Algebraic geometry and algebraic topology

voevodsky connecting two worlds of math bringing intuitions from each area to the other

coding and frobenius

quantum information theory and quantum mechanics

joint ... Asok
and Jean Fasel and
Mike Hill

1890s-1970s: Many problems in mathematics were understood to be problems in algebraic topology/homotopy theory.

This was due in large measure to the homotopy invariance of bundle theory.

long entwined
relation between fields allowing radically different intuitions to be brought to bear

eg Frobenius vs complex analysis
coding and eigenvalues of Frobenius
quantum physics and quantum computing

Story of Milnor in Dec. Theme of conference. Joyal's talk.

Milnor of conj -> cat thy -> equations summarized by diagrams a la Joyal

you can take that as an affirmation that we are all connected in this great universe, that the things that seem to divide us are illusions and whether by virtue of low entropy or a spirit in the sky all beings are but one being.

and sometimes I have the reaction, of

1890s–1970s: Many problems in mathematics were understood to be problems in algebraic topology/homotopy theory.

1970s: Abstract homotopy theory

Today: Thanks in large measure to Voevodsky, more and more mathematical questions are being understood to have a homotopy theoretic component, often rooted in abstract homotopy theory.

Question: Which complex vector bundles on complex algebraic varieties have algebraic structures?

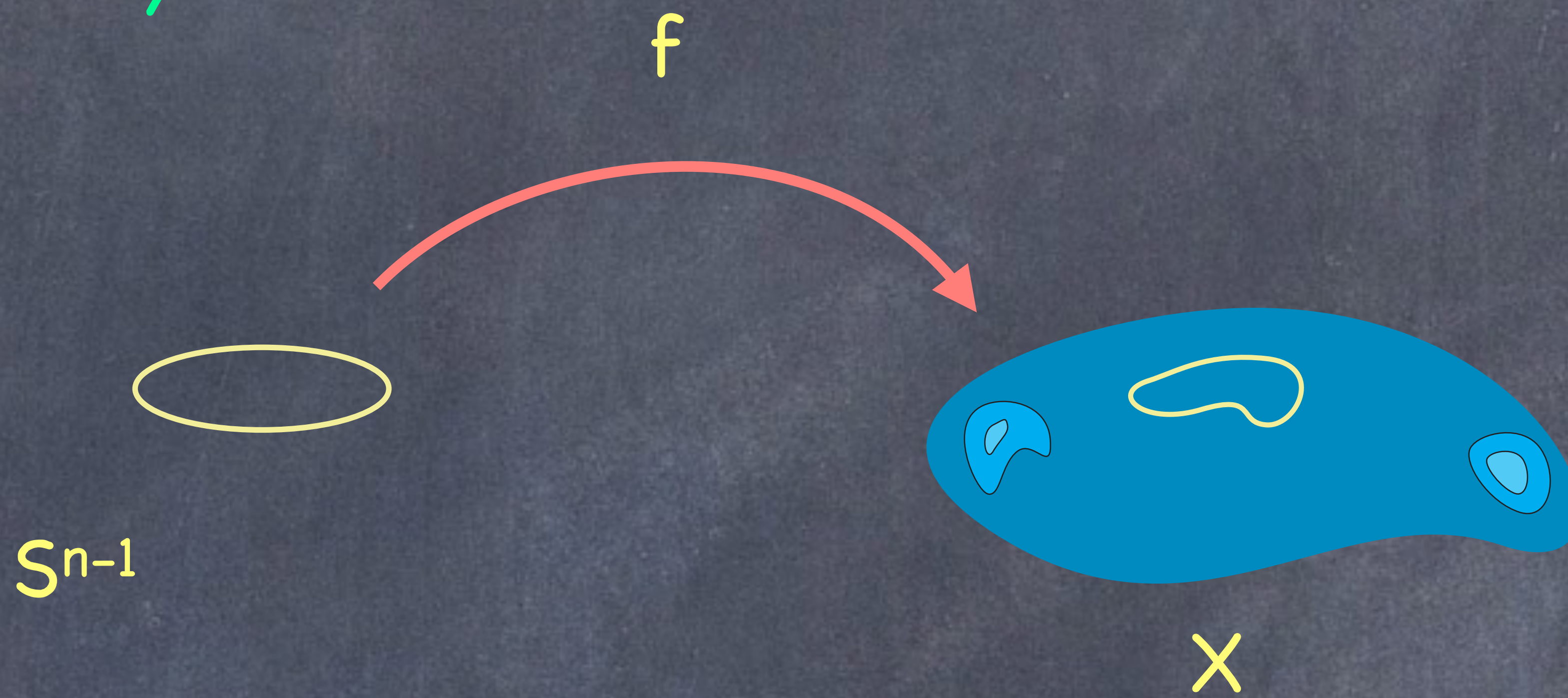
1970s: Griffiths investigated this question on smooth affine varieties in connection with the Hodge Conjecture.

