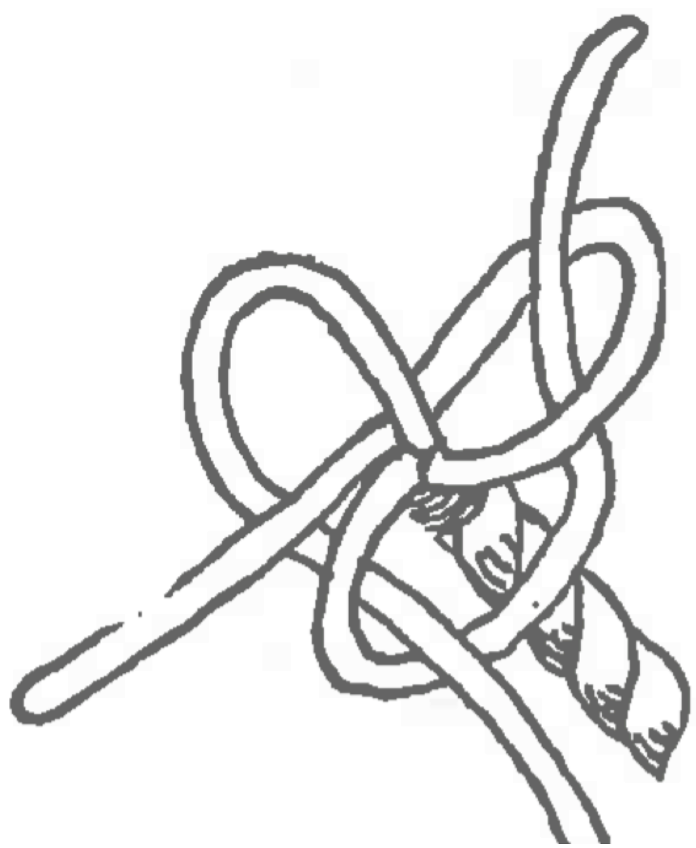
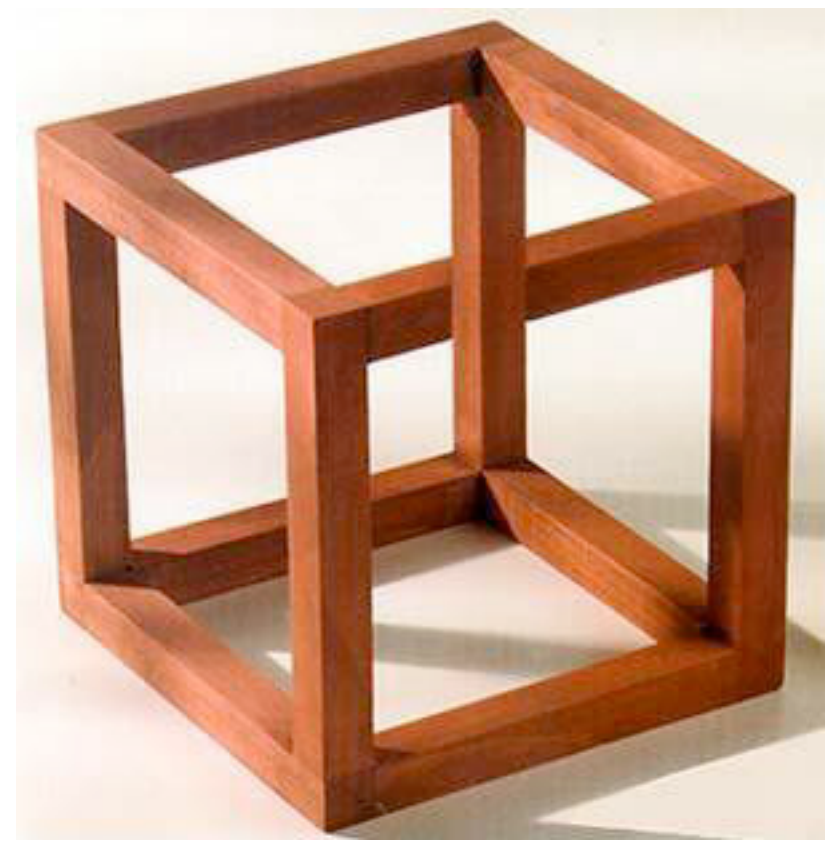
Entanglement of embedded graphs

Toen Castle

with Myf Evans, Vanessa Robins AND Stephen Hyde





Workshop on Topology: Identifying Order in Complex Systems

Saturday, 18 April 2015

The Institute for Advanced Study

Penn

SIMONS FOUNDATION

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Ταλκ στρυχτυρε

1. Εμβεδδεδ γραπησ ανδ μεασυρεσ οφ τηειρ εντανγλεμεντ.

2. Εξιστενχε οφ νον−κνοτ ανδ νον−λινκ εντανγλεμεντ (ραϖελσ).

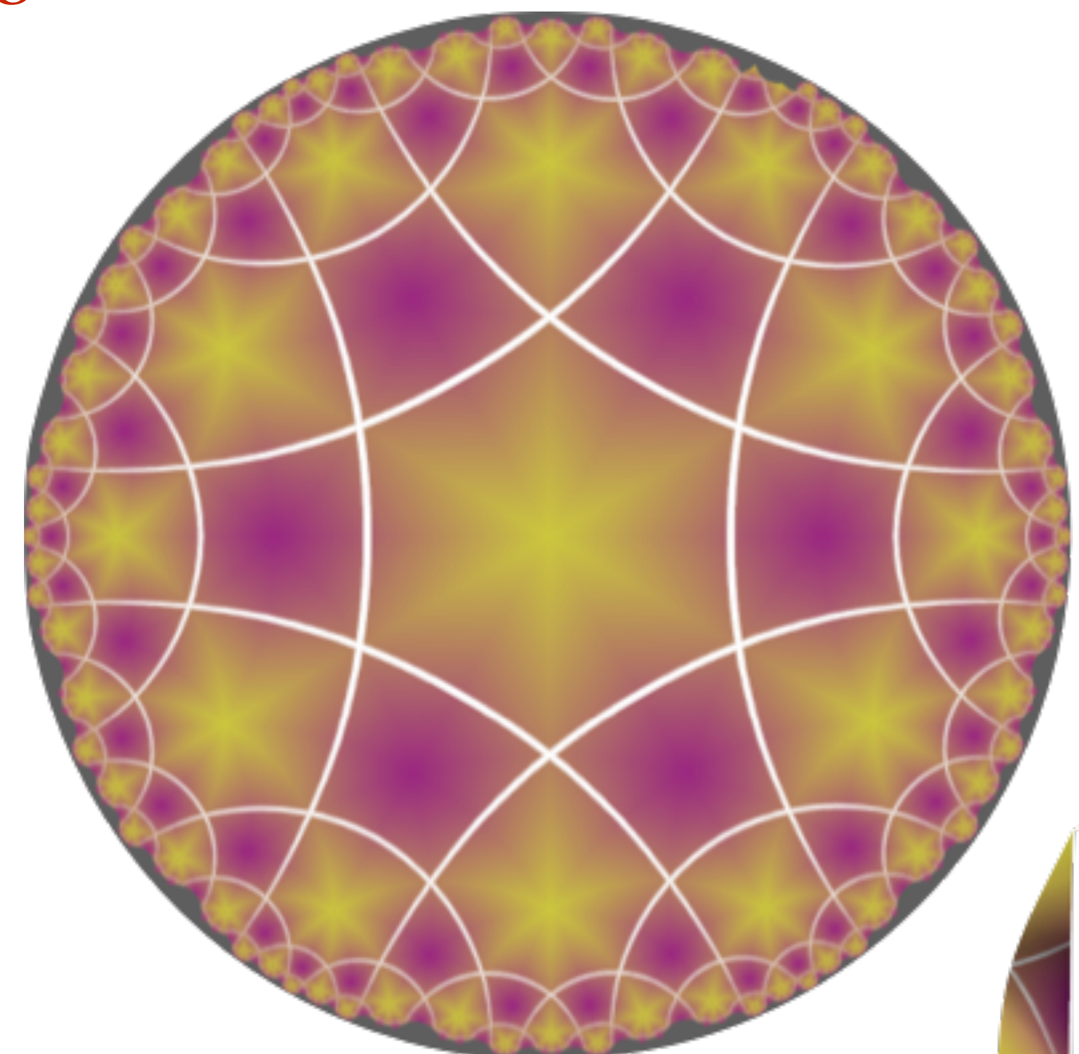
3. Χομπλεξ ραϖελσ ανδ τηειρ προπερτιεσ.

4. Ρελατινγ εντανγλεμεντ ανδ πλαναριτψ, τηεν χλασσιφψινγ εντανγλεμεντ τψπεσ.

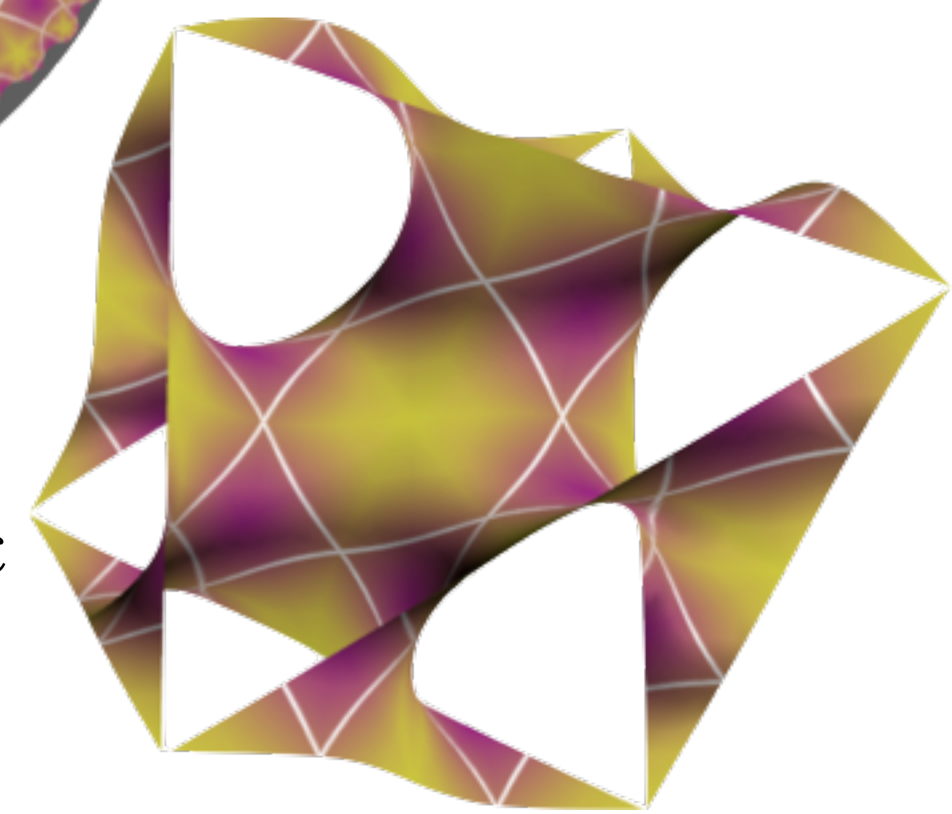
1. Σταρτινγ τηε στορψ – μακινγ εμβεδδεδ γραπησ φρομ συρφαχε τιλινγσ

επινετ

Ευχλιδεαν
Παττερνσ
Ιν
Νον
Ευχλιδεαν
Τιλινγσ



Τριπλψ Περιοδιχ
Μινιμαλ Συρφαχε



επινετ.ανυ.εδυ.αυ

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Στεπηεν Ηψδε, Στυαρτ Ραμσδεν, ςανεσσα Ροβινσ, Ολαφ Δελγαδο Φρειδριχησ

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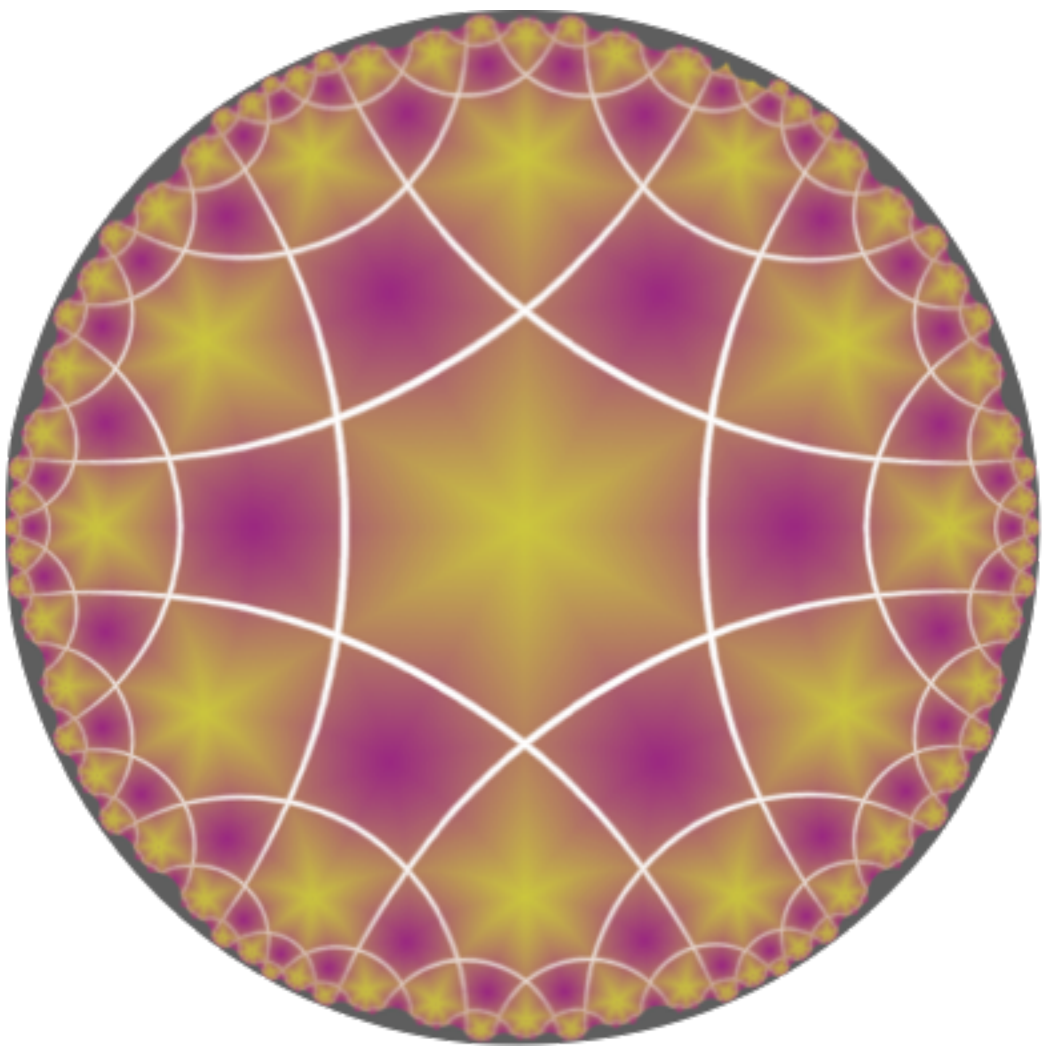

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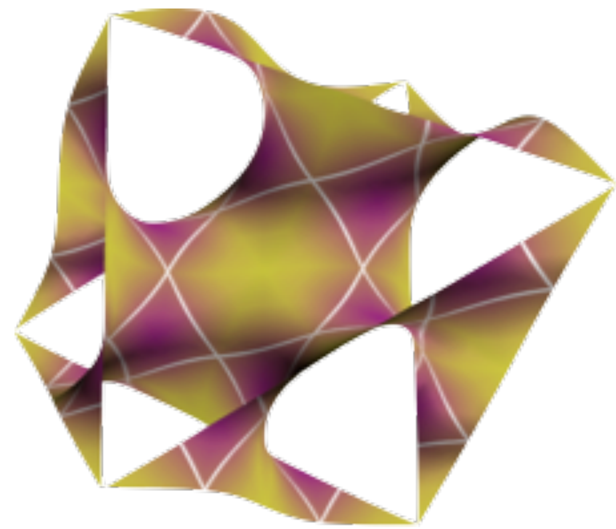

