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The synthetic theory of ∞-categories vs the synthetic theory of ∞-categories

joint with Dominic Verity and Michael Shulman

Vladimir Voevodsky Memorial Conference

The motivation for ∞-categories

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$\rightsquigarrow$ Thus, we want to extend 1-category theory (e.g., adjunctions, limits and colimits, universal properties, Kan extensions) to ∞-category theory.

First problem: it is hard to say exactly what an ∞-category is.

The idea of an ∞-category

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But this definition is tricky to make precise.

Models of ∞-categories

The notion of ∞-category is made precise by several models:

$$\begin{tikzcd}
 & \mathcal{R}\mathrm{ezk} \arrow[r] \arrow[ddr, shift left] & \mathcal{S}\mathrm{egal} \arrow[dd, shift left] & \\
\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \arrow[ur] \arrow[dr] \arrow[drr] & & & \mathcal{T}\mathrm{op}\text{-}\mathcal{C}\mathrm{at} \arrow[dl] \\
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- topological categories and relative categories are the simplest to define but do not have enough maps between them
-
  - quasi-categories (nee. weak Kan complexes),
  - Rezk spaces (nee. complete Segal spaces),
  - Segal categories, and
  - (saturated 1-trivial weak) 1-complicial sets

  each have enough maps and also an internal hom, and in fact any of these categories can be enriched over any of the others

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- work analytically to give categorical definitions and prove theorems using the combinatorics of one model

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- work synthetically to give categorical definitions and prove theorems in all four models $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ at once

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- work synthetically in a simplicial type theory augmenting HoTT to prove theorems in $\mathcal{R}ezk$

  (R-Shulman: an ∞-category is a type with unique binary composites in which isomorphism is equivalent to identity)

Plan

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2. The synthetic theory of ∞-categories (in homotopy type theory)

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①

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Henceforth ∞-category and ∞-functor are technical terms that mean the objects and morphisms of some ∞-cosmos.

The homotopy 2-category

The homotopy 2-category of an ∞-cosmos is a strict 2-category whose:

- objects are the ∞-categories A, B in the ∞-cosmos
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- 2-cells we call ∞-natural transformations $$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$$ which are defined to be homotopy classes of 1-simplices in $\mathrm{Fun}(A, B)$

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Prop. Equivalences in the homotopy 2-category

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{id}_A"{name=U}] \arrow[r, bend right, "gf"'{name=D}] & A \arrow[from=U, to=D, Rightarrow, "\cong"] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "\mathrm{id}_B"{name=U}] \arrow[r, bend right, "fg"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\cong"] \end{tikzcd}$$

coincide with equivalences in the ∞-cosmos.

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Thus, non-evil 2-categorical definitions are "homotopically correct."

Adjunctions between ∞-categories

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defn. An adjunction between ∞-categories is an adjunction in the homotopy 2-category, consisting of:

- ∞-categories A and B
- ∞-functors $u \colon A \to B$, $f \colon B \to A$
- ∞-natural transformations $\begin{tikzcd} B \arrow[r, bend left, "\mathrm{id}_B"] \arrow[r, bend right, "uf"'] \arrow[r, phantom, "\Downarrow\eta"] & B \end{tikzcd}$ and $\begin{tikzcd} A \arrow[r, bend left, "fu"] \arrow[r, bend right, "\mathrm{id}_A"'] \arrow[r, phantom, "\Downarrow\epsilon"] & A \end{tikzcd}$

satisfying the triangle equalities

$$\begin{tikzcd} & B \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \arrow[d, phantom, "\Downarrow\eta"] & B \\ A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \arrow[ur, "u"'] & \end{tikzcd} = \begin{tikzcd} B \\ A \arrow[u, bend left, "u"] \arrow[u, bend right, "u"'] \arrow[u, phantom, "="] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\ & A \arrow[ur, "u"] \arrow[rr, equal] & {} & A \end{tikzcd} = \begin{tikzcd} B \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\ A \end{tikzcd}$$

Write $f \dashv u$ to indicate that f is the left adjoint and u is the right adjoint.

The 2-category theory of adjunctions

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Prop. Adjunctions compose:

$$\begin{tikzcd} C \arrow[r, bend left, "f'"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "u'"] \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u"] & \rightsquigarrow & C \arrow[r, bend left, "ff'"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "u'u"] \end{tikzcd}$$

Prop. Adjoints to a given functor $u\colon A \to B$ are unique up to canonical isomorphism: if $f \dashv u$ and $f' \dashv u$ then $f \cong f'$.

Prop. Any equivalence can be promoted to an adjoint equivalence: if $u\colon A \xrightarrow{\sim} B$ then u is left and right adjoint to its equivalence inverse.