Griffiths' approach was via complex analysis. By a theorem of Grauert, every complex vector bundle on a Stein manifold has a unique holomorphic connection. The obstruction to the existence of an algebraic structure can be expressed in terms of the growth rate of the connection form. Griffiths used value distribution theory (Nevanlinna theory) to express this obstruction.

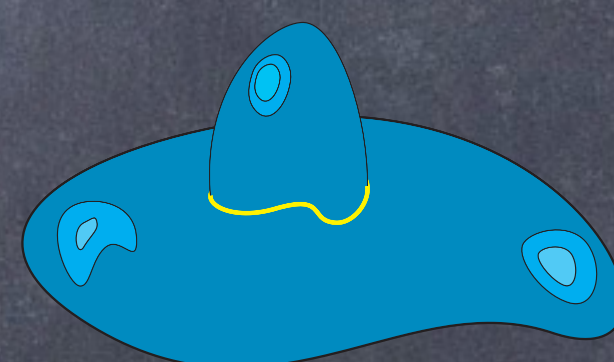
1990s - present: The work of Morel and Voevodsky let's us approach this question using the methods of homotopy theory.

Methods in homotopy theory

Homotopy groups



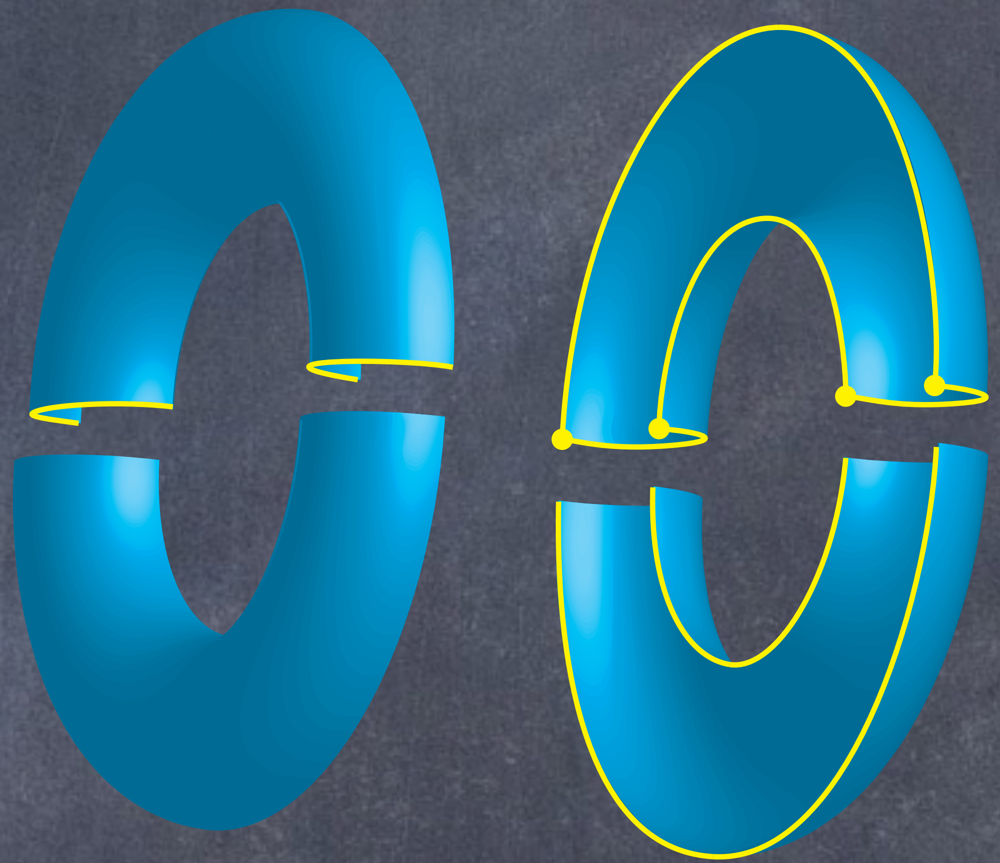
Cell attachment



$X \cup D^n / v \sim f(v)$

Methods in homotopy theory

Cell decompositions



Methods in homotopy theory

Eilenberg-MacLane spaces

Postnikov towers

Spectra

Brown-Gitler spectra

Dyer-Lashof algebra

Steenrod algebra

First examples

Algebra vs topology

P^n n-dimensional projective space over C

$[x_0, \ldots, x_n]$ $|x_i| = 1$

$\mathcal{O}(d)$ line bundle whose local sections are homogeneous rational functions of degree d

