Mirror symmetry for minuscule flag varieties

with Thomas Lam (U. of Michigan)

“Everything should be made as simple as possible, but not simpler.”

Deligne's purity theorem

Let p be a prime number, the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and $n \geqslant 2$. Define the hyper Kloosterman sums by

$$Kl_n = \sum_{x_1, \cdots, x_{n-1} \in \mathbb{F}_p^\times} e^{\frac{2\pi i}{p}\left(x_1 + \cdots + x_{n-1} + \frac{1}{x_1 \cdots x_{n-1}}\right)}$$

Each term is a p^{th}-root of unity and there are $(p-1)^{n-1}$ terms. Very important in number theory is that there is "square-root cancellation"

$$\text{Deligne's bound:} \quad |Kl_n| \leqslant np^{\frac{n-1}{2}}$$

This can be thought as the Riemann Hypothesis over finite fields. For $n = 2$, this is Weil's bound for Kloosterman sums.

Kloosterman sums, brief timeline.

- First appears in Poincare 1912: Fourier expansion of Poincare series.

$$\mathrm{Kl}(a) := \sum_{x \in \mathbb{F}_p^\times} e\left(x + \frac{a}{x}\right)$$

- First application by Kloosterman 1926: quadratic forms in four variables.
- Weil's bound, 1948, consequence of RH for curves: $|\mathrm{Kl}(a)| \leq 2\sqrt{p}$
- Deligne SGA41/2, hyper-Kloosterman sums:

$$\mathrm{Kl}_n(a) := \sum_{x_1 x_2 \cdots x_n = a} e\left(x_1 + x_2 + \cdots + x_n\right) \qquad |\mathrm{Kl}_n(a)| \leq np^{\frac{n-1}{2}}$$

- Katz's proof. Rigid local systems. Monodromy.
- Bump-Friedberg-Goldfeld: Fourier expansion of Poincare series and Peterson trace formula for GL(n).
- Jacquet-Ye fundamental lemma, Ngo Ph.D. 1997
- Heinloth-Ngo-Yun 2010: generalized Kloosterman sums.

Omitted here: Linnik-Selberg, Laumon, Iwaniec, Fouvry-Michel, Voronoi summation.

Recall: $$Kl_n(a) = \sum_{x_1 \cdots x_{n-1} \in \mathbb{F}_p^\times} e_p\left(x_1 + \cdots + x_{n-1} + \frac{a}{x_1 \cdots x_{n-1}}\right)$$

THM (Deligne SGA $4\frac{1}{2}$, Sperber 77) (n=2: Weil 48, Dwork 74)
Wan 04

for every $a \in \mathbb{F}_p^\times$ (i) $Kl_n(a) = \alpha_1 + \cdots + \alpha_n$ is a sum of Weil numbers of wt $n-1$

(ii) $v_p(\alpha_1) = 0$, $v_p(\alpha_2) = 1$ ... $v_p(\alpha_n) = n-1$.

The proof is deep: Weil's conjecture, Dwork p-adic cohomology.

(i) is Deligne purity: cohomological calculation.

(ii) are the slopes of the Newton polygon of the L-function.

$$\left(\mathbb{G}_m^2 , \; x_1 + x_2 + \frac{a}{x_1 x_2} \right) \xleftrightarrow{\text{MIRROR}} \mathbb{P}^2$$

"Hodge numbers for $Kl_3(a)$"

$$\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 1 & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{array}$$

$$h^{pq} = \begin{cases} 1 & \text{if } p+q = 2 \\ 0 & \text{o/w} \end{cases}$$

Hodge diamond of $\mathbb{P}^2$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

generated by the hyperplane class σ (purely algebraic)

$$H^{p,q}(\mathbb{P}^2) = \begin{cases} \mathbb{C} \cdot \sigma^p & \text{if } p = q \\ 0 & \text{o/w} \end{cases}$$

purity $\xleftrightarrow{\text{MIRROR}}$ Hodge-Tate type

slopes $\xleftrightarrow{\text{MIRROR}}$ degrees

$\mathbb{F}_p$ $\mathbb{R}, \mathbb{C}$

l-adic sheaf vs D-module

Kloosterman sum Bessel function

$$a\frac{d}{da} - \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$$

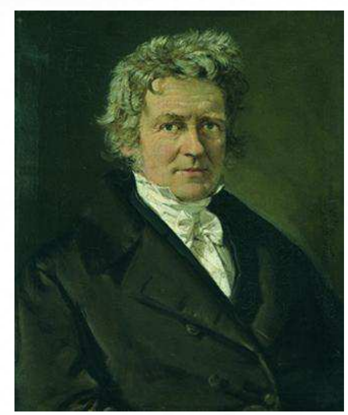
$$\oint_{S^1} e^{z+\frac{a}{z}} \frac{dz}{2i\pi z} = \sum_{r=0}^{\infty} \frac{a^r}{(r!)^2} = I_0(2\sqrt{a})$$

$$\int_{-\infty}^{0} e^{z+\frac{a}{z}} \frac{dz}{z} = 2K_0(2\sqrt{a})$$

I_0, K_0 are in the kernel of the Bessel operator

$$\left(a\frac{d}{da}\right)^2 - a$$

Friedrich Wilhelm Bessel (1784-1846)



$$\mathbb{C}\mathbb{P}^1 = GL(2) \Big/ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