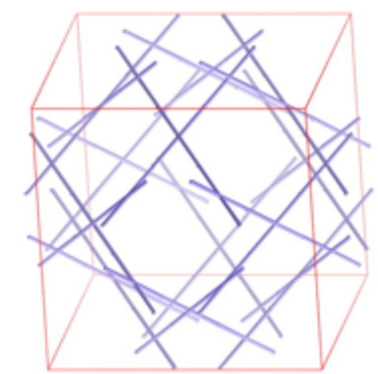

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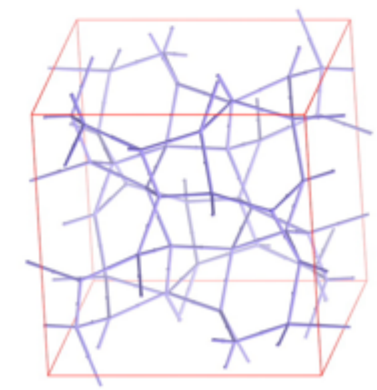

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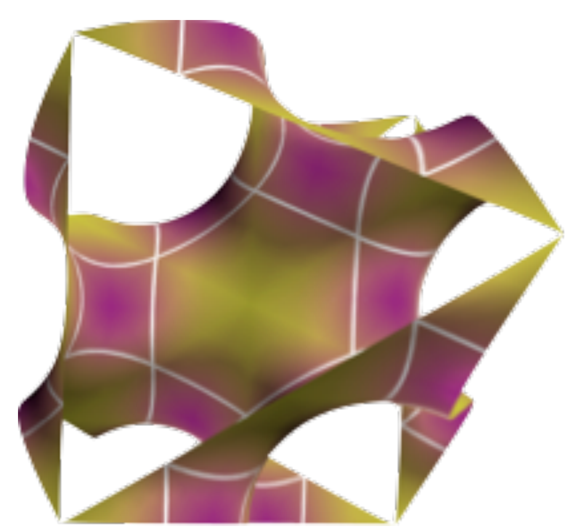

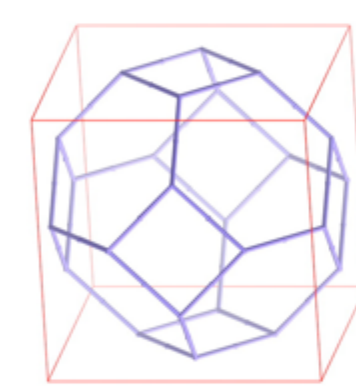

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Στεπηεν Ηψδε, ςανεσσα Ροβινσ, Ολαφ Δελγαδο Φρειδριχησ, Στυαρτ Ραμσδεν

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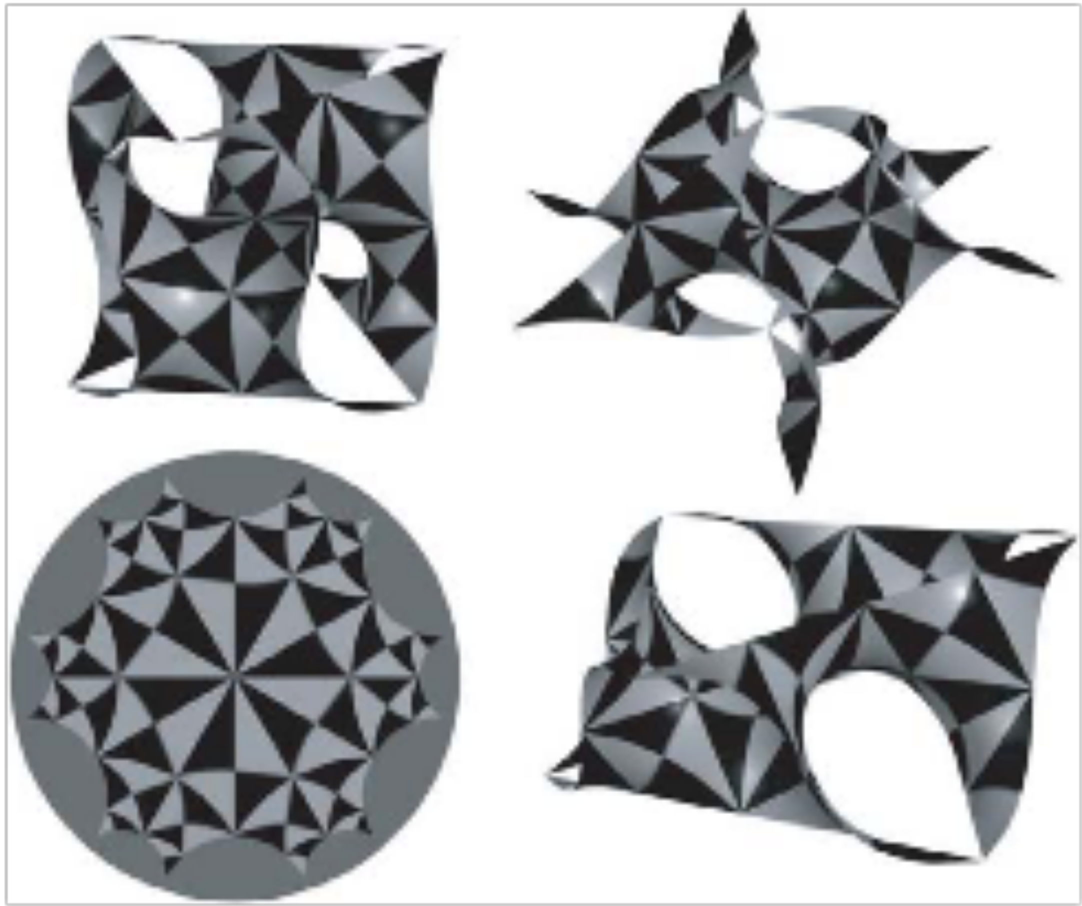


Σтανδαρδ μαπ φρομ H^2 το ΤΠΜΣ

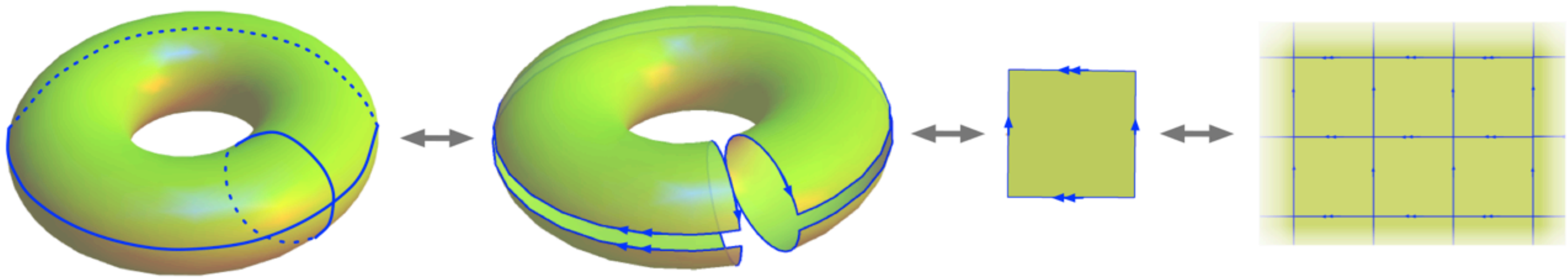
?

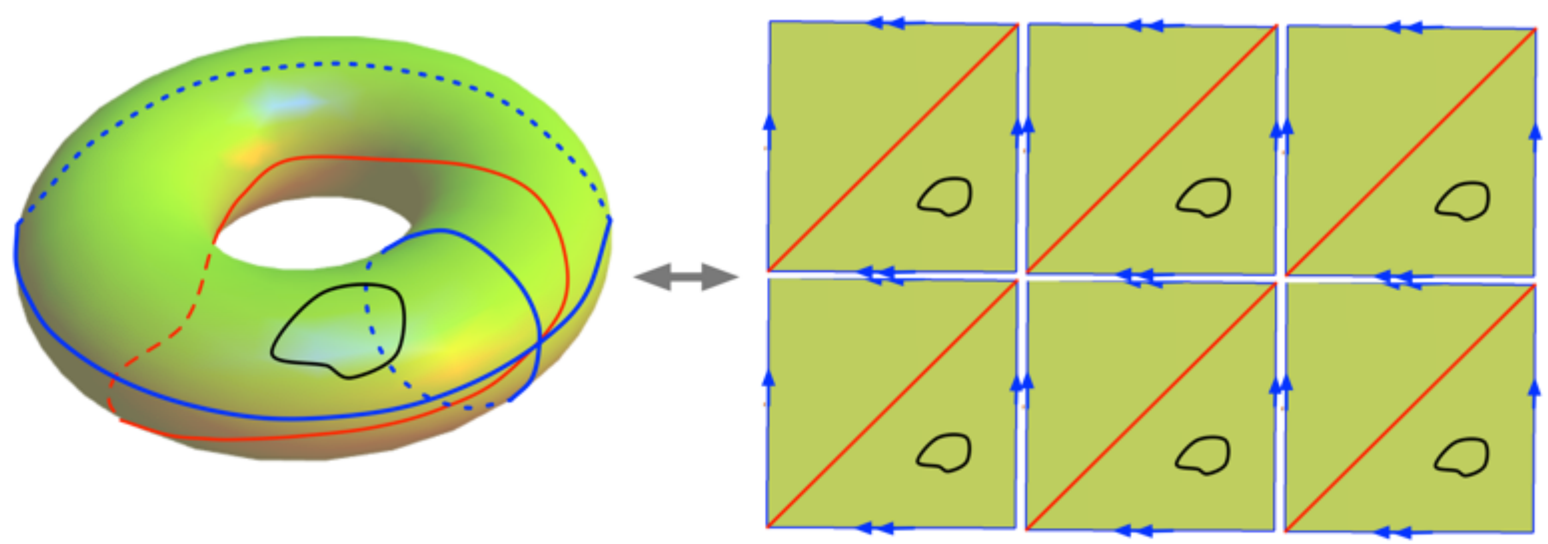
Μαππινγ H^2 το ΤΠΜΣ ωιτη α σηεαρεδ υνιτ χελλ

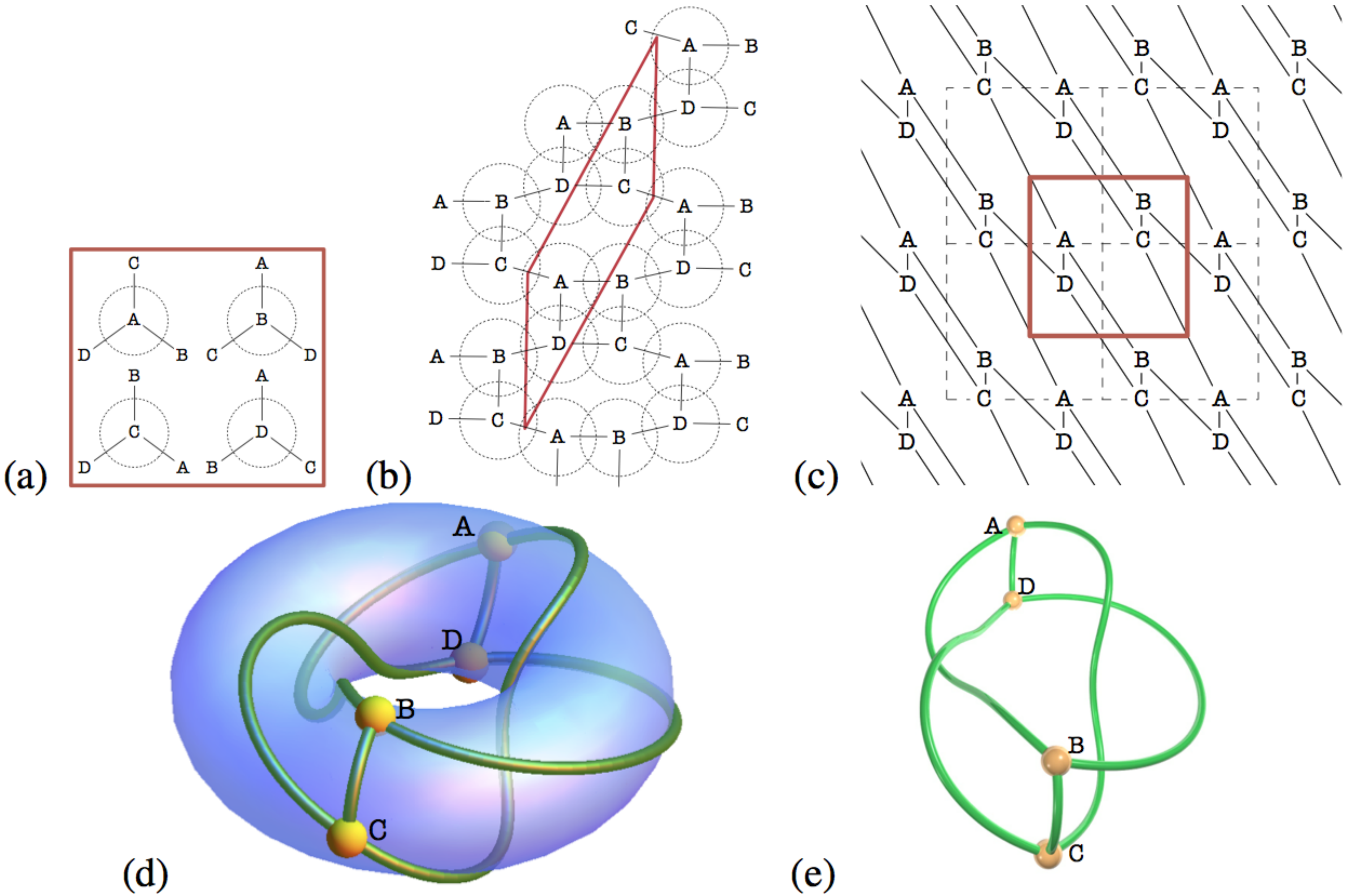
A note on the two symmetry-preserving covering maps of the gyroid minimal surface

V. Robins[1,a], S.J. Ramsden[1], and S.T. Hyde[1]

Τηε μαπ βετωεεν τηε τορυσ ανδ ιτσ υνιϖερσαλ χοϖερ, τηε Ευχλιδεαν πλανε.







Remove the torus and you've got a graph embedding in space.