Composing adjunctions

Prop. Adjunctions compose:

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\end{tikzcd}$$

Proof: The composite 2-cells

$$\begin{tikzcd}
C \arrow[rrrr, equal] \arrow[dr, "f'"'] & {} \arrow[to=2-4, phantom, "\Downarrow\eta'"] & & & C \\
& B \arrow[rr, equal] \arrow[dr, "f"'] \arrow[to=3-4, phantom, "\Downarrow\eta"] & & B \arrow[ur, "u'"'] & \\
& & A \arrow[ur, "u"'] & {} &
\end{tikzcd}
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A \arrow[ur, "u"] \arrow[rrrr, equal] & & & {} & A
\end{tikzcd}$$

define the unit and counit of $ff' \dashv u'u$ satisfying the triangle equalities.

Initial and terminal elements in an ∞-category

defn. An ∞-category A has a terminal element iff $$\begin{tikzcd} 1 \arrow[r, bend right, dashed, "t"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "!"'] \end{tikzcd}$$.

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Prop. Right adjoints preserve terminal elements.

Proof: Compose the adjunctions
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More generally:

Prop. Right adjoints preserve limits and left adjoints preserve colimits.

Proof: The usual one!

The universal property of adjunctions

defn. Any ∞-category A has an ∞-category of arrows $A^{\mathbb{2}}$, pulling back

to define the comma ∞-category:

$$\begin{tikzcd}
\mathrm{Hom}_A(f,g) \arrow[r] \arrow[d, two heads, "(\mathrm{cod},\mathrm{dom})"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}} \arrow[d, two heads, "(\mathrm{cod},\mathrm{dom})"] \\
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Prop. $\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \end{tikzcd}$ if and only if $\mathrm{Hom}_A(f, A) \simeq_{A\times B} \mathrm{Hom}_B(B, u)$.

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Prop. If $f \dashv u$ with unit η and counit ϵ then

- η is initial in $\mathrm{Hom}_B(B,u)$ over B.
- ϵ is terminal in $\mathrm{Hom}_A(f,A)$ over A.

2

The synthetic theory of ∞-categories (in homotopy type theory)

The Curry-Howard-Voevodsky correspondence

type theory	set theory	logic	homotopy theory
A	set	proposition	space
$x : A$	element	proof	point
$\emptyset, 1$	$\emptyset, \{\emptyset\}$	$\bot, \top$	$\emptyset, *$
$A \times B$	set of pairs	A and B	product space
$A + B$	disjoint union	A or B	coproduct
$A \to B$	set of functions	A implies B	function space
$x : A \vdash B(x)$	family of sets	predicate	fibration
$x : A \vdash b : B(x)$	fam. of elements	conditional proof	section
$\prod_{x:A} B(x)$	product	$\forall x.B(x)$	space of sections
$\sum_{x:A} B(x)$	disjoint sum	$\exists x.B(x)$	total space
$p : x =_A y$	$x = y$	proof of equality	path from x to y
$\sum_{x,y:A} x =_A y$	diagonal	equality relation	path space for A

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$$\mathsf{path\text{-}ind} : \left(\prod_{x:A} B(x, x, \mathsf{refl}_x) \right) \to \left(\prod_{x,y:A} \prod_{p:x=_A y} B(x, y, p) \right).$$

The intended model

$\mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}$	$\supset$	$\mathcal{R}eedy$	$\supset$	$\mathcal{S}egal$	$\supset$	$\mathcal{R}ezk$
$\Vert$		$\Vert$		$\Vert$		$\Vert$
bisimplicial sets		types		types with composition		types with composition & univalence

The intended model

$$\begin{array}{ccccccc} \mathcal{S}et^{\mathbf{\Delta}^{op}\times\mathbf{\Delta}^{op}} & \supset & \mathcal{R}eedy & \supset & \mathcal{S}egal & \supset & \mathcal{R}ezk \\ \| & & \| & & \| & & \| \\ \text{bisimplicial sets} & & \text{types} & & \begin{array}{c}\text{types with}\\ \text{composition}\end{array} & & \begin{array}{c}\text{types with}\\ \text{composition}\\ \text{\& univalence}\end{array} \end{array}$$

Theorem (Shulman). Homotopy type theory is modeled by the category of Reedy fibrant bisimplicial sets.

The intended model

$$\begin{tikzcd}
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\text{bisimplicial sets} & \text{types} & \begin{array}{c}\text{types with}\\ \text{composition}\end{array} & \begin{array}{c}\text{types with}\\ \text{composition}\\ \text{\& univalence}\end{array}
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Theorem (Shulman). Homotopy type theory is modeled by the category of Reedy fibrant bisimplicial sets.