Gelfond - Tsetlin

$$(\mathbb{C}^\times, f_a)$$

$$\downarrow$$

$$z \longrightarrow a$$

quantum connection is

$$a\frac{d}{da} - \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$$

$$(*) \quad f_a(z) = \sum_{\text{arrows}} \frac{\text{head}}{\text{tail}} = z + \frac{a}{z}$$

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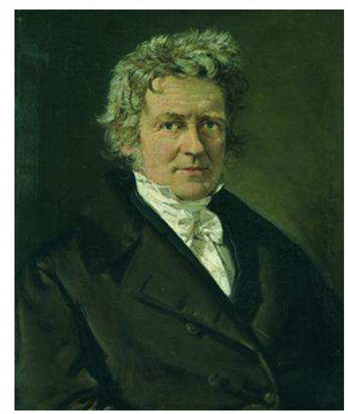
MIRROR $\Longrightarrow$ I_0, K_0 are in the kernel of the Bessel operator

$$\left(a\frac{d}{da}\right)^2 - a$$



Friedrich Wilhelm Bessel (1784-1846)

$\mathbb{F}_p$ $\mathbb{R}, \mathbb{C}$

l-adic sheaf vs D-module

Kloosterman sum

Bessel function


Landau-Ginzburg
=
Exponential Sum

$\mathbb{F}_p$ ··························· $\mathbb{R}, \mathbb{C}$

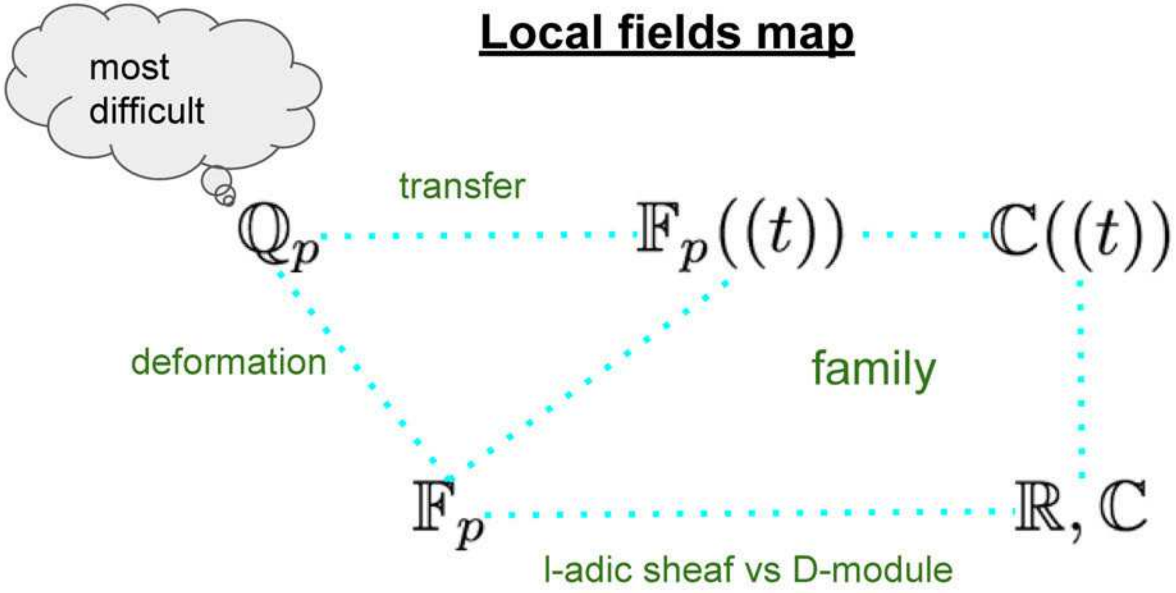
l-adic sheaf vs D-module

exponential sum attached to (Y,f)

Landau-Ginzburg model (Y,f)

This analogy is implicit in the influential work of Kontsevich-Soibelman. However our approach with Lam is even more direct and made explicit in its relation to the Langlands program.

