL STABILITY(OS). ψ bounded for all time.
2. ASYMPT STABILITY(AS). $\psi \longrightarrow 0$ as $t \to \infty$.

LINEARIZED EQUATIONS. $\mathcal{N}'[\phi_0]\psi = 0.$

1. MODE STABILITY (MS). No growing modes.
2. BOUNDEDNESS.
3. QUANTITATIVE DECAY.

GENERAL STABILITY PROBLEM $\mathcal{N}[\phi_0] = 0$

STATIONARY CASE. **Possible instabilities for** $\mathcal{N}'[\phi_0]\psi = 0$:

- Family of stationary solutions ϕ_λ, $\lambda \in (-\epsilon, \epsilon)$,

$$\mathcal{N}[\phi_\lambda] = 0 \Longrightarrow \mathcal{N}'[\phi_0](\frac{d}{d\lambda}\Phi_\lambda)|_{\lambda=0} = 0.$$

- Mappings $\Psi_\lambda : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$, $\Psi_0 = I$ taking solutions to solutions.

$$\mathcal{N}[\phi_0 \circ \Psi_\lambda] = 0 \Longrightarrow \mathcal{N}'[\phi_0]\frac{d}{d\lambda}(\phi_0 \circ \Psi_\lambda)|_{\lambda=0} = 0.$$

- Intrinsic instability of ϕ_0. Negative eigenvalues of $\mathcal{N}'(\phi_0)$.

GENERAL STABILITY PROBLEM $\mathcal{N}[\phi_0] = 0$

STATIONARY CASE. Expected linear instabilities due to non-decaying states in the kernel of $\mathcal{N}'[\phi_0]$:

1. Presence of continuous family of stationary solutions ϕ_λ implies that the final state ϕ_f may differ from initial state ϕ_0

2. Presence of a continuous family of invariant diffeomorphism requires us to track dynamically the gauge condition to insure decay of solutions towards the final state.

QUANTITATIVE LINEAR STABILITY. After accounting for (1) and (2), all solutions of $\mathcal{N}'[\phi_0]\psi = 0$ decay sufficiently fast.

MODULATION. Method of constructing solutions to the nonlinear problem by tracking (1) and (2).

STABILITY OF MINKOWSKI SPACE

THEOREM[Global Stability of Minkowski] Any asymptotically flat initial data set which is sufficiently close to the trivial one has a regular, complete, maximal development. Christodoulou-K

The result provides detailed information about the decay properties of the curvature tensor, **peeling**, and a rigorous derivations of the laws of gravitational radiation.