$\mathcal{O} = \mathcal{O}(0)$

Algebra vs topology

On $\mathbf{P}^1$ the vector bundles

$$\mathcal{O}(1) \oplus \mathcal{O}(-1) \qquad \text{and} \qquad \mathcal{O} \oplus \mathcal{O}$$

are not isomorphic:

$$\mathrm{Hom}(\mathcal{O}(1) \oplus \mathcal{O}(-1), \mathcal{O}(-1)) = \mathbf{C}$$

$$\mathrm{Hom}(\mathcal{O} \oplus \mathcal{O}, \mathcal{O}(-1)) = 0$$

Algebra vs topology

However they have the same first Chern class, so topologically they are isomorphic.

If we add variables y_0 and y_1 of degree -1 satisfying

$$x_0 y_0 + x_1 y_1 = 1$$

then there is an isomorphism

$$\mathcal{O}(-1) \oplus \mathcal{O}(1) \xrightarrow{\begin{bmatrix} x_0 & -y_1 \\ x_1 & y_0 \end{bmatrix}} \mathcal{O} \oplus \mathcal{O}$$

Jouanoulou's Device

R^n_* $C[x_0, \cdots, x_n, y_0, \ldots, y_n]/(x_0y_0 + \cdots + x_ny_n = 1)$

$$|x_i| = 1$$

$$|y_i| = -1$$

J^n Spec R^n_0

J^n is the space of rank 1 projection operators on C^{n+1}

The map $J^n \to P^n$ sending a projection operator to its image is an (algebraic) homotopy equivalence.

Algebra vs topology

The pullbacks of

$$\mathcal{O}(1) \oplus \mathcal{O}(-1) \quad \text{and} \quad \mathcal{O} \oplus \mathcal{O}$$

to $\mathbb{J}^2$ are isomorphic, so the classification of algebraic vector bundles is not a problem in "homotopy theory."

But maybe it has to do with splitting exact sequences of projective modules, and it is homotopy theoretic when restricted to affine varieties.

Algebra vs topology (dimension 2)

Let I be the ideal sheaf of $[0,0,1] \in P^2$

$$I = (x_0, x_1) \subset C[x_0, x_1, x_2]$$

Since $\dim_C \mathrm{Ext}^1(\mathcal{O}(2), I) = 1$

there is a unique non-trivial extension

$$0 \to \mathcal{O}(1) \to V \to I \otimes \mathcal{O}(-1) \to 0$$

defining a rank 2 vector bundle V on P^2.

Algebra vs topology

V is non-trivial:

$$\text{Hom}(\mathcal{O}(1), \mathcal{O}^2) = 0$$

V is topologically trivial: $(c_1(V) = c_2(V) = 0)$

Question: Is V trivial on J^2?