Theorem (Rezk). ∞-categories are modeled by Rezk spaces aka complete Segal spaces.

Shapes in the theory of the directed interval

Our types may depend on other types and also on shapes $\Phi \subset \mathbb{2}^n$, polytopes embedded in a directed cube, defined in a language

$$\top, \bot, \wedge, \vee, \equiv \qquad \text{and} \qquad 0, 1, \leq$$

satisfying intuitionistic logic and strict interval axioms.

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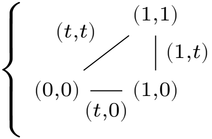
$$\Delta^n := \{(t_1, \ldots, t_n) : \mathbb{2}^n \mid t_n \leq \cdots \leq t_1\} \qquad \text{e.g.} \qquad \Delta^1 := \mathbb{2}$$

$\Delta^2 :=$



$$\partial\Delta^2 := \{(t_1, t_2) : \mathbb{2}^2 \mid (t_2 \leq t_1) \wedge ((0 = t_2) \vee (t_2 = t_1) \vee (t_1 = 1))\}$$

$$\Lambda^2_1 := \{(t_1, t_2) : \mathbb{2}^2 \mid (t_2 \leq t_1) \wedge ((0 = t_2) \vee (t_1 = 1))\}$$

Extension types

Formation rule for extension types

$$\dfrac{\Phi \subset \Psi \text{ shape} \qquad A \text{ type} \qquad a : \Phi \to A}{\left\langle \begin{tikzcd} \Phi \arrow[r, "a"] \arrow[d, tail] & A \\ \Psi \arrow[ur, dashed] & \end{tikzcd} \right\rangle \text{ type}}$$

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The simplicial type theory allows us to *prove* equivalences between extension types along composites or products of shape inclusions.

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The hom type for A depends on two terms in A:

$$x, y : A \vdash \mathrm{Hom}_A(x, y)$$

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Semantically, $\sum_{x,y:A} \mathrm{Hom}_A(x, y)$ recovers the ∞-category of arrows A^2 in the ∞-cosmos $\mathcal{R}$ezk and $\mathrm{Hom}_A(x, y)$ recovers the comma ∞-category from x to y.

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By contractibility, $\left\langle \begin{tikzcd} \Lambda^2_1 \arrow[r, "{[f,g]}"] \arrow[d, tail] & A \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd} \right\rangle$ has a unique inhabitant.

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Identity arrows

For any $x : A$, the constant function defines a term

$$\mathrm{id}_x := \lambda t.x : \mathrm{Hom}_A(x,x) := \left\langle \begin{tikzcd} \partial\Delta^1 \arrow[r, "{[x,x]}"] \arrow[d, tail] & A \\ \Delta^1 \arrow[ur, dashed] & \end{tikzcd} \right\rangle,$$

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For any $f : \mathrm{Hom}_A(x,y)$ in a Segal type A, the term

$$\lambda(s,t).f(t) : \left\langle \begin{tikzcd} \Lambda^2_1 \arrow[r, "{[\mathrm{id}_x, f]}"] \arrow[d, hook] & A \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd} \right\rangle$$

witnesses the unit axiom $f = f \circ \mathrm{id}_x$.

Associativity of composition

Let A be a Segal type with arrows

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Proof: Consider the composable arrows in the Segal type $\Delta^1 \to A$:

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Composing defines a term in the type $\Delta^2 \to (\Delta^1 \to A)$ which yields a term $\ell : \mathrm{Hom}_A(x,w)$ so that $\ell = h \circ (g \circ f)$ and $\ell = (h \circ g) \circ f$.

Isomorphisms

An arrow $f \colon \mathrm{Hom}_A(x, y)$ in a Segal type is an isomorphism if it has a two-sided inverse $g \colon \mathrm{Hom}_A(y, x)$. However, the type

$$\sum_{g \colon \mathrm{Hom}_A(y,x)} (g \circ f = \mathrm{id}_x) \times (f \circ g = \mathrm{id}_y)$$

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$$\mathrm{isiso}(f) := \left(\sum_{g \colon \mathrm{Hom}_A(y,x)} g \circ f = \mathrm{id}_x \right) \times \left(\sum_{h \colon \mathrm{Hom}_A(y,x)} f \circ h = \mathrm{id}_y \right).$$

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For $x, y : A$, the type of isomorphisms from x to y is:

$$x \cong_A y := \sum_{f : \mathrm{Hom}_A(x,y)} \mathrm{isiso}(f).$$

Rezk types $\equiv$ ∞-categories

By path induction, to define a map

$$\text{path-to-iso} \colon (x =_A y) \to (x \cong_A y)$$

for all $x, y : A$ it suffices to define

$$\text{path-to-iso}(\text{refl}_x) := \text{id}_x.$$

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Discrete types $\equiv$ ∞-groupoids

Similarly by path induction define

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A type A is discrete iff every arrow is an identity, i.e., iff path-to-arr is an equivalence.

