Some problems in Random Matrix theory

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April 3, 2014

Random Matrices with iid entries ξ_{ij}.

M_n^{Het} : Hermitian with complex entries.
M_n^{Sym} : Symmetric with real entries.
A_n : IID model. Matrix with n^2 iid entries.

Real case: ξ_{ij} mean 0 and variance 1.
Complex case: $\xi_{ij} = \tau_{ij} + \mathbf{i}\eta_{ij}$; mean 0 and variance $1/2$.
(Plus some tail decay condition.)

Examples:
Gaussian: GUE, GOE, Ginibre.
Bernoulli: ξ_{ij}, τ_{ij}, η_{ij} are ± 1.

The Big Picture

(1) One can compute many statistics for Gaussian models using direct methods.

(2) We expect these statistics hold for general models (Universality).

(3) Bernoulli seems to be a good representative toy model.

We will present many results and conjectures with the Bernoulli ensemble for simplicity, with the understudying that these hold in much more general setting.

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Other models: Random Covariance, Random Band, Adjacency matrix of random graphs, Random Sparse, Random Unitary, β-ensemble, Rectangular, General Wigner matrix, Invariant models, etc.

Global Distributions

Hermitian/Symmetric case.

Theorem (Wigner semi-circle law)

Assume that ξ_{ij} have mean and variance 1, then the limiting spectral distribution of $\frac{1}{\sqrt{n}}M_n$ has semi-circle density.

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Matrices with dependent entries: Single ring theorem (Guionnet-Krishnapur-Zeitouni; Rudelson-Vershynin); Elliptic Law (Naumov, Nguyen-O'rourke).

Joint distribution of Individual eigenvalues

Gustavsson (2005)

Theorem

(Gaussian fluctuation of eigenvalues) For GUE has

$$\frac{\lambda_i(A_n) - \mu_{i,n}}{\sigma(i,n)} \to N(0,1)_{\mathbb{R}}.$$

Similar result for joint distribution of k eigenvalues, for any fixed k.

O'rourke (2009): GOE.

Universality

Definition. [Matching moments] Two random matrices match to order k if the first k moments of the entries are the same.
Tao-V (2009)

Theorem (Four Moment Theorem)

If two (Hermitian or Symmetric) matrices match to the fourth order, then the joint distribution of any k eigenvalues are asymptotically the same. One can take k to be as large as $n^{.0001}$.

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Open question. Do Bernoulli matrices have Gaussian fluctuation ?

Correlation Functions: Hermitian/Symmetric case

Problem

Given a k variable function $f(x_1, \ldots, x_k)$. Determine

$$\lim_{n\to\infty} \int_{\mathbb{R}^k} f(x_1, \ldots, x_k)\rho_n^k(x_1, \ldots, x_k)dx_1 \ldots dx_k.$$

If $k = 1$, I is an interval and f is the indicator function of I, then

$$\int f(x)\rho_n^1(x)dx = \sum_{i=1}^{n} \mathbf{E}\mathbf{I}_{\lambda_i \in I} \qquad (1)$$

which is just simply the expectation of number of eigenvalues in I.
For $k = 2$ and f being the indicator of a rectangle $I \times J$, then

$$\int f(x_1, x_2)\rho_n^2(x_1, x_2)dx_1 dx_2 = \sum_{1\le i,j\le n} \mathbf{E}\mathbf{I}_{\lambda_i \in I}\mathbf{I}_{\lambda_j \in J} \qquad (2)$$

which is the expectation of the number of pairs (i, j) such that $\lambda_i \in I, \lambda_j \in J$.

An application

Computing the distribution of N_I, the number of eigenvalues in the short interval I.

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In general: Occupational probability $\mathbf{P}(N_I = k)$ for any fixed k.

Local version

Localized to some point u,

$$\rho^{(k)}_{n,u}(t_1,\dots,t_k) := \frac{1}{(n\rho_{sc}(u))^k}\rho^{(k)}_n\Big(u+\frac{t_1}{n\rho_{sc}(u)},\dots,u+\frac{t_k}{n\rho_{sc}(u)}\Big). \tag{3}$$

Explicit computation for gaussian case: Gaudin-Mehta, Dyson, Tracy-Widom. For GUE

$$\lim_{n\to\infty}\rho^{(k)}_{n,u}(x_1,\dots,x_k) = \rho^{(k)}_{\mathrm{Sine}}(x_1,\dots,x_k) \tag{4}$$

locally uniformly in $x_1,\dots,x_k$ where

$$\rho^{(k)}_{\mathrm{Sine}}(x_1,\dots,x_k) := \det(K_{\mathrm{Sine}}(x_i,x_j))_{1\le i,j\le k}$$

and K_{Sine} is the *sine kernel*

$$K_{\mathrm{Sine}}(x,y) := \frac{\sin(\pi(x-y))}{\pi(x-y)}$$

(with the usual convention that $\frac{\sin x}{x}$ equals 1 at the origin).