Algebra vs topology

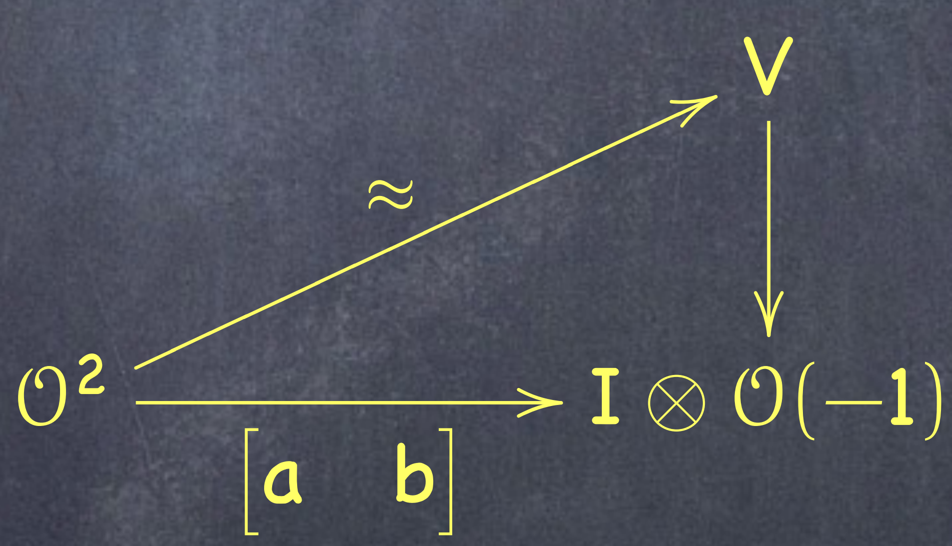
Observation: In the ring

$$\mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2]/(\underline{x} \cdot \underline{y} - 1)$$

the ideal

$$(x_0, x_1)$$

can be generated by two elements a and b of degree -1.



Algebra vs topology

Choose $r, s, t \in R^n_*$

satisfying $r\,x_0 + s\,x_1 + t\,x_2^2 = 1$

$|r| = |s| = -1$

$|t| = -2$

set

$$S = \begin{bmatrix} r \\ s \end{bmatrix} \cdot \begin{bmatrix} r & s \end{bmatrix} + \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} = \begin{bmatrix} r^2 & rs + t \\ rs - t & s^2 \end{bmatrix}$$

and $\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} x_0 & x_1 \end{bmatrix} \cdot S$

One can check that a and b generate (x_0, x_1) and have degree -1.

Question: Is every topologically trivial vector bundle on J^n trivial?

Question: Is the map

$$\mathrm{Vect}_k^{alg}(J^n) \to \mathrm{Vect}_k^{top}(P^n)$$

a bijection?

Homotopy invariance of bundle theory in algebraic geometry

Serre (1955)

Faisceaux algébriques cohérents

Vector bundles on X ⟷ projective modules over O_X

"Note that when $X=K^r$ (so that $O_X=K[x_1,\ldots,x_r]$) it is unknown if there exist finitely generated projective modules which are not free."

Serre's problem

Serre (1955) Faisceaux algébriques cohérents

Serre's problem was solved independently by Quillen and Suslin

Theorem (Quillen, Suslin 1977): If k is a field then every finitely generated projective module over

$$k[x_1,\dots,x_n]$$

is free.

Bass-Quillen conjecture

Bass-Quillen Conjecture: If A is a commutative regular ring of finite Krull dimension, then every finitely generated projective module over $A[x_1,\ldots,x_n]$ is extended from A, ie

$$P \approx P_0 \bigotimes_A A[x_1, \ldots, x_n]$$

where

$$P_0 = P \bigotimes_{A[x_1,\ldots,x_n]} A$$

Bass-Quillen conjecture

Theorem (Lindel, 1981): The Bass-Quillen conjecture is true when A is a finitely generated (commutative regular) algebra over a field.

(the geometric case)

Lam, T. Y. "Serre's problem on projective modules"

Algebraic and motivic homotopy Theory

Classical homotopy theory

$\mathcal{C}$ the category of topological spaces

$\mathcal{W}$ the class of (weak) homotopy equivalences

$\mathcal{C}^{top}$ Classical homotopy theory

$$[X, Y]_{top} = \mathrm{ho}\, \mathcal{C}^{top}(X, Y)$$

Algebraic homotopy theory

Ken Brown ('73), "Abstract homotopy theory and generalized sheaf cohomology"

Sm the category of smooth varieties (over $\mathbb{C}$)

$\mathcal{C}$ the category of (simplicial) presheaves on Sm

Algebraic homotopy theory

Ken Brown ('73), "Abstract homotopy theory and generalized sheaf cohomology"

Sm the category of smooth varieties (over $\mathbb{C}$)

C the category of (simplicial) presheaves on Sm

W the class of Nisnevich (hyper-)coverings.