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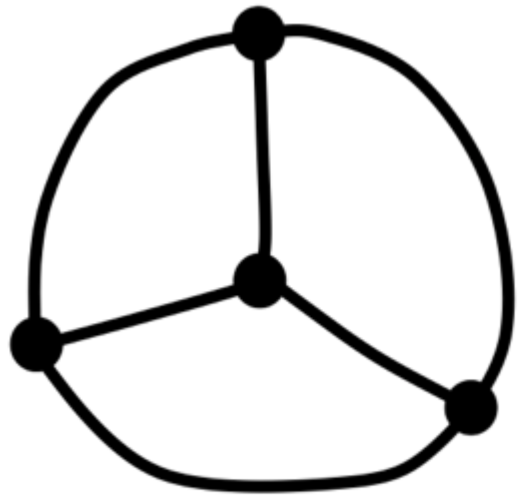
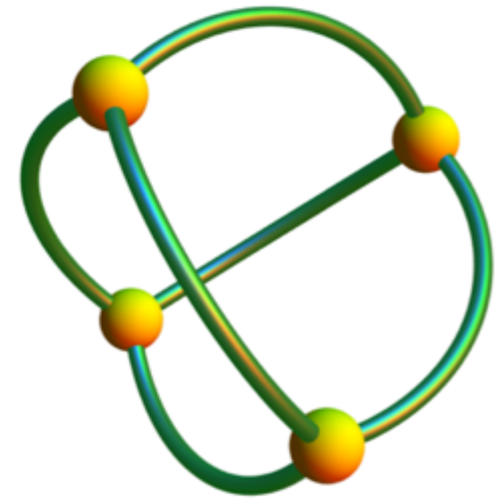
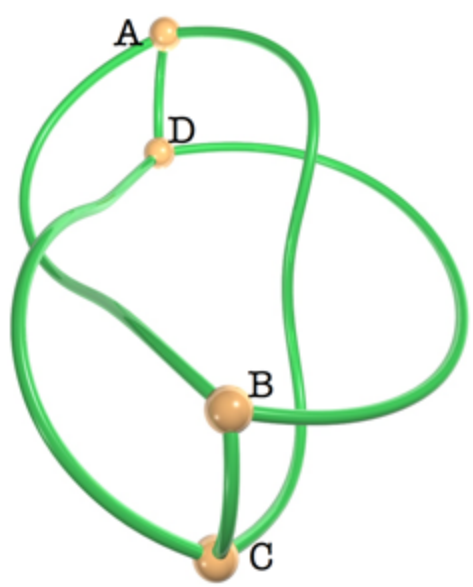
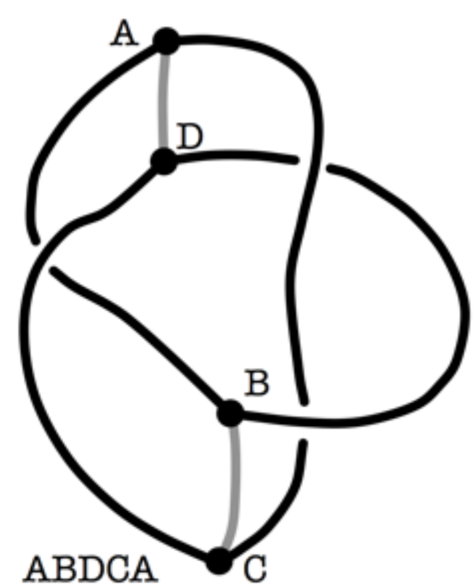
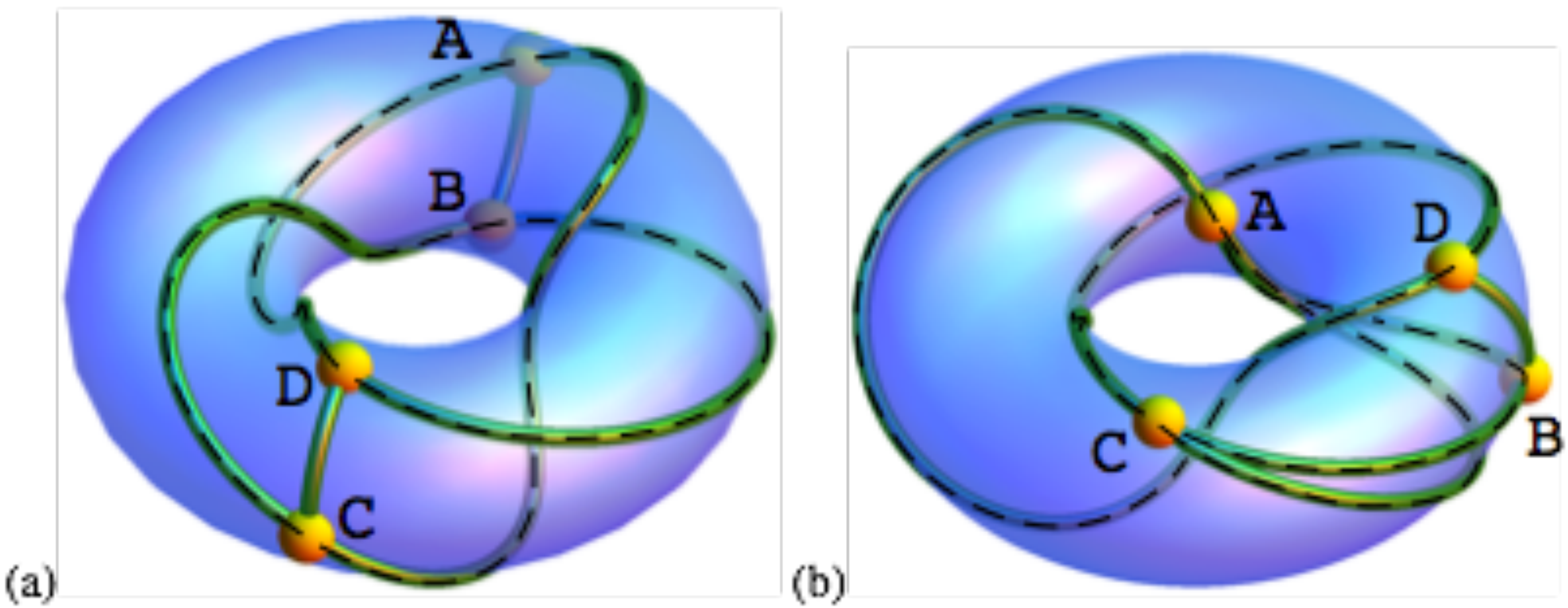



K_4, the tetrahedral graph, embeds on the sphere in the standard tetrahedral way (without entanglements).



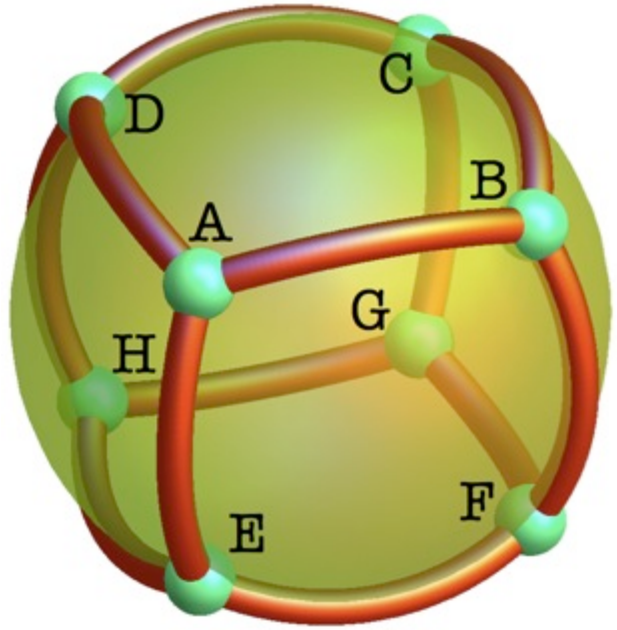


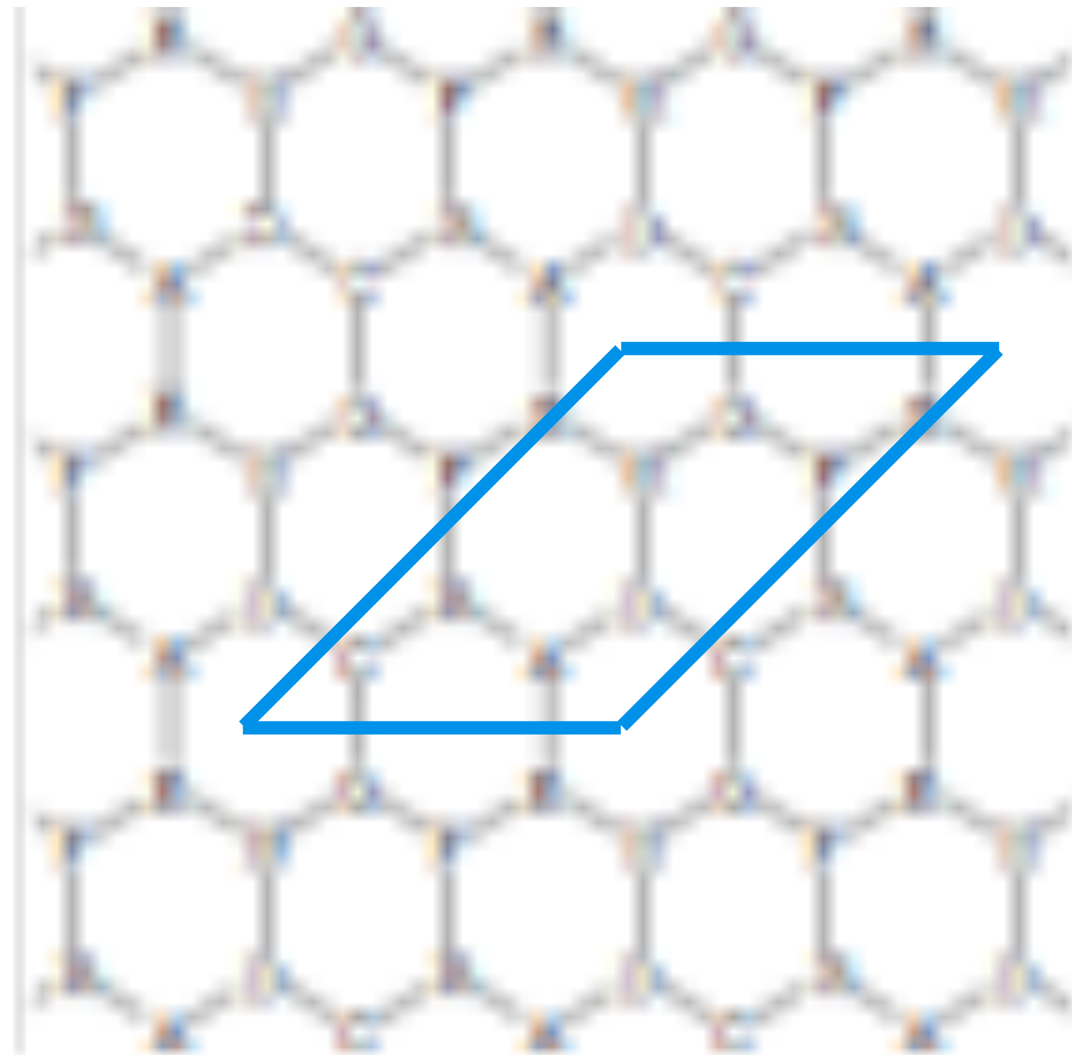
On the torus, this graph can be more topologically interesting.

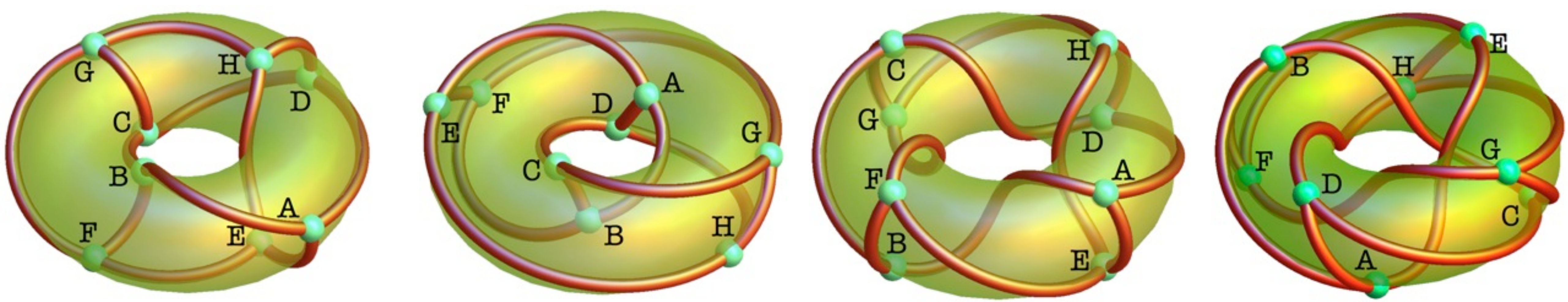


Two embeddings of K_4 on the torus, with one and two (respectively) trefoils.

Cube examples

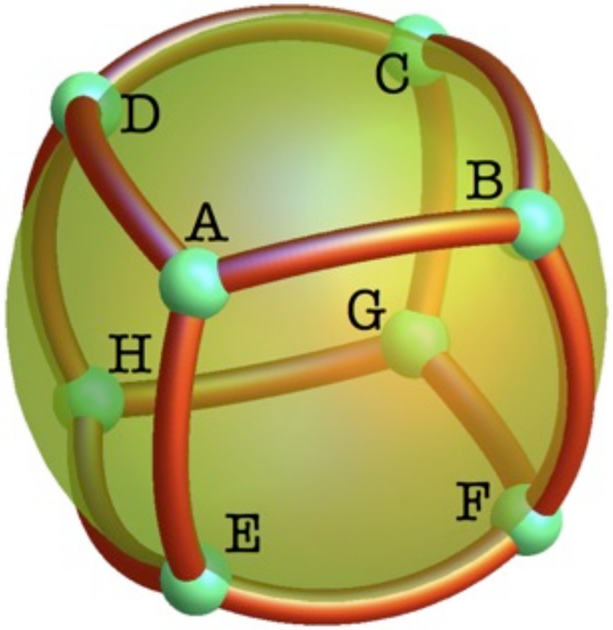




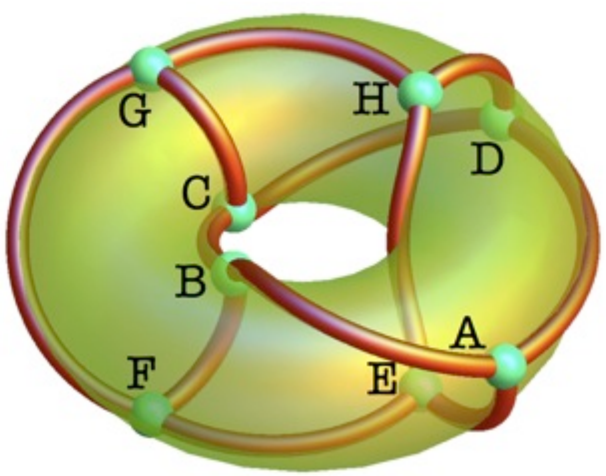


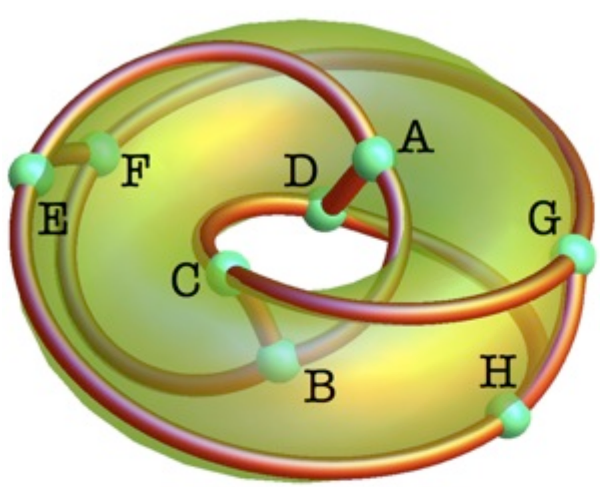
Torus examples with slightly stretched unit cells.

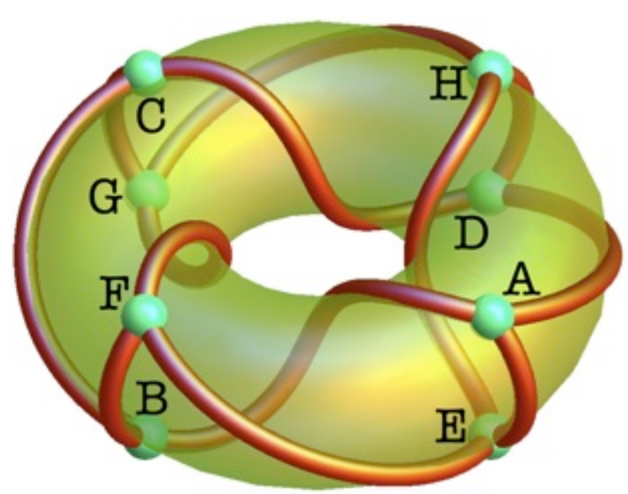
Tabulating entanglement via constituent (torus) knots and (torus) links.