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Vague Convergence.

$$\lim_{n\to\infty} \int_{\mathbb{R}^k} F(x_1, \dots, x_k)\rho_{n,u}^{(k)}(x_1, \dots, x_k)\, dx_1 \dots dx_k = \int_{\mathbb{R}^k} F(x_1, \dots, x_k)\rho_{\mathrm{Sin}}^{(k)} \tag{5}$$

for all continuous, compactly supported functions $F : \mathbb{R}^k \to \mathbb{R}$.

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Average Vague Convergence. Erdős-Yau et. al. ; averaging over u,

$$\lim_{b\to 0}\lim_{n\to\infty}\frac{1}{2b}\int_{E-b}^{E+b}\int_{\mathbb{R}^k} F(x_1,\ldots,x_k)\rho_{n,u}^{(k)}(x_1,\ldots,x_k)\,dx_1\ldots dx_k du$$
$$= \int_{\mathbb{R}^k} F(x_1,\ldots,x_k)\rho_{\text{Sine}}^{(k)}(x_1,\ldots,x_k)\,dx_1\ldots dx_k \tag{6}$$

for all $-2<E<2$.

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(2) Universality for the hole probability $\mathbf{P}([-t/n, t/n] \textit{ is empty})$? Distribution of the least singular value; Goldstein-von Neumann 1940s. Can one avoid going through correlation ?

Beyond the semi-circle law

External source model: $\frac{1}{\sqrt{n}}M_n + D$, where D is deterministic and diagonal.

Global distribution is no longer semi-circle, often disconnected. Bleher-Kuijlaars (2004) considered $GUE + D$, where D has two values $\pm a$ and showed

$$\lim_{n\to\infty} \frac{1}{n\rho(x_0)}\hat{K}_n\left(x_0 + \frac{u}{n\rho(x_0)}, x_0 + \frac{v}{n\rho(x_0)}\right) = \frac{\sin \pi(u-v)}{\pi(u-v)}, \quad (7)$$

where $\hat{K}_n(x,y) = e^{n(h(x)-h(y))}K_n(x,y)$ for some explicit function h.

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Local statistics is more consistent !

Question. Universality ?
True (with Vague convergence) under 4 moment matching assumption (O'rourke-V 2013; related work Lee-Schnelli)
Question. Universality for gauss divisible models ?

Local Semi-circle Law

Erdös-Schlein-Yau: With high probability,

$$|N_I - n\int_I \rho_{sc}(x)\,dx| \le \delta n|I|,$$

for any *short* interval I and fixed $\delta > 0$, where N_I denotes the number of eigenvalues of $W_n := \frac{1}{\sqrt{n}} M_n$ in the interval I.

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Formally, we say that $f(n)$ is the threshold scale if with probability $1 - o(1)$

$$|N_I - n\int_I \rho_{sc}(x)\ dx| \leq \delta n|I|,$$

for any interval I in the bulk of length $\omega(f(n))$ (and any fixed > 0) and does not hold at length $o(f(n))$.

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Question. Is there a direct way to prove that there is no long empty interval ($n^{-1+\epsilon}$) ?

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Correlation functions:

$$\int_{\mathbb{R}^k}\int_{\mathbb{C}_+^l} F(x_1,\dots,x_k,z_1,\dots,z_l)\rho_n^{(k,l)}(x_1,\dots,x_k,z_1,\dots,z_l)\, dx_1\dots dx_k dz_1$$

$$= \mathbf{E} \sum_{1\le i_1<\dots<i_k\le N_{\mathbb{R}}[M_n]}\ \sum_{1\le j_1<\dots<j_l\le N_{\mathbb{C}_+}[M_n]} F(\lambda_{i_1,\mathbb{R}}(M_n),\dots,\lambda_{i_k,\mathbb{R}}(M_n),\lambda_{j_1,\mathbb{C}_+}(M_n),\dots,\lambda_{j_l,\mathbb{C}_+}(M_n)). \tag{8}$$

Explicit formulae for Gaussian case: Ginibre, Sommers, Sinclair, Borodin, May...

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Question. With high probability, Bernoulli matrix has 2 real roots ?

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Recent works for non-hermitian matrices: Bourgade's talk.

Universality of Determinant; Central limit theorems

Delannay-La Caer (2000)

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Tao-V. (2011): Universality under four moment assumption.

Question. Central limit theorem for Bernoulli matrices ?

$$\log|\det M_n| = \sum_{i=1}^{n} \log|\lambda_i|$$

Question. Central limit theorem for linear statistic with some larger class of test functions ? (Duits' talk)