C^{alg} Algebraic homotopy theory

$$[X, Y]_{alg} := \mathrm{ho}\, C^{alg}(X, Y)$$

Motivic Homotopy Theory (Morel-Voevodsky 2001)

Sm — the category of smooth varieties (over $\mathbb{C}$)

C — the category of (simplicial) presheaves on Sm

W — the class of Nisnevich (hyper-)coverings together with the maps $X \times A^1 \to X$

C^{mot} — Motivic homotopy theory

$$[X, Y]_{mot} := \text{ho}\, C^{mot}(X, Y)$$

Abstract homotopy theory

$$C^{alg} \to C^{mot} \to C^{top}$$

realization functors

Algebraic and Motivic Vector Bundles

Classifying spaces

Classifying space for principal G-bundles: BG

$$\cdots G \times G \rightrightarrows G \rightrightarrows *$$

Bundle theory Adams was referring to

Topology

$$[X, BGL_k(\mathbb{C})]_{top} \approx [X, BU(k)]_{top} \approx Vect_k(X)$$

Classifying spaces

For X a smooth variety, there is an isomorphism

$$[X, BGL_k]_{alg} \approx \mathrm{Vect}_k^{alg}(X)$$

(local description of vector bundles in terms of charts and transition functions)

Algebraic and motivic vector bundles

$\mathrm{Vect}_k^{top}(X) = [X, BGL_k(\mathbb{C})]_{top}$ $\mathrm{Vect}_k^{alg}(X) = [X, BGL_k]_{alg}$

$$\mathrm{Vect}_k^{mot}(X) = [X, BGL_k]_{mot}$$

motivic vector bundles

Realization maps

$$\mathrm{Vect}_k^{alg}(X) \to \mathrm{Vect}_k^{mot}(X) \to \mathrm{Vect}_k^{top}(X)$$

Vector bundles on affine varieties

Fabien Morel (2012) M. Schlichting (2015)

Aravand Asok, Marc Hoyois, Matthias Wendt (2015)

Theorem: For smooth affine X, the map

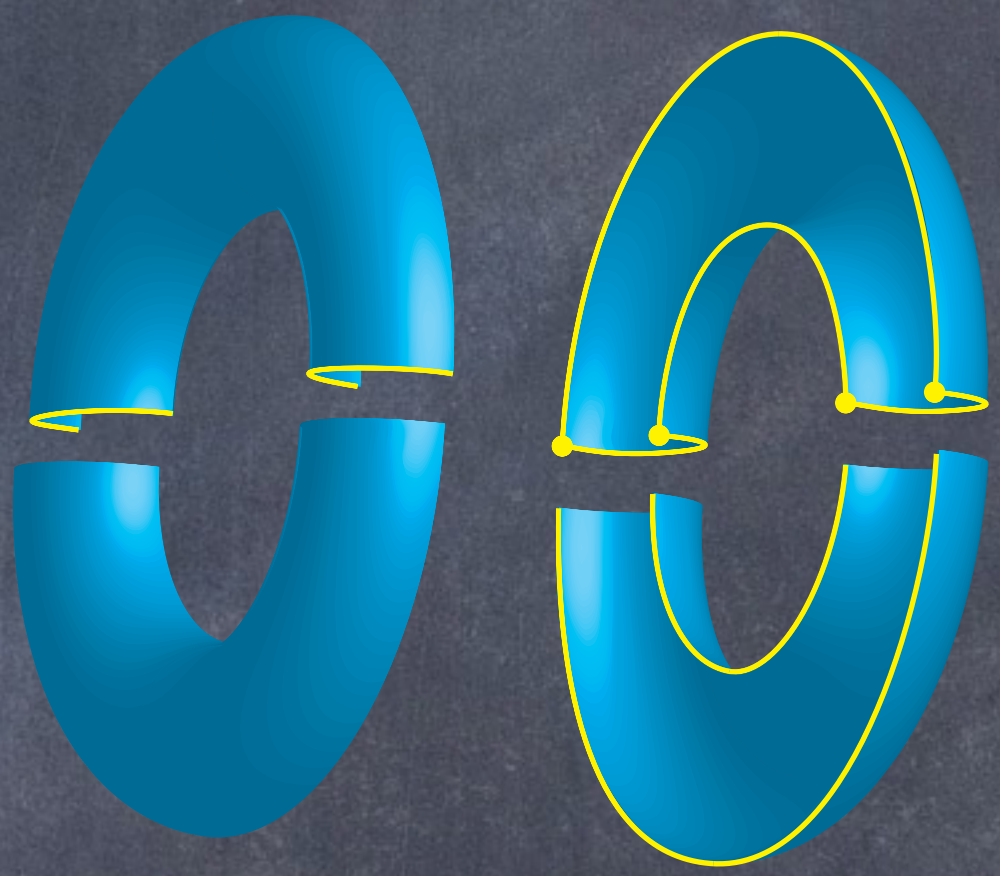
$$\mathrm{Vect}_k^{alg}(X) \to \mathrm{Vect}_k^{mot}(X)$$

is an isomorphism.