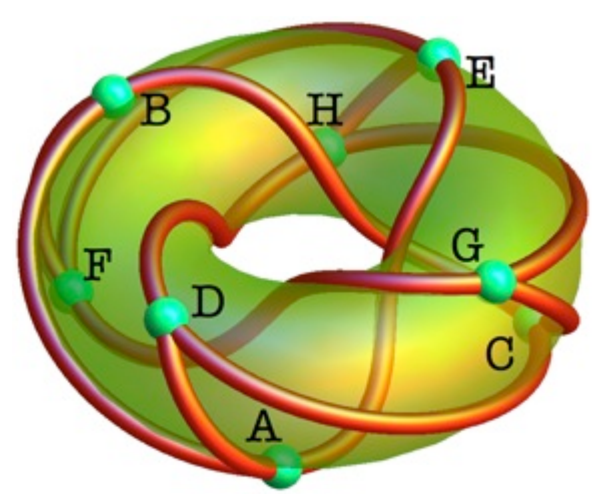


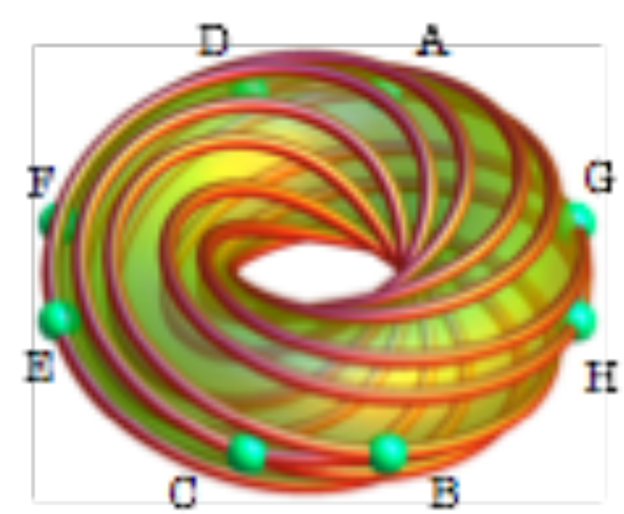
Energy	Name	Tiling	Side vectors	Knots	Links
0.83	-	⟨4, 4, 4, 12⟩	(1,0), (0,1)	-	-
1.45	-	Sphere	-	-	-
1.33	*A*	Honeycomb	(1,0), (0,1)	-	(2,2)
1.76	*C*	Brick wall	(1,0), (1,1)	(3,2)	2*(2,2)
2.20	*B*	Brick wall	(1,2), (0,1)	-	(2,4)
2.31	*D*	Honeycomb	(1,1), (1,0)	2*(2,3)	(2,2),(2,4)
2.82	-	⟨4, 4, 6, 10⟩	(1,1), (1,2)	4*(3,2)	(2,2)
2.91	-	Brick wall	(1,0),(2,1)	(3,2), (5,2)	2*(4,2)
2.98	-	⟨4, 4, 6, 10⟩	(1,1), (2,1)	4*(3,2)	(4,2)
3.14	-	⟨4, 4, 6, 10⟩	(3,1),(1,0)	-	(6,2)
3.16	-	⟨4, 4, 8, 8⟩	(0,1),(1,3)	4*(3,2)	(2,2), (4,2)
3.53	*E*	Brick wall	(1,2), (1,1)	4*(2,3),(3,4)	2*(2,2),(2,4)











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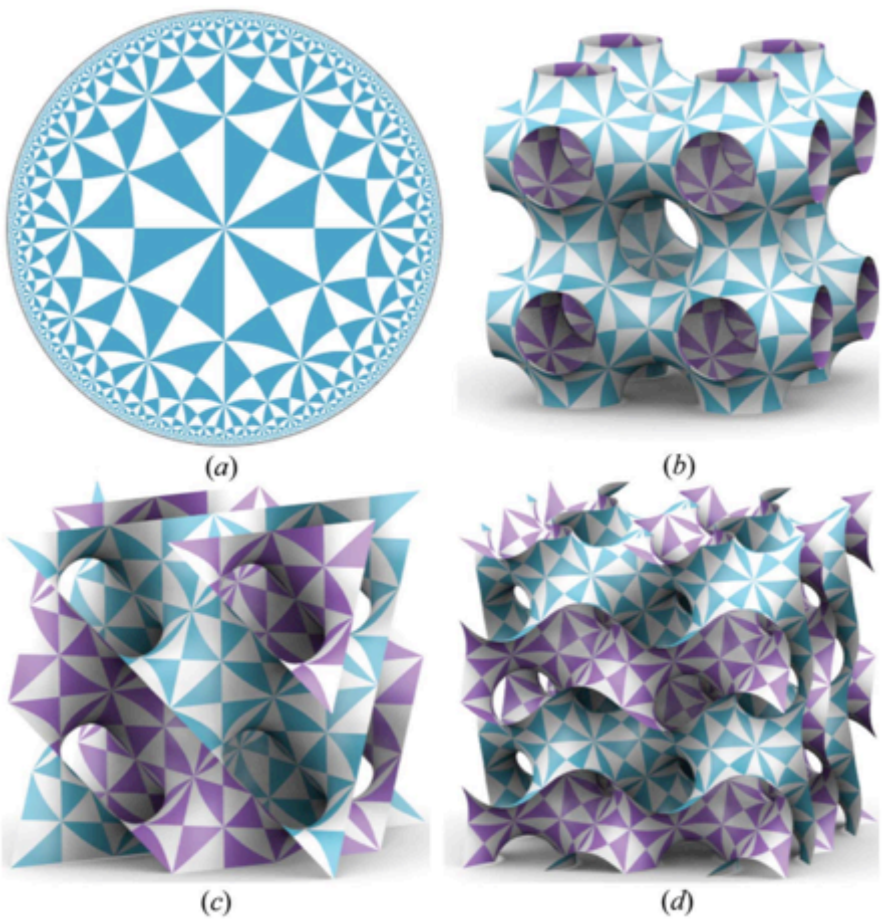
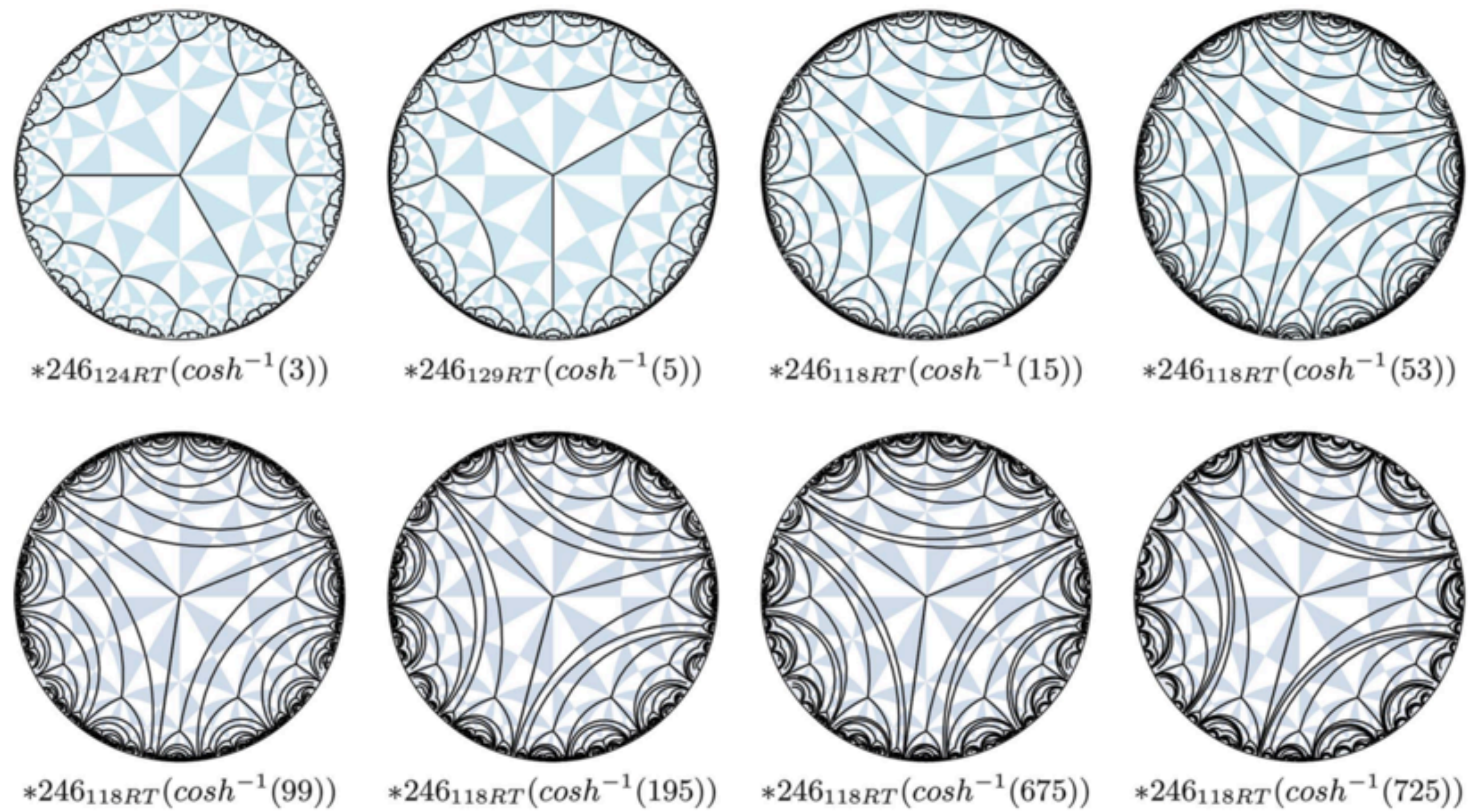
Φινισηινγ τηε στορψ – σηεαρινγ τηε υνιτ χελλσ ιν Η^3 ανδ ον ΤΠΜΣ'σ





Periodic entanglement I: networks from hyperbolic reticulations

Myfanwy E. Evans, Vanessa Robins and Stephen T. Hyde

Acta Cryst. (2013). A**69**, 241–261

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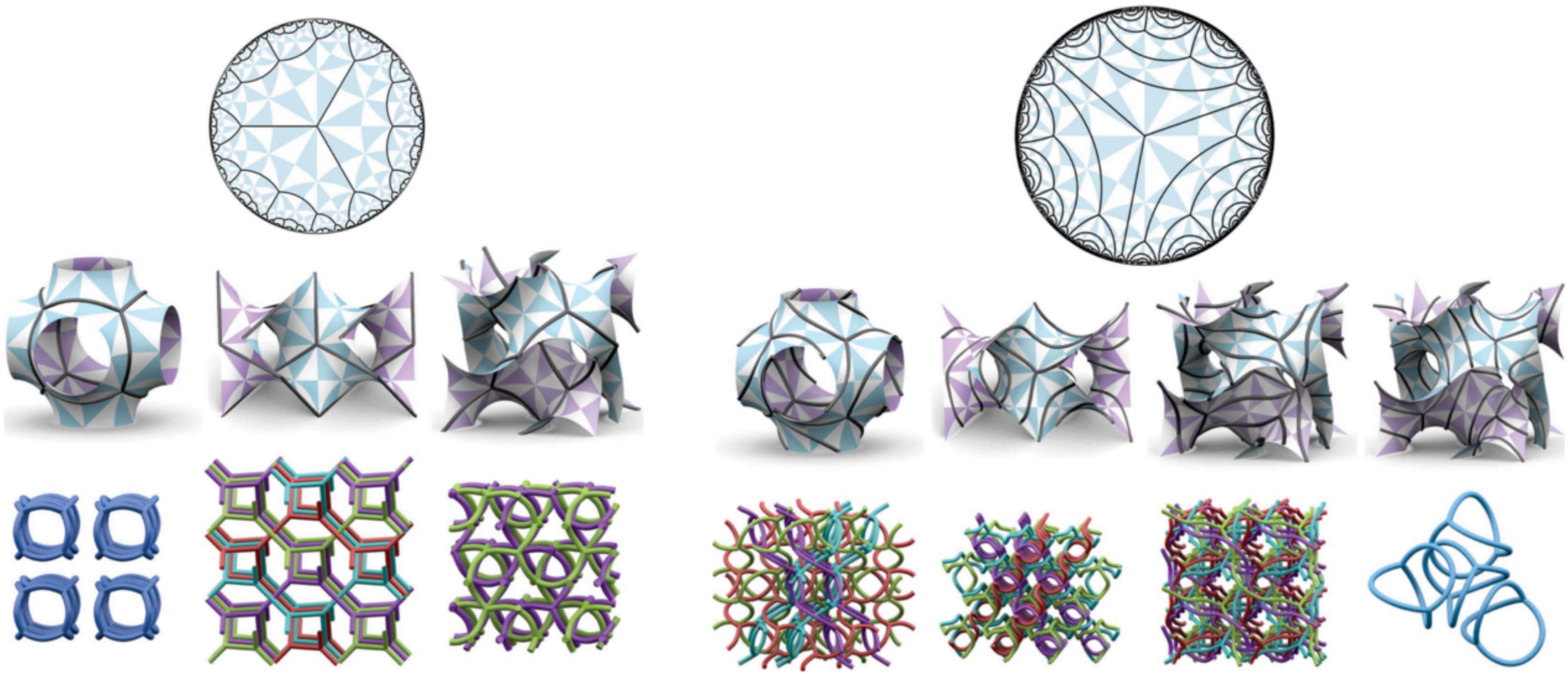

Periodic entanglement I: networks from hyperbolic reticulations

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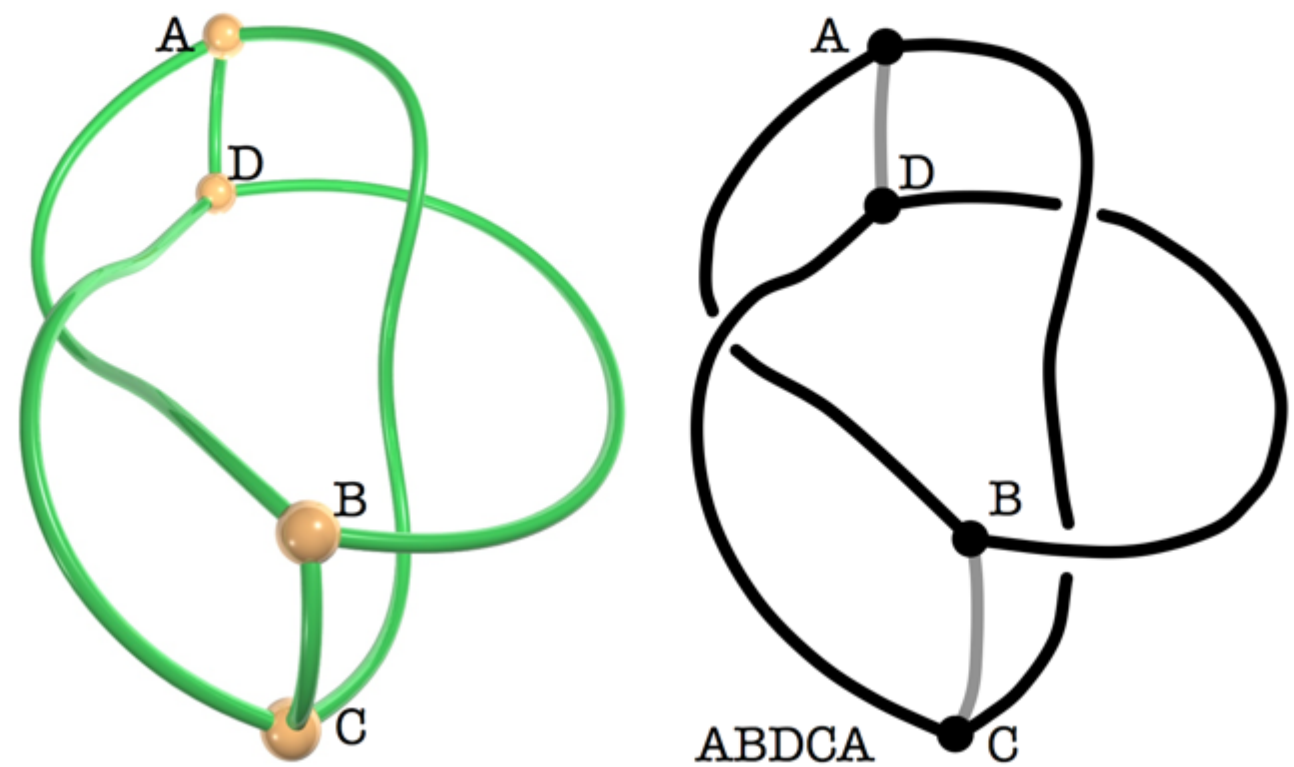
Analysis of entanglement

So far: looking for knots and links in cycles



There are other measures too, inspired by knot theory.
Some are topological or have topological flavours:
- Minimum crossing number of a planar embedding
- Graph embedding polynomials

Others are geometric:
- Line fattening á la Jon Simon's 'ideal knots' (Myf's SONO algorithm),
- KnotPlot-style energy minimisations,
- Self-illumination measures,
- Average crossing number when viewed from different angles,
etc.