Prop. A type is discrete if and only if it is Rezk and all of its arrows are isomorphisms.

Proof:

$$\begin{tikzcd}
x =_A y \arrow[rr, "\text{path-to-arr}"] \arrow[dr, "\text{path-to-iso}"'] & & \mathrm{Hom}_A(x, y) \\
& x \cong_A y \arrow[ur, hook] &
\end{tikzcd}$$

∞-categories for undergraduates

defn. An ∞-groupoid is a type in which arrows are equivalent to identities:

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Covariant type families $\equiv$ categorical fibrations

A type family $x : A \vdash B(x)$ over a Segal type A is covariant if for every $f : \mathrm{Hom}_A(x, y)$ and $u : B(x)$ there is a unique lift of f with domain u.

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Prop. For $u : B(x)$, $f : \mathrm{Hom}_A(x, y)$, and $g : \mathrm{Hom}_A(y, z)$,

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Prop. If $x : A \vdash B(x)$ is covariant then for each $x : A$ the fiber $B(x)$ is discrete. Thus covariant type families are fibered in ∞-groupoids.

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Prop. Fix $a : A$. The type family $x : A \vdash \mathrm{Hom}_A(a, x)$ is covariant.

The Yoneda lemma

Let $x : A \vdash B(x)$ be a covariant family over a Segal type and fix $a : A$.

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Yoneda lemma. The maps

$$\text{ev-id} := \lambda\phi.\phi(a, \text{id}_a) : \left(\prod_{x:A} \text{Hom}_A(a, x) \to B(x)\right) \to B(a)$$

and

$$\text{yon} := \lambda u.\lambda x.\lambda f.f_* u : B(a) \to \left(\prod_{x:A} \text{Hom}_A(a, x) \to B(x)\right)$$

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are inverse equivalences.

Corollary. A natural *isomorphism* $\phi : \prod_{x:A} \text{Hom}_A(a, x) \cong \text{Hom}_A(b, x)$ induces an *identity* $\text{ev-id}(\phi) : b =_A a$ if the type A is Rezk.

The dependent Yoneda lemma

Yoneda lemma. If A is a Segal type and $B(x)$ is a covariant family dependent on $x : A$, then evaluation at (a, id_a) defines an equivalence

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The Yoneda lemma is a “directed” version of the “transport” operation for identity types, suggesting a dependently-typed generalization analogous to the full induction principle for identity types.

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The Yoneda lemma is a "directed" version of the "transport" operation for identity types, suggesting a dependently-typed generalization analogous to the full induction principle for identity types.

Dependent Yoneda lemma. If A is a Segal type and $B(x, y, f)$ is a covariant family dependent on $x, y : A$ and $f : \mathsf{Hom}_A(x, y)$, then evaluation at (x, x, id_x) defines an equivalence

$$\text{ev-id} : \left(\prod_{x,y:A} \prod_{f:\mathsf{Hom}_A(x,y)} B(x, y, f) \right) \to \prod_{x:A} B(x, x, \mathsf{id}_x)$$

Dependent Yoneda is directed path induction

Slogan: the dependent Yoneda lemma is directed path induction.

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Path induction. If $B(x, y, p)$ is a type family dependent on $x, y : A$ and $p : x =_A y$, then to prove $B(x, y, p)$ it suffices to assume y is x and p is refl_x. I.e., there is a function

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Arrow induction. If $B(x, y, f)$ is a covariant family dependent on $x, y : A$ and $f : \mathsf{Hom}_A(x, y)$ and A is Segal, then to prove $B(x, y, f)$ it suffices to assume y is x and f is id_x. I.e., there is a function

$$\mathsf{id\text{-}ind} : \left(\prod_{x:A} B(x, x, \mathsf{id}_x)\right) \to \left(\prod_{x,y:A} \prod_{f:\mathsf{Hom}_A(x,y)} B(x, y, f)\right).$$

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- compatible with new foundations: synthetic constructions can easily be adapted to simplicial HoTT, which yields further streamlining.

References

For more on the synthetic theories of ∞-categories, see:

Emily Riehl and Dominic Verity

- draft book in progress:

  Elements of ∞-Category Theory
  www.math.jhu.edu/~eriehl/elements.pdf

- mini-course lecture notes:

  ∞-Category Theory from Scratch
  arXiv:1608.05314

Emily Riehl and Michael Shulman

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Thank you!