affine homotopy invariance of algebraic bundle theory

Obstruction Theory

Cell decompositions



Obstruction theory

$$\bigoplus H^i(X; \pi_i Y) \Rightarrow [X, Y]$$

Algebraic Homotopy Theory:

$$\pi_i BGL_k = \begin{cases} GL_k & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H^1(X; GL_k) \approx Vect_k(X)$$

Obstruction theory

Motivic Homotopy Theory:

$$\pi_0 BGL_k = *$$

go to town here…mention our example (rational chern classes) relation with griffiths,

Interesting obstruction theory

Morel and Asok-Fasel have investigated splitting free summands off of projective modules over regular rings and classification of projective modules over smooth algebras of Krull dimension less than or equal to 3.

The Rees Bundles

The Rees bundles

$$\mathrm{Vect}_2^{top}(CP^n) \approx [CP^n, BU(2)]_{top}$$

Obstruction theory

$$\bigoplus_i H^{2i}(CP^n; \pi_{2i} BU(2)) \Rightarrow \mathrm{Vect}_2(CP^n)$$

one needs to understand

$$\begin{aligned} \pi_{2i} BU(2) &\approx \pi_{2i-1} U(2) \\ &\approx \pi_{2i-1} SU(2) \qquad (i > 1) \\ &\approx \pi_{2i-1} S^3 \end{aligned}$$

odd homotopy groups of S^3

The Rees bundles

The first non-zero odd homotopy group of S^3 (localized at p) is

$$\pi_{4p-3} S^3 \approx Z/p$$

Rees considers the bundle ξ_p corresponding to

$$CP^{2p-1} \to CP^{2p-1}/CP^{2p-2} = S^{4p-2} \to BU(2)$$

Theorem (Rees): The bundle ξ_p is non-trivial for all p.

Note: The bundle ξ_p has no Chern classes.

The Rees bundles

Question: Do the Rees bundles exist over J^{2p-1}?

The Rees bundles

Answer (Asok, Fasel, H.): Yes.

Motivic Rees bundles

The motivic Rees bundle corresponds to a rank 2 projective module over the degree zero part, R_0 , of the ring

$$C[x_0, \cdots, x_{2p-1}, y_0, \cdots, y_{2p-1}]/\underline{x} \cdot \underline{y} = 1$$

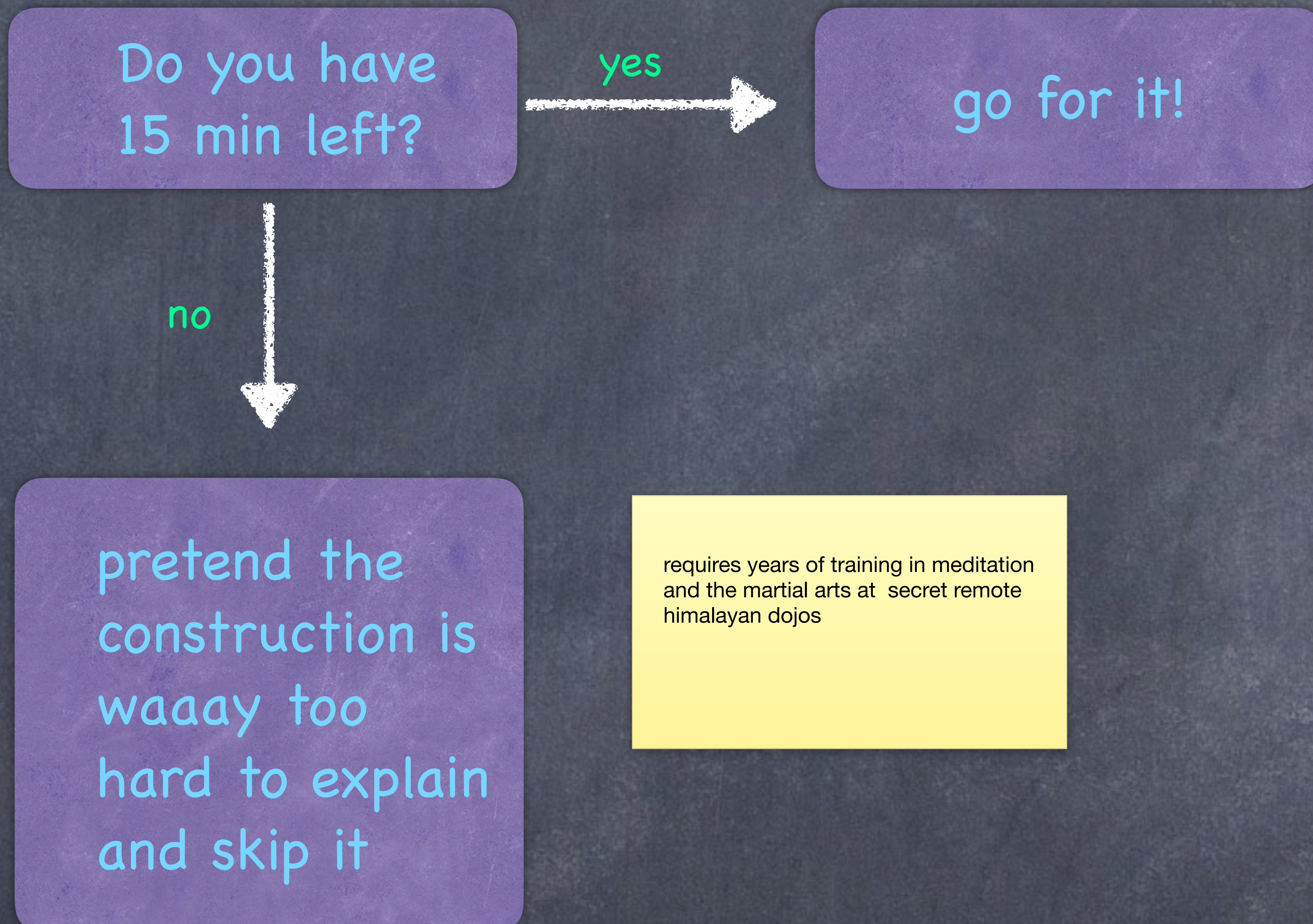
Homotopy theory implies that it arises as the kernel of a map