I care about topology.

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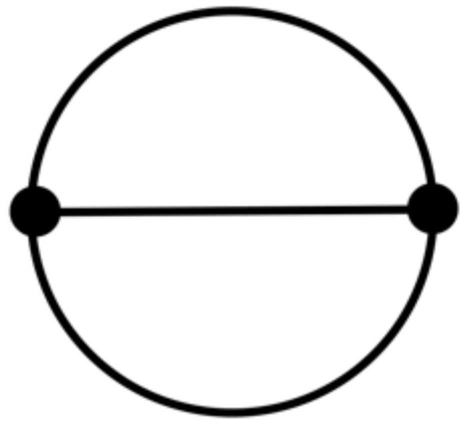
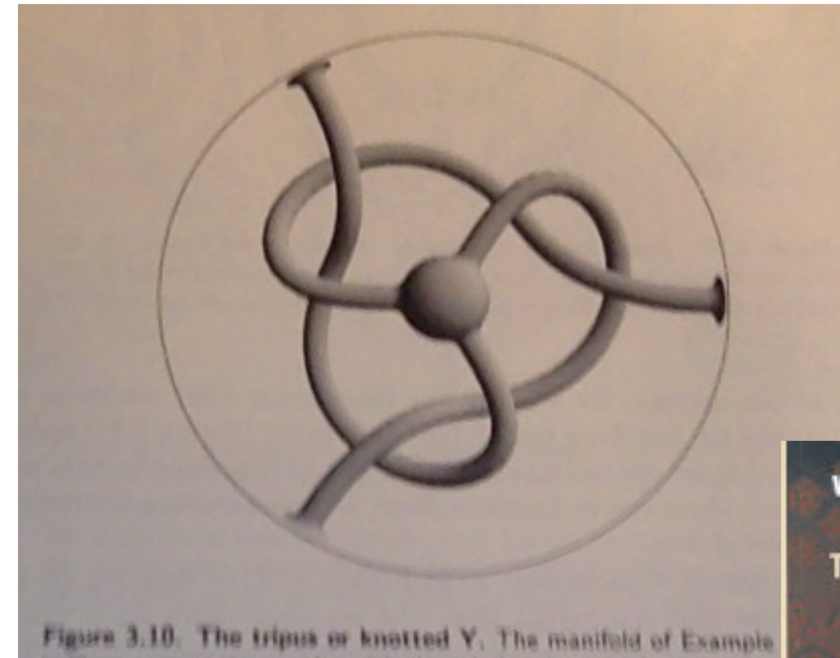
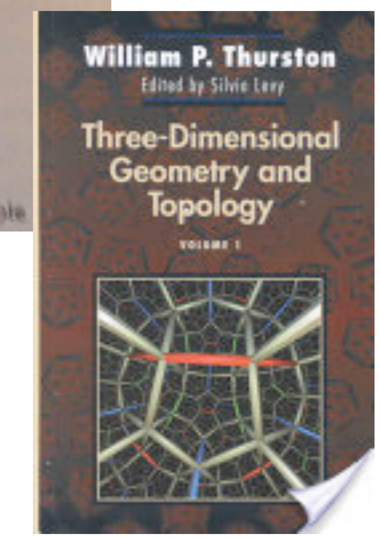
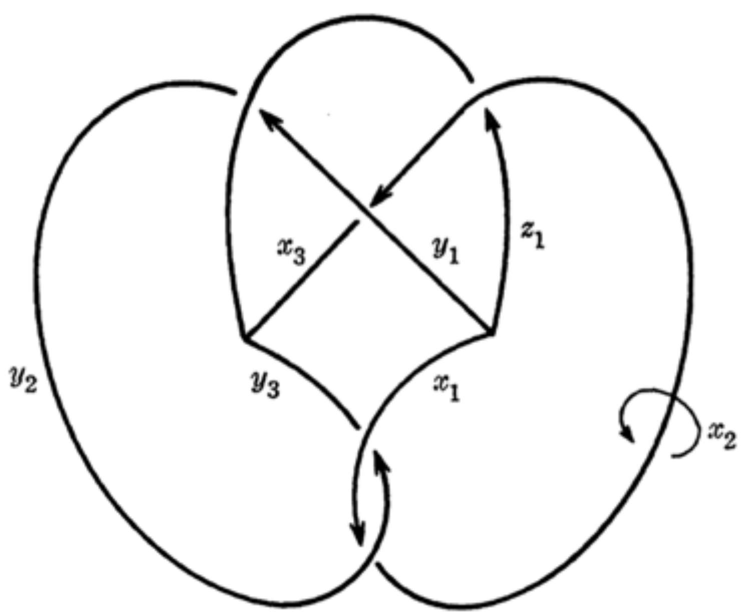
But what about these guys?









Kinsohita, 1972

Splicing a rope (The Ashley Book of Knots)

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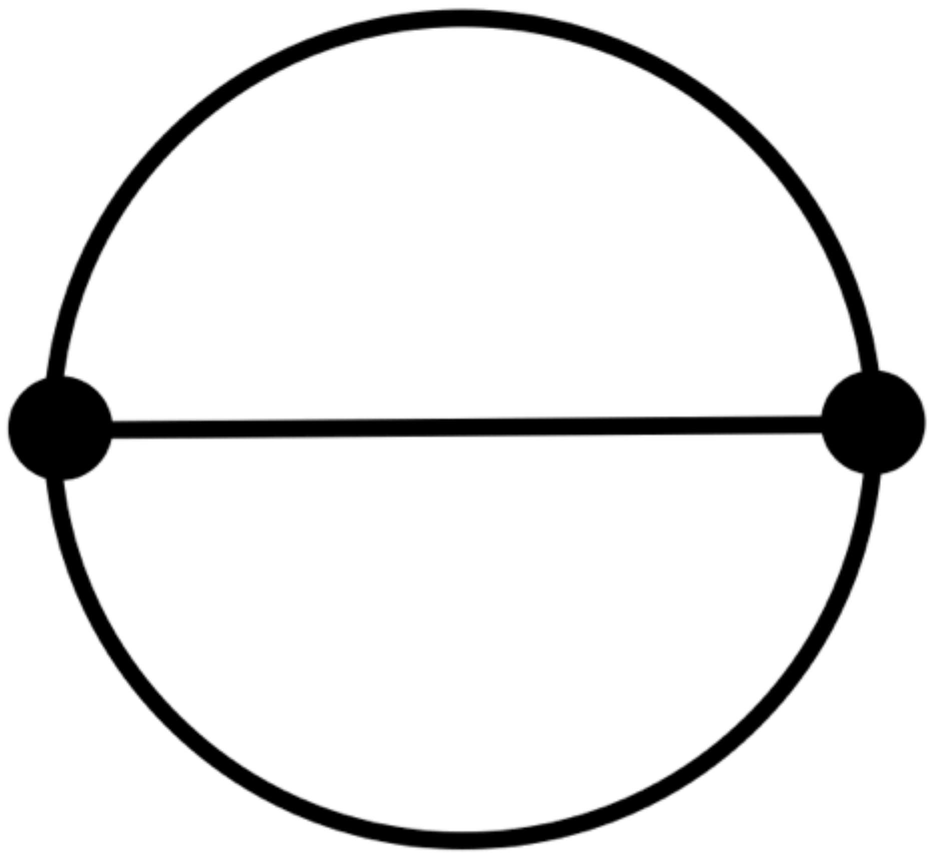
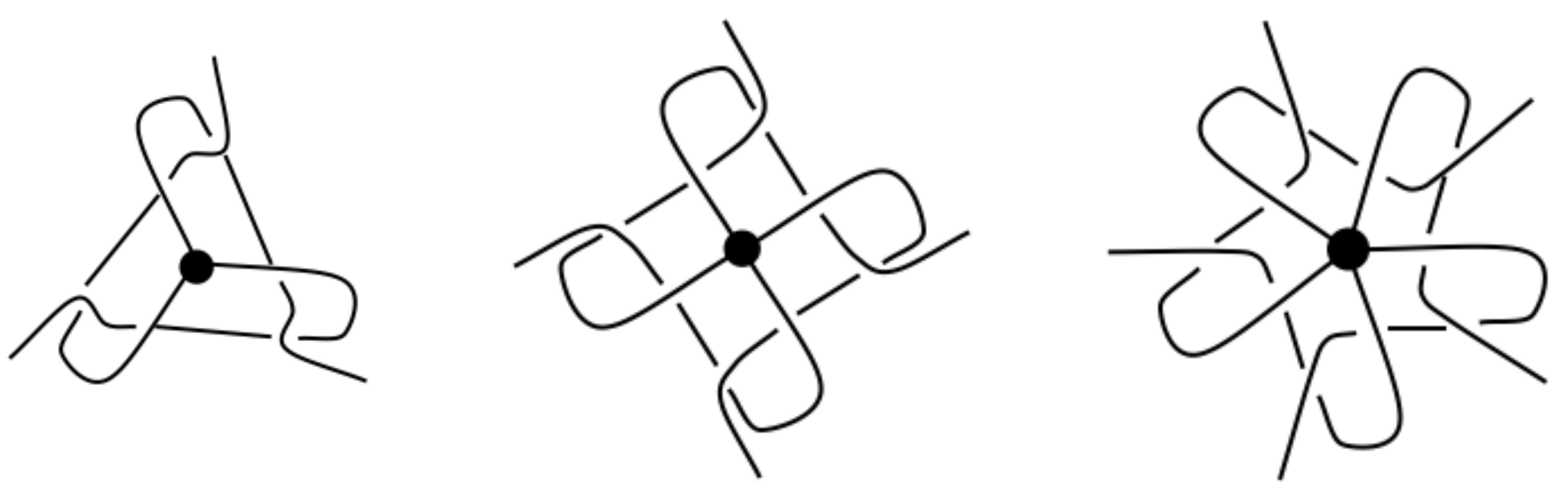
2. Εξιστενχε οφ νον−κνοτ ανδ νον−λινκ εντανγλεμεντ (ραϖελσ).



Vertex ravels are examples of entanglements that contain neither knots nor links.

They are entanglements in the edges centred around a vertex. Any number of edges can be involved.



Ravels: Knot free but not free, Toen Castle, Myfanwy E. Evans, and S. T. Hyde. New Journal Of Chemistry, 2008

Σομεονε ελσε'σ δεφινιτιον

Definition 1.1. Let B be a ball containing a graph G consisting of a vertex with n edges whose second vertices lie in ∂B. Let Γ denote the graph obtained by bringing these n vertices together within ∂B. If Γ is a θ_n graph which is non-planar but contains no knots then the pair (B, G) is said to be an n-***ravel*** and the embedded graph Γ is said to be ***raveled***.

UNRAVELLING TANGLED GRAPHS 2012

CATHERINE FARKAS, ERICA FLAPAN* and WYNN SULLIVAN

Προβλεμ (σελεχτιϖε ραϖελ):

Χονσιδερ α 'χυτ ποιντ' ορ "βριδγε−ϖερτεξ"

Definition 1.1. Let B be a ball containing a graph G consisting of a vertex with n edges whose second vertices lie in ∂B. Let Γ denote the graph obtained by bringing these n vertices together within ∂B. If Γ is a θ_n graph which is non-planar but contains no knots then the pair (B, G) is said to be an n-*ravel* and the embedded graph Γ is said to be *raveled*.

UNRAVELLING TANGLED GRAPHS 2012

CATHERINE FARKAS, ERICA FLAPAN* and WYNN SULLIVAN

Definition 1.2: Let B be a ball containing a graph G consisting of an n-valent vertex v connected by straight edges to vertices lying in ∂B. Let Γ denote the graph obtained by bringing together the pathwise connected vertices of G-v in ∂B. If Γ is non-planar but contains no knots then the pair (B,G) is said to be a *vertex n-ravel* and the embedded graph Γ is said to be ravelled.

Ugly definition from my thesis

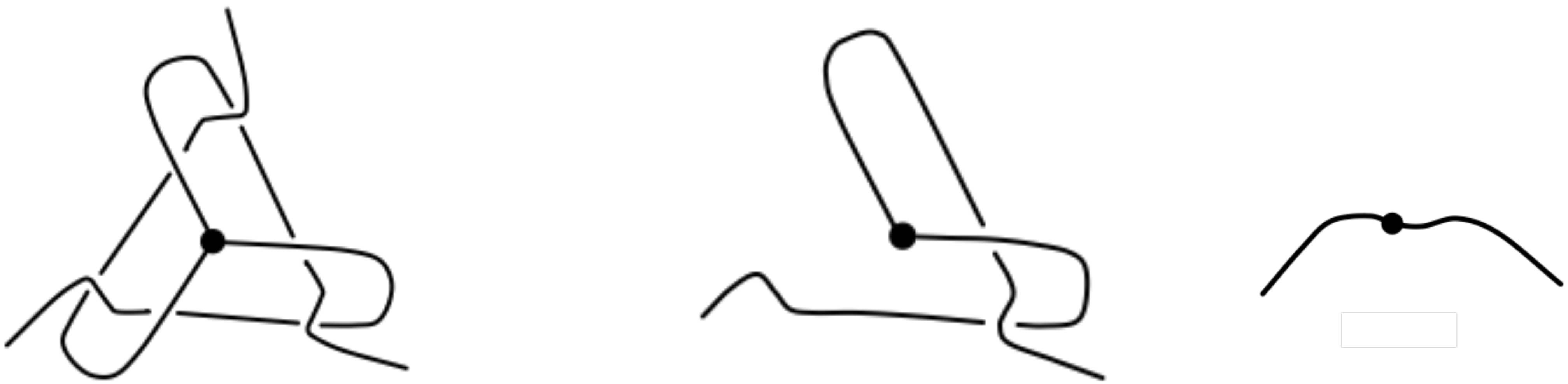
Definition 1.2: Consider a graph G with embedding E(G) in S^3. Let there be a simply connected domain D containing an n-valent vertex v of G such that $E(G)$ and δD intersect at only n points, one for each connected edge of v. A new graph G' and graph embedding $R(G')$ can be created if there exists more than one 'closure points'

that can be added to δD (the boundary of D), and the edges of $E(G)$ can be brought together without crossing within δD to connect to these closure points such that at least two edges terminate at each closure point. If $R(G')$ is non-planar while containing no knots, then such a graph embedding is an example of a selective vertex n-ravel around vertex v. If there is only one required closure point, then the vertex ravel satisfies the more stringent requirements of a universal n-ravel.

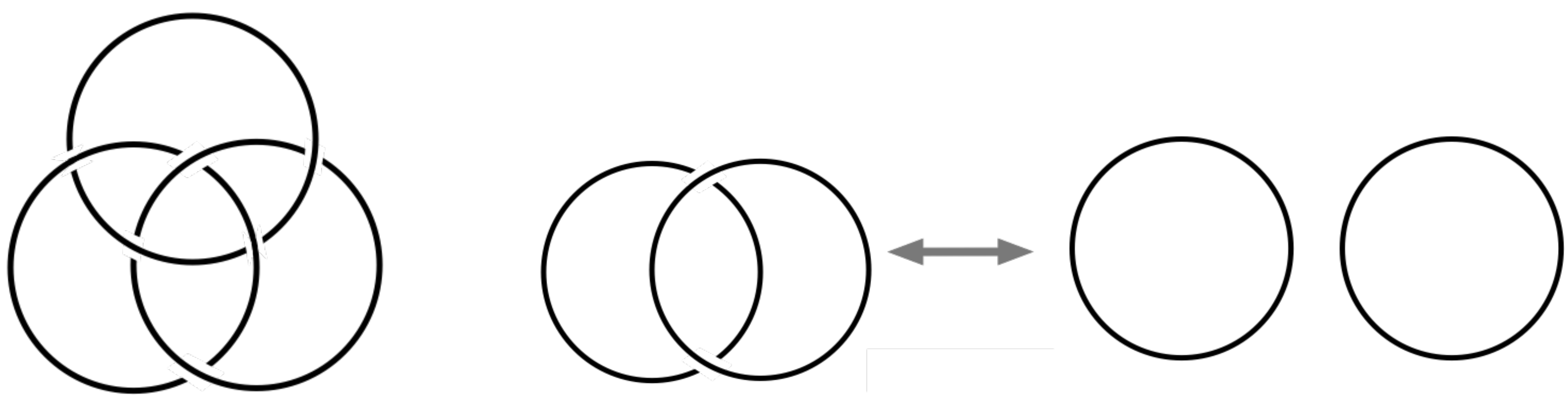
Features of vertex ravels

A ‘shelled’ vertex ravel.

Vertex ravels are similar to the 'Borromean rings': if any component is removed, the entanglement falls apart.

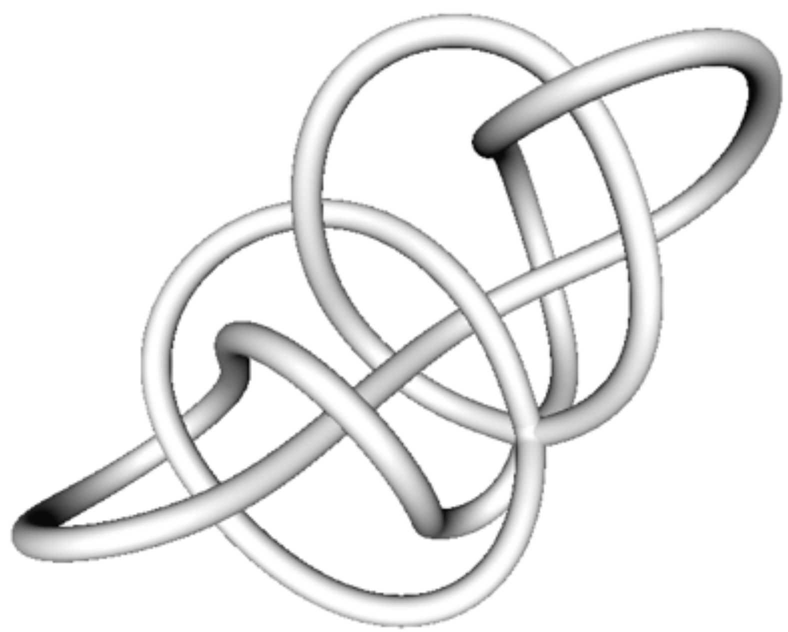


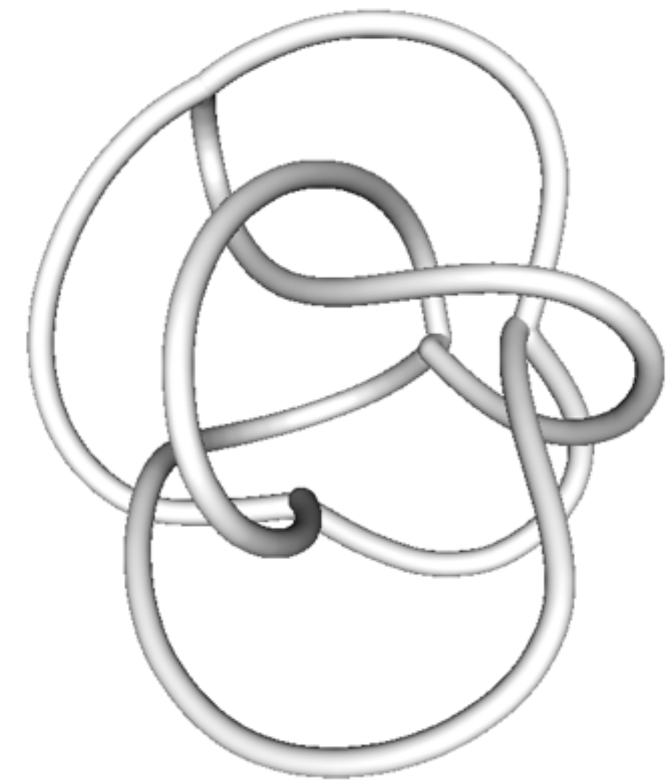
Vertex ravel

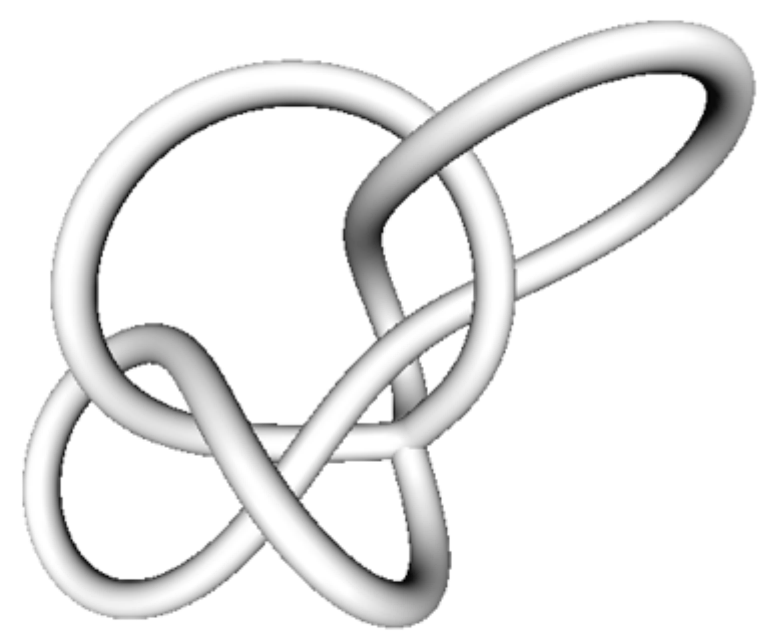


Borromean rings

Vertex ravels in small graphs

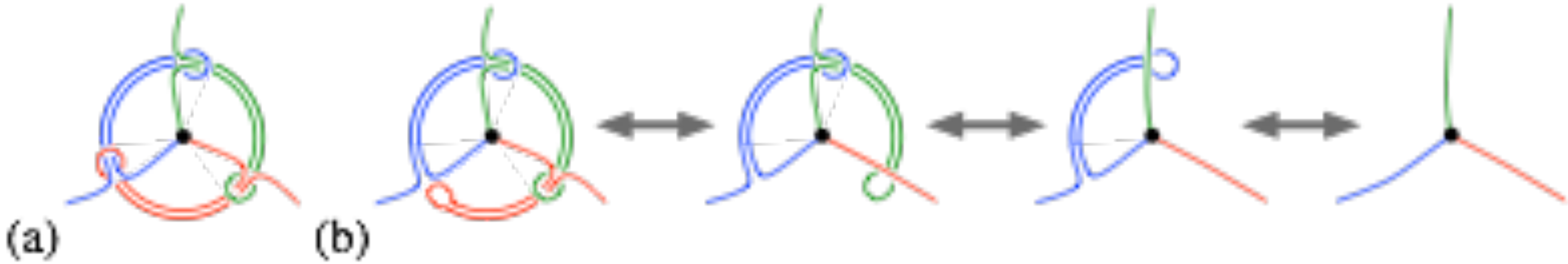




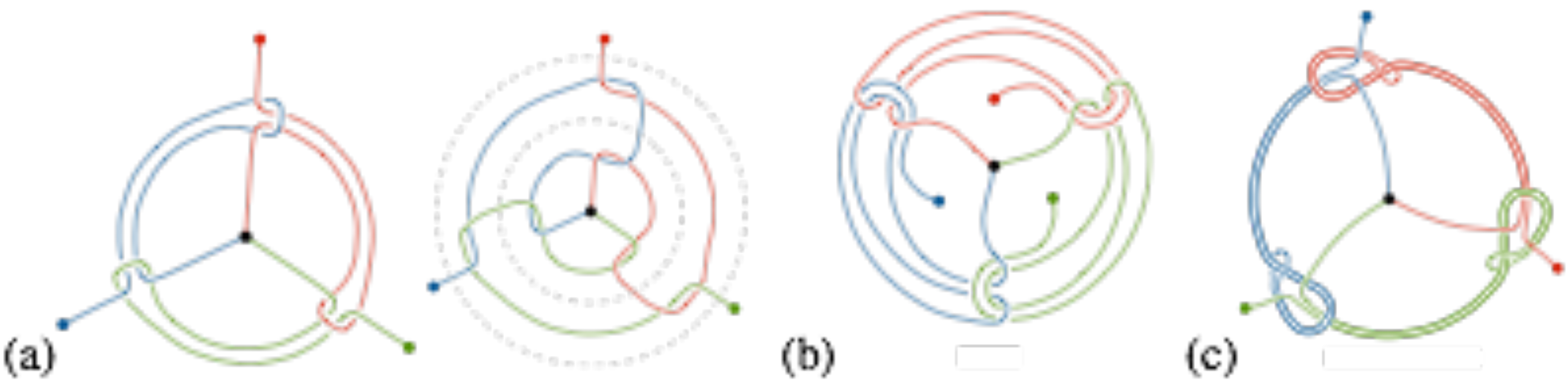


3. Χομπλεξ ραϖελσ ανδ τηειρ προπερτιεσ.

Introduction to ‘wandering ravels’

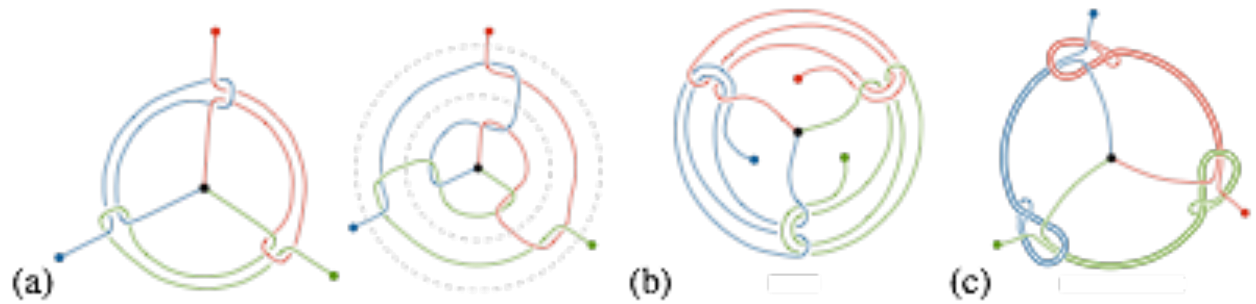


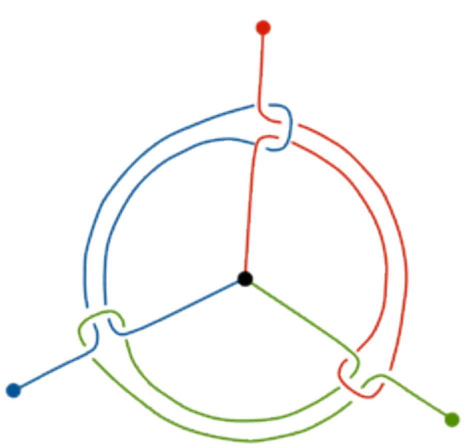
(a) A ravelled vertex (b) An unravelled vertex

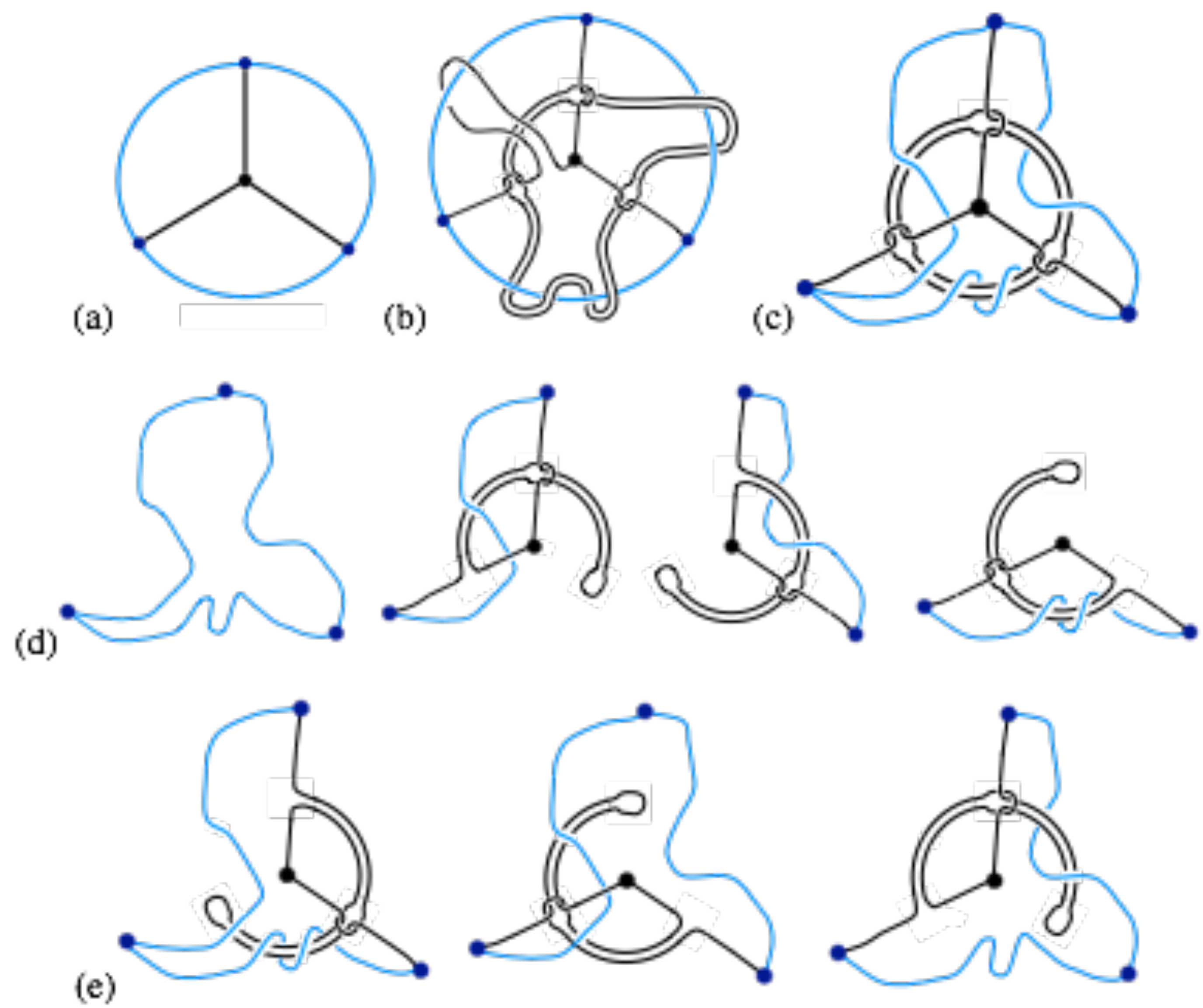


vertex ravels

Wandering ravels are extensions of vertex ravels which can travel through the graph embedding