$$(R_0)^3 \to R_0$$

$$(s, t, u) \mapsto as + bt + cu$$

for some unimodular row [a,b,c] in R_0 .

Construction of the motivic Rees bundles



requires years of training in meditation and the martial arts at secret remote himalayan dojos

Construction of ζ_p

Key step

$$\begin{array}{ccc} & & GL_n(\mathbb{C}) \\ & \nearrow & \downarrow \\ S^{2n-1} & \xrightarrow[(n-1)!]{} & S^{2n-1} \end{array}$$

leads to a map

$$\alpha_p : S^{2p} \to S^3$$

which may be composed with its suspension to give

$$S^{4p-3} \xrightarrow{S^{2p-3}\alpha_p} S^{2p} \xrightarrow{\alpha_p} S^3$$

Motivic spheres

"dimension"

$S^{p,q}$ $(p \geq q)$

Hodge weight

$S^{1,1} = A^1 - \{0\}$

$S^{1,0} = A^1/\{0,1\}$ $S^{p,q} = (S^{1,1})^{\wedge q} \wedge (S^{1,0})^{\wedge p-q}$

Other examples

$$A^n - \{0\} \approx S^{2n-1,n}$$

$$P^n/P^{n-1} \approx S^{2n,n}$$

Motivic homotopy groups

$$\pi_{a,b}(X) = [S^{a,b},X]_{mot}$$

topological realization

$$C^{mot} \to C^{top}$$

$$S^{1,1} \mapsto S^1 \qquad S^{1,0} \mapsto S^1 \qquad S^{p,q} \mapsto S^p$$

$$\pi_{a,b}X \to \pi_a X$$

Movitic α_p and ζ_p

We need a diagram

$$
\begin{array}{ccccc}
 & & & & GL_n \\
 & & & \nearrow & \downarrow \\
S^{2n-1,n} \approx & J(\mathbb{A}^n - \{0\}) & & \xrightarrow[(n-1)!]{} & \mathbb{A}^n - \{0\} \approx S^{2n-1,n}
\end{array}
$$

Suslin's n! theorem (unimodular rows)

Theorem (Suslin): Let R be a commutative ring, and

$$x_1, \cdots, x_n \quad \text{and} \quad y_1, \cdots, y_n$$

elements of R, satisfying

$$x_1y_1 + \cdots + x_ny_n = 1$$

If

$$\epsilon_1, \cdots, \epsilon_n$$

is a sequence of positive integers whose product

$$\epsilon_1 \cdots \epsilon_n$$

is divisible by (n-1)!, then there exists a unimodular matrix whose first row is

$$[x_1^{\epsilon_1} \ \dots \ x_n^{\epsilon_n}]$$

Movitic α_p and ζ_p

(requires (-1) to be a sum of squares)

Suslin's theorem gives a diagram

$$\begin{array}{ccccc} & & & & GL_n \\ & & & \nearrow & \downarrow \\ S^{2n-1,n} \approx J(A^n - \{0\}) & & \xrightarrow[(n-1)!]{} & & A^n - \{0\} \approx S^{2n-1,n} \end{array}$$

Leads, as in topology, to a map (the case n=p+1)

$$\alpha_p^{mot} : S^{2p,p+1} \to S^{3,2}$$

which when composed with its suspension gives

$$\zeta_p^{mot} : S^{4p-3,2p} \to S^{3,2}$$

(having order p)

Rank 2 vector bundles on P^n

the motivic Rees bundle should arise as

$$P^{2p-1} \to P^{2p-1}/P^{2p-2} = S^{4p-2,2p-1} \to BGL_2$$

from a map

$$S^{4p-3,2p-1} \to A^2 - \{0\} = S^{3,2}$$

But we produced

$$\zeta_p^{mot} : S^{4p-3,2p} \to S^{3,2}$$

The Hodge weight is wrong.

Motivic ρ

There is a map in motivic homotopy theory

$$\rho : (\pi_{a,b} X)_{\text{torsion}} \to (\pi_{a,b-1} X)_{\text{torsion}}$$

which changes weight, and realizes to the identity

Motivic Rees bundles

The motivic map ζ_p is killed by p, so it can be multiplied by ρ, and we can form

$$P^{2p-1} \to P^{2p-1}/P^{2p-2} \approx S^{4p-2,p-1} \xrightarrow{\rho\zeta_p} BGL_2$$

and lift the Rees bundles to motivic vector bundles.

Motivic Rees bundles

The motivic Rees bundle corresponds to a rank 2 projective module over the degree zero part, R_0, of the ring

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Question: Is the map

$$\mathrm{Vect}_k^{alg}(J^n) \to \mathrm{Vect}_k^{top}(P^n)$$

a bijection?

Unstable Adams-Novikov resolution

$$X \to \Omega^\infty MU \wedge X \cdots \to (\Omega^\infty MU \wedge (\,-\,))^{m+1} X \to \cdots$$

The Wilson Space Hypothesis

Homologically even spaces

Many naturally occurring spaces have cell decompositions with only even dimensional cells: homologically even spaces

For such a space X

i) $H_{2n-1}X = 0$

ii) $H_{2n}(X)$ is a free abelian group.

Homotopically even spaces

Definition: A space X is homotopically even if

i) $\pi_{2n-1}X = 0$

ii) $\pi_{2n}X$ is a free abelian group.

Wilson spaces (even spaces)

Definition: An even space is a space which is both homologically and homotopically even.

Examples: CP^∞ BU BSU

Wilson's Theorem

Theorem (Steve Wilson): The classifying spaces for complex cobordism

$$\Omega^\infty S^{2n} \wedge MU$$

are even spaces.

Consequence. If X is homologically even then

$$\Omega^\infty MU \wedge X$$

is even.

Unstable Adams-Novikov resolution

Start with X homologically even

$$X \to \Omega^\infty MU \wedge X \cdots \to (\Omega^\infty MU \wedge (-))^{m+1} X \to \cdots$$

even

Leads to a resolution of X by K(Z,2n)

Motivic spaces

Notation: $S^{2n,n} = A^n/A^n-\{0\}$

$S^{2n-1,n} = A^n-\{0\}$

Definition: A motivic space X is homologically even if

$$HZ \wedge X \approx \bigvee HZ \wedge S^{2n,n}$$

Definition: A motivic space X is homotopically even if

$\pi_{2n,n}X$ is a free abelian group.

$$\pi_{2n-1,n}X = 0$$

Motivic spaces

Definition: A motivic space X is even if it is both homologically and homtopically even.

Wilson space hypothesis: For every n, the motivic space

$$\Omega^\infty S^{2n,n} \wedge MGL$$

is even.

Equivariant Wilson spaces

Theorem (Hill, H.): For every integer n, the real space

$$\Omega^\infty S^{n\rho} \wedge MU_{\mathbb{R}}$$

is an even real space.

complex cobordism with
complex conjugation

A friend sent me this yesterday. I know it's not possible but I take some comfort in imagining it to be a "sighting."


MATH PROFESSOR