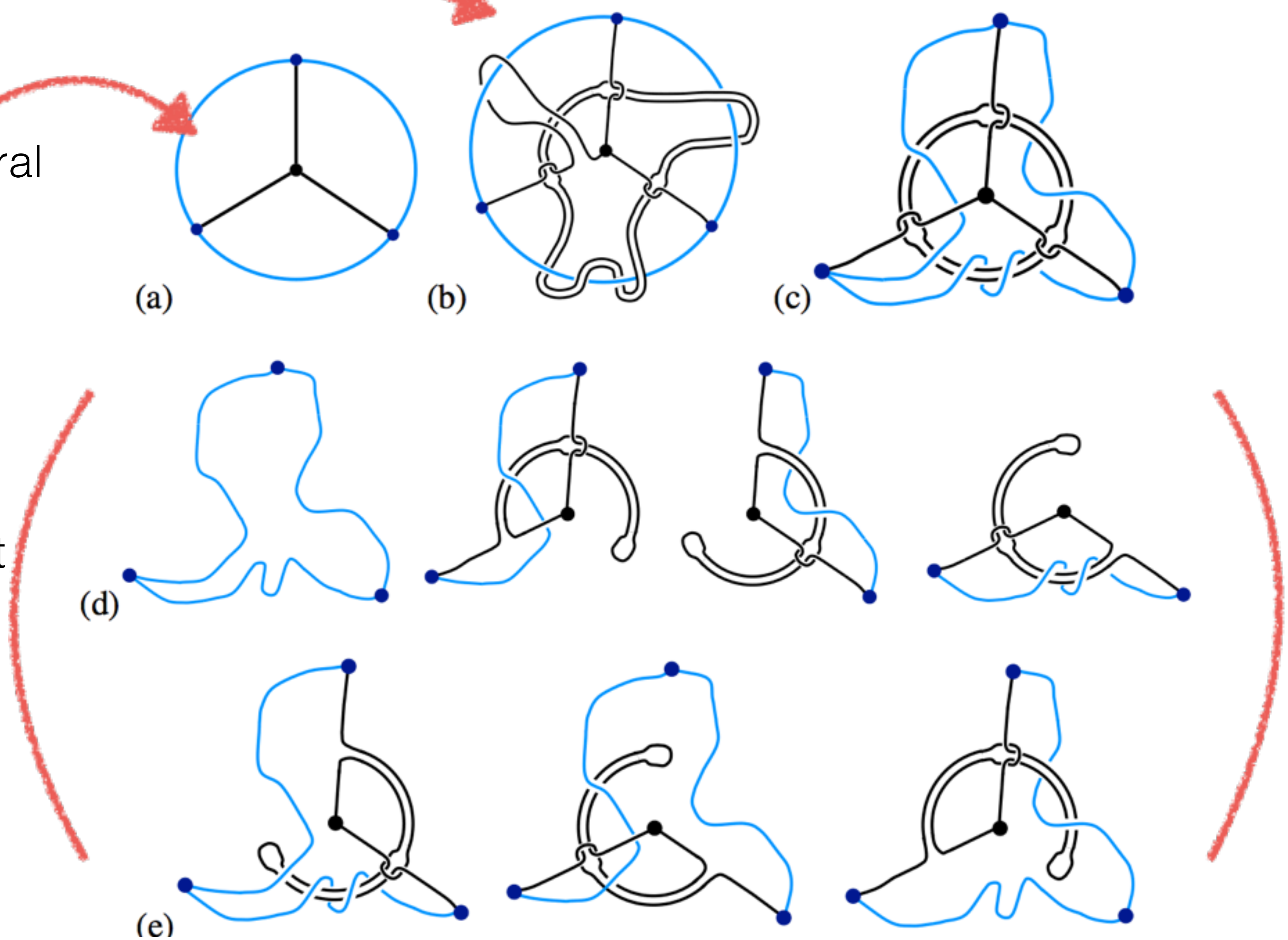
A more complex ravel, not restricted to the neighbourhood of a single vertex

The ravelled graph embedding

The ravelled embedding re-drawn

The tetrahedral graph

The constituent cycles



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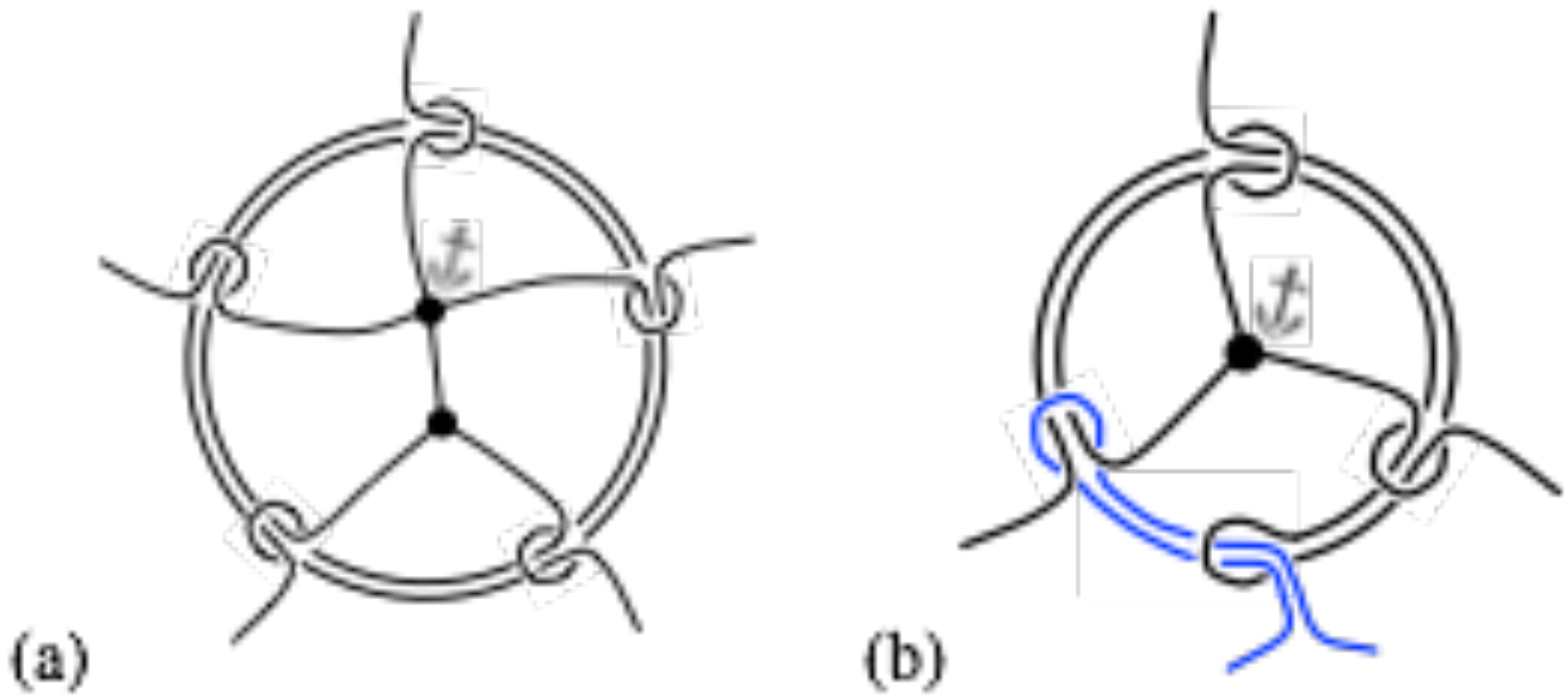
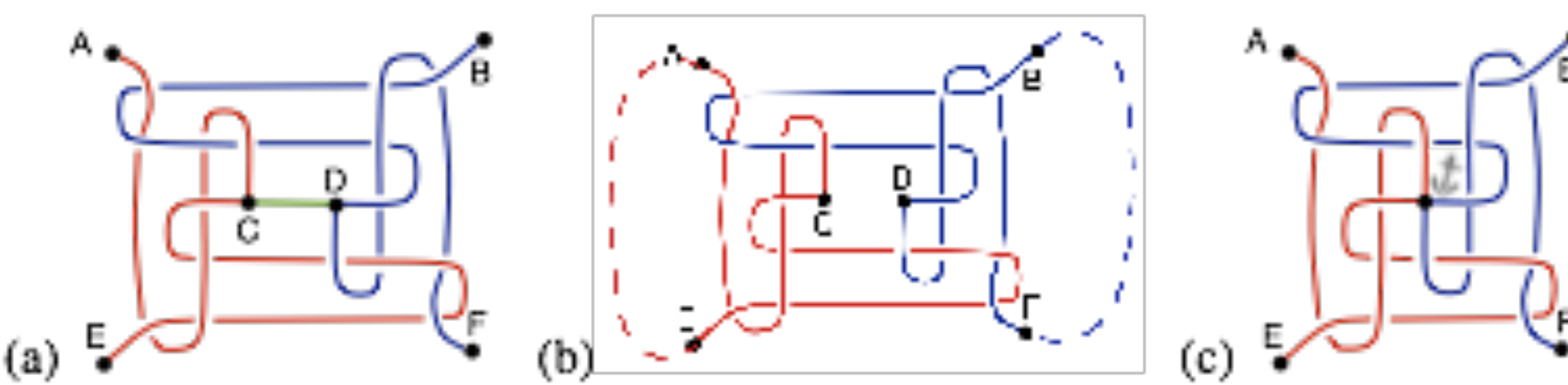
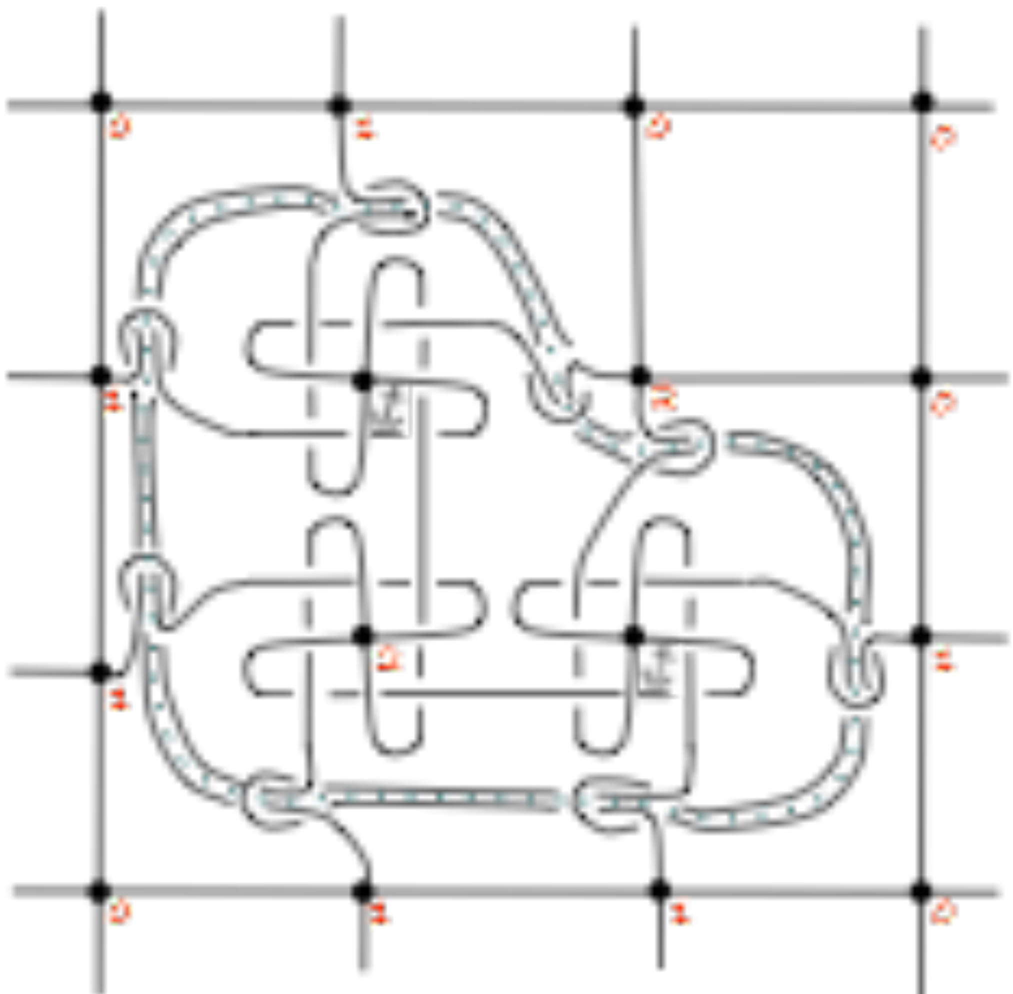
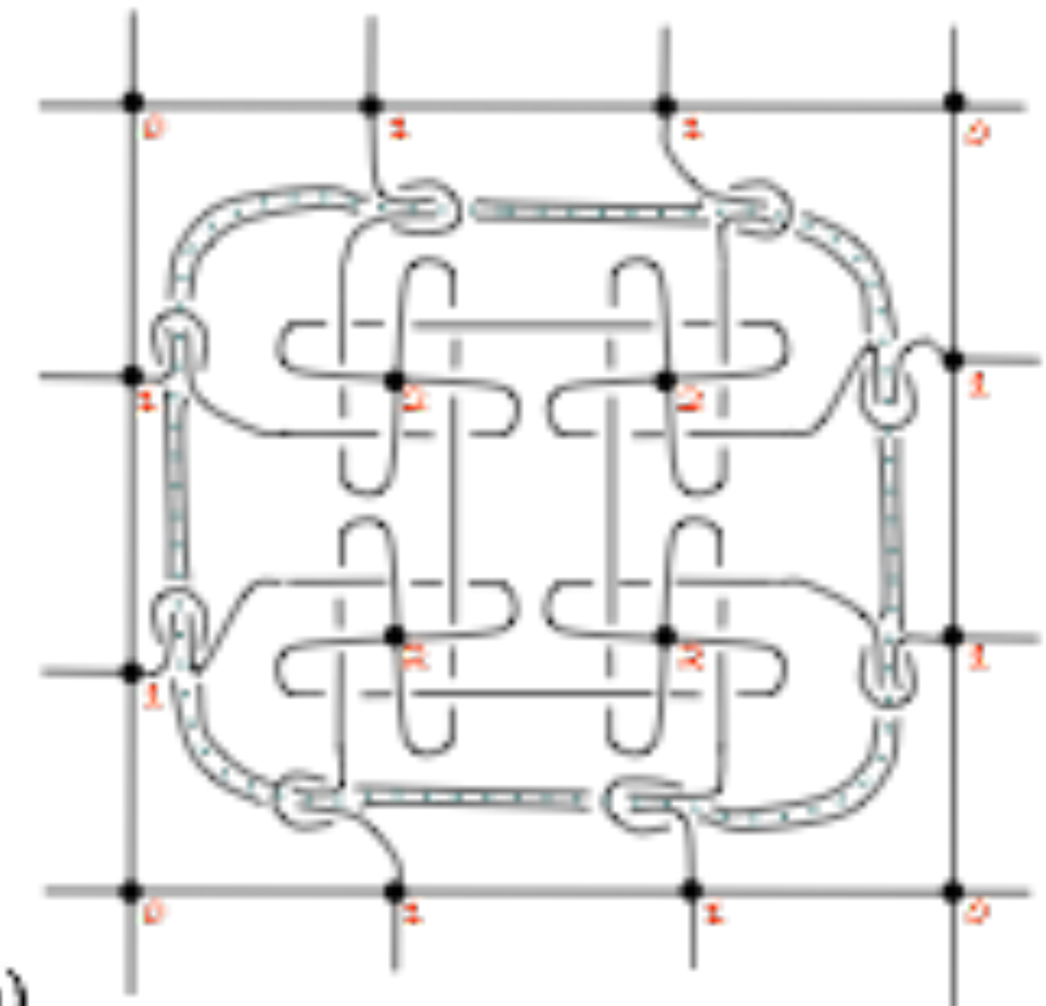
The importance of anchors





More anchors

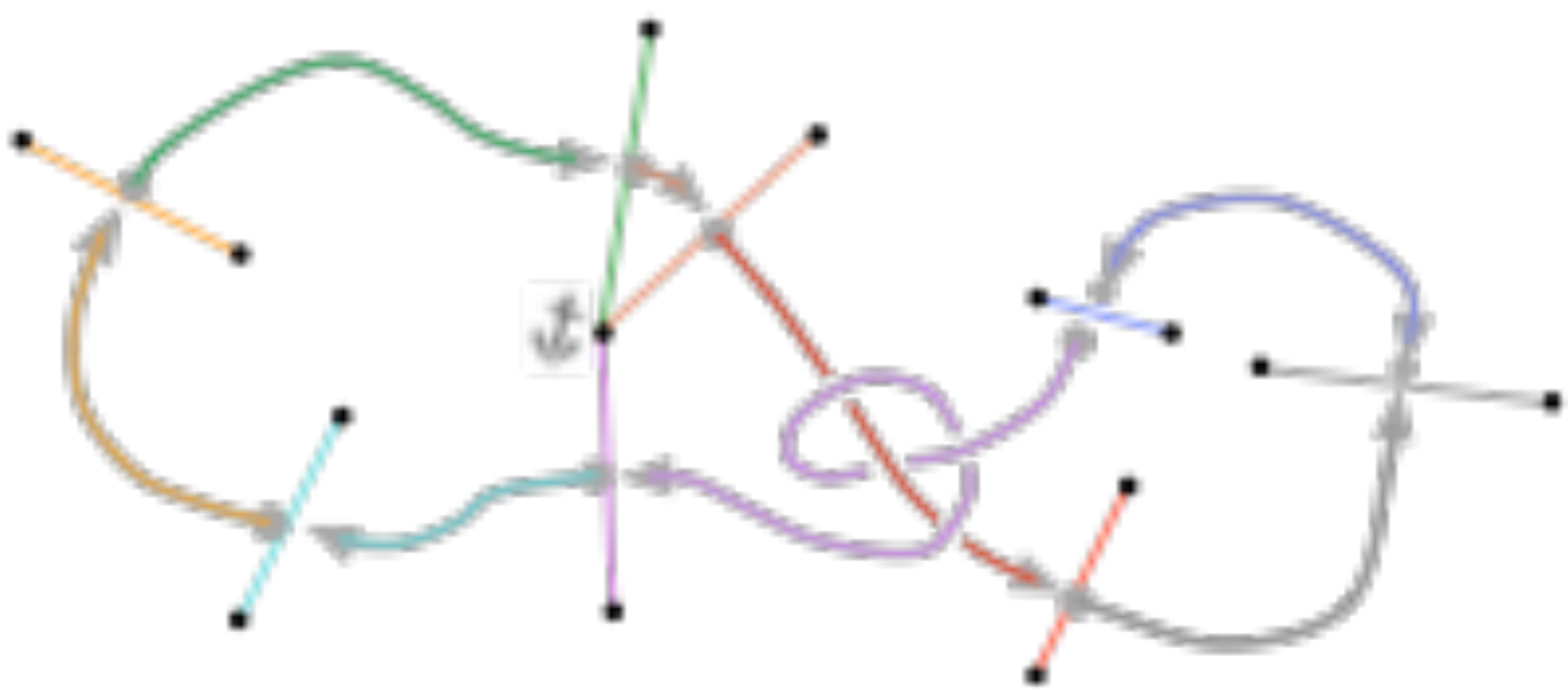

(a)


(b)

Ravel

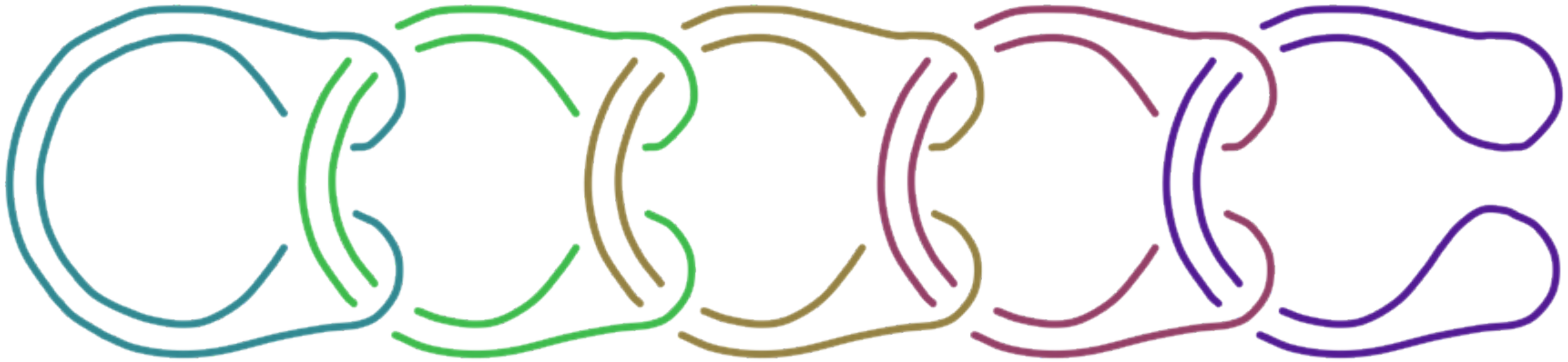
Not a ravel

Wandering ravels have a (knot free and link free) chain of entanglement travelling through the network.

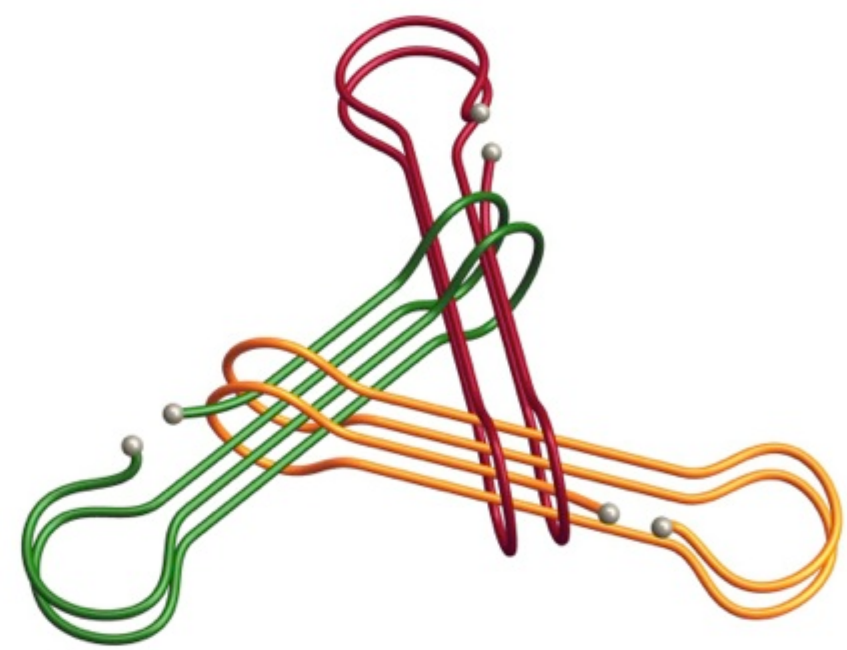


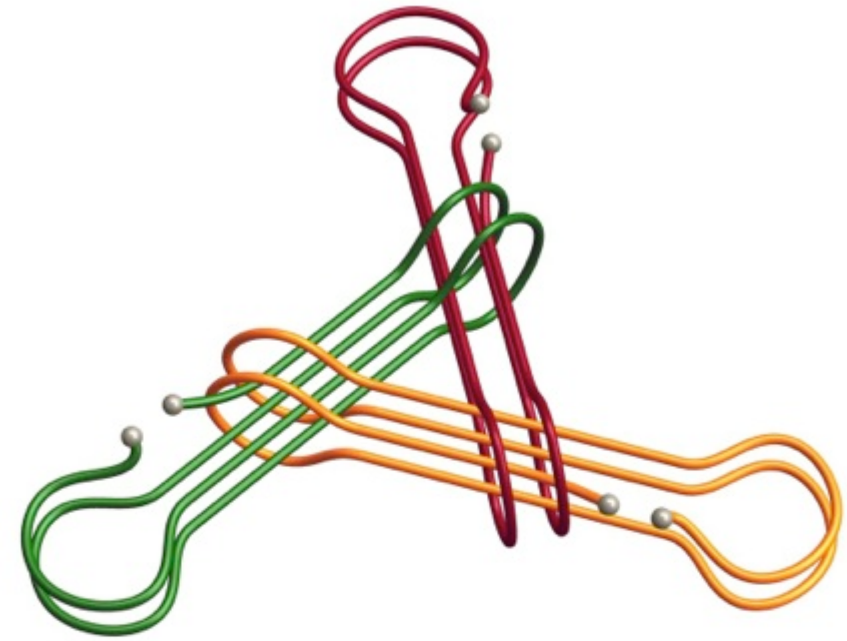
Polytropic ravels are more complex.

They are conceptually based on the set of links:



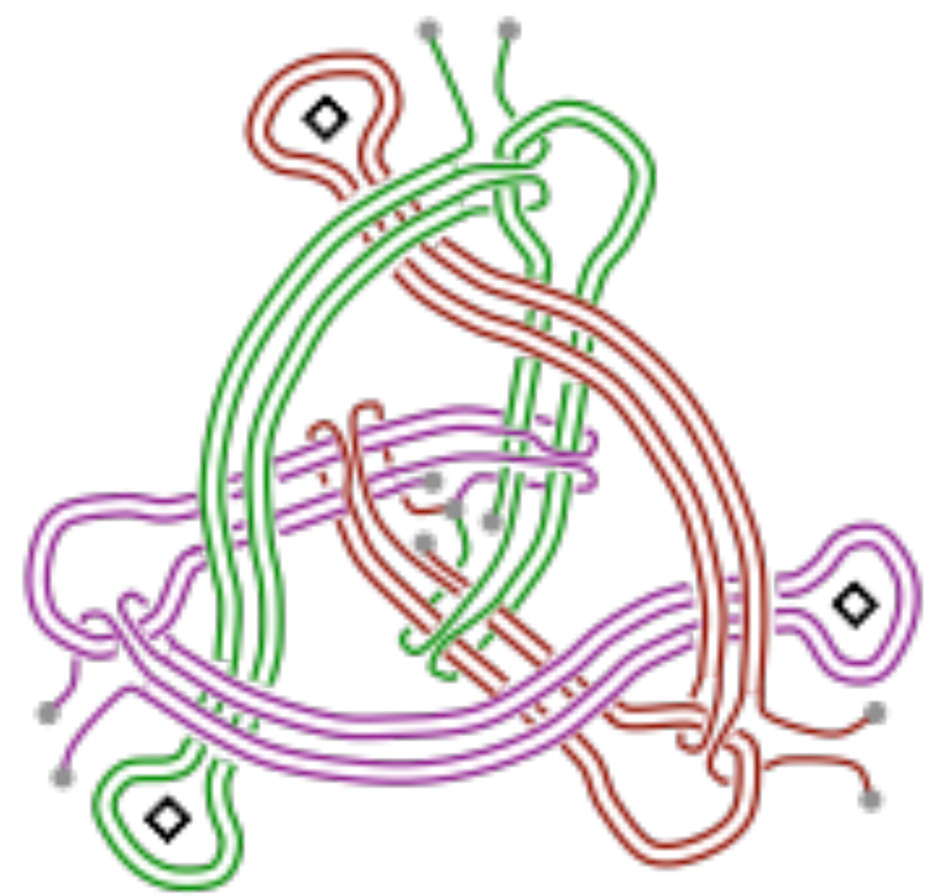
A polytropic ravel can contain branches, and so allows branching chains of entangled edges to travel through the network.





Polytropic ravels have a graph structure. They contain analogues of edges and vertices, embedded in space.

So they are a kind of meta-graph embedding, living within the entanglement in a graph embedding.



A vertex ravel tied in the “meta structure” of a polytropic ravel.

Penn

ANU
THE AUSTRALIAN NATIONAL UNIVERSITY

Ρελατινγ εντανγλεμεντ ανδ πλαναριτψ, τηεν χλασσιφψινγ εντανγλεμεντ τψπεσ (ϖερψ μυχη ωορκ ιν προγρεσσ).

Ρεχαλλ:

Definition 1.2: Let B be a ball containing a graph G consisting of an n-valent vertex v connected by straight edges to vertices lying in ∂B. Let Γ denote the graph obtained by bringing together the pathwise connected vertices of G-v in ∂B. **If Γ is non-planar** but contains no knots then the pair (B,G) is said to be a *vertex n-ravel* and the embedded graph Γ is said to be ravelled.

Entanglement is quite a fraught concept, especially as it relates to graphs with no “untangled” embedding.

Periodicity adds a further issues.

Selective ravels i.e. Ladder graph

On Planarity of Graphs in 3-manifolds *

Ying-Qing Wu

Definition. *Suppose Γ is embedded in a 3-manifold M. Then a cycle C of Γ is trivial (with respect to (M, Γ)) if it bounds a disk with interior disjoint from Γ.*

Theorem 2 *An abstractly planar graph Γ in M is planar if and only if all cycles of Γ are trivial.*

My hypothesised taxonomy of entanglement

The planet of a lazy man.

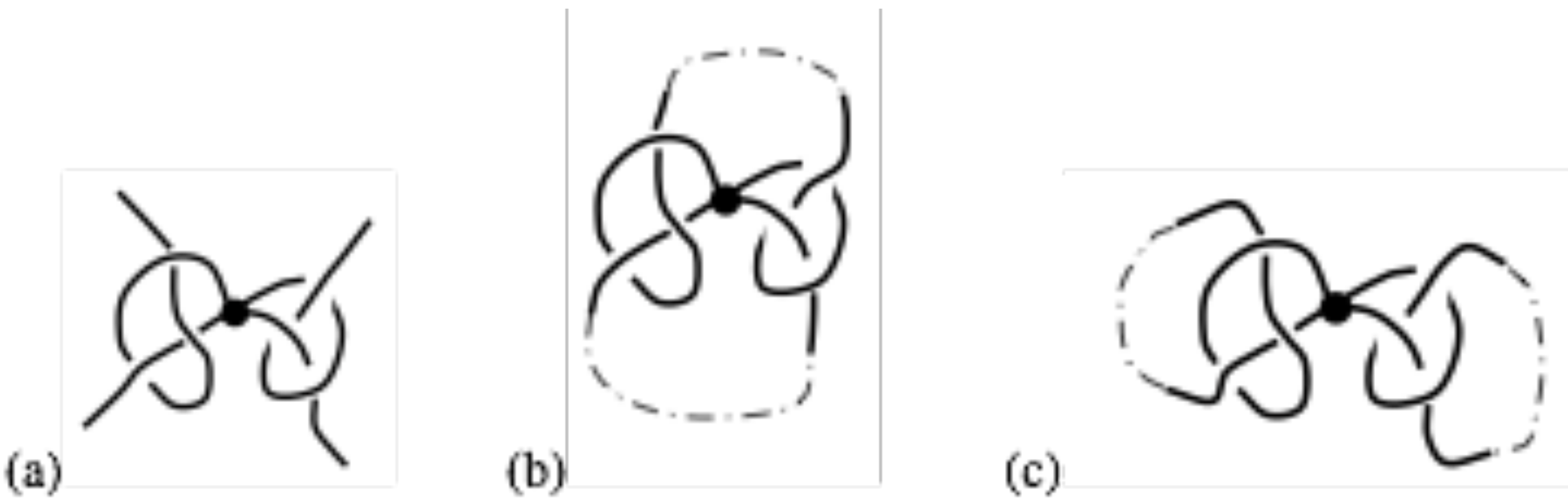
Thanks to Stephen Hyde, Vanessa Robins and Myf Evans for their earlier work with me, and the Australian taxpayer for allowing long, free, government funded PhDs without any duties.

Honourable mention:

THERE EXIST NO MINIMALLY KNOTTED PLANAR SPATIAL GRAPHS ON THE TORUS

SENJA BARTHEL

arxiv


(a)
(b)
(c)