Local fields map



$$
\begin{tikzcd}
\text{smooth projective Fano} \arrow[d] & & \begin{array}{c}\text{Landau-Ginzburg model} \\ \text{(quasi-projective Calabi-Yau, potential)}\end{array} \arrow[d] \\
\text{small quantum differential equation} \arrow[rr, phantom, "\overset{?}{\cong}"' , "\text{MIRROR}" below] & & \text{pushforward } D\text{-module}
\end{tikzcd}
$$

Projective homogeneous spaces: G/P with P parabolic subgroup.

Grassmannian: Gr(k,n) = { k-subspaces in $\mathbf{C}^n$}
There is a transitive action by G=GL(n,$\mathbf{C}$). Restricting the action to the torus T of diagonal matrices inside GL(n,$\mathbf{C}$), the fixed points are the coordinate subspaces spanned by the choice of k coordinate vectors. So there are $\binom{n}{k}$ fixed points. This is the dimension of cohomology, and also the number of Plücker coordinates p_I.

The stabilizer of a standard coordinate subspace is a parabolic subgroup P, i.e. the subgroup (k,n-k) block triangular matrices.

Example: Gr(1,n) = {lines in $\mathbf{C}^n$}= projective space $\mathbf{P}^{n-1}$
Example: Gr(2,4) = {planes in $\mathbf{C}^4$} = Klein 4-dimensional quadric = $GL(4,\mathbf{C})/\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$
Plücker embedding inside $\mathbf{P}(\Lambda^2\mathbf{C}^4)$
Plücker relation: $p_{12}p_{34} + p_{14}p_{23} = p_{13}p_{24}$

Motivation to study flag manifolds appear in many different fields.

Lie theory: G/B parametrizes the Borel subgroups of G.

Algebraic geometry: Tautological vector bundle. Characteristic classes.

Geometric representation theory: Borel-Weil-Bott theorem.

Combinatorics: generalizations of toric varieties. Replace torus action by group action.

Enumerative geometry: Schubert calculus.

Number theory: Geometry at the boundary of Shimura varieties. Harish-Chandra structure theory.

Theorem (Lam–T '16)

If $P^\vee$ is a minuscule parabolic, then there is an isomorphism

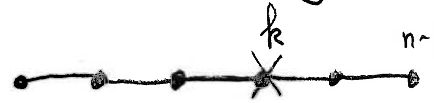
$$\begin{array}{ccc} \text{quantum connection} & & \text{crystal } \mathcal{D}\text{-module} \\ \text{for } G^\vee/P^\vee & \simeq & \text{for } \left(\overset{\circ}{G}/P,\, f_a\right) \end{array}$$

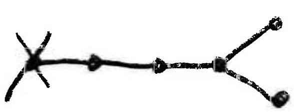
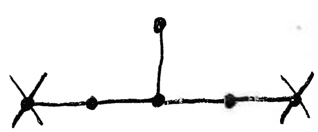
$$a\frac{d}{da} - \sigma *_a \qquad\qquad\qquad \int_{\overset{\circ}{G}/P} e^{f_a}$$

connection 1-form = quantum multiplication by σ, where $\mathrm{Pic}(G^\vee/P^\vee) = \mathbb{Z}\sigma$

pushforward $\mathcal{D}$-module $\mathcal{D} = \mathbb{C}[a, a^{-1}]\langle a\frac{d}{da}\rangle$

<u>List of minuscule flag varieties</u> ($\subset$ compact Hermitian symmetric spaces)

- $\mathbb{P}^n$ and Grassmannian $Gr(k,n)$ 

- even-dimensional quadric 
- Spinor variety = orthogonal Grassmannian $OG(n,2n)$ 
- Cayley plane = projective Octonions (dim = 16) 
- Freudenthal variety (dim = 27) 

<u>Notation</u>. G complex reductive Lie group $\supset P$ parabolic subgroup.

$G^\vee$ dual group $\supset P^\vee$ parabolic with same nodes as P.

Partial flag variety $G^\vee/P^\vee$ — homogeneous, smooth, projective, Fano.

<u>Question</u>: Mirror symmetry for flag varieties?

The open projected Richardson $\mathring{G/P}$ — smooth, affine Calabi-Yau, cluster variety.

$\boxed{\text{Conjecture (Rietsch '08)} \quad G^\vee/P^\vee \text{ is mirror to } (\mathring{G/P}, f).}$

Kim-Givental '95: complete flag varieties, i.e. $P, P^\vee =$ Borel.

The conjecture emerged in relation with works by Lusztig, Zelevinsky, Fomin and others on crystals, Peterson and others on quantum Schubert calculus, Witten, Vafa and others on Landau-Ginzburg models.

In work with T. Lam we approach the problem via <u>automorphic forms</u>

W_P: Weyl group of the Levi subgroup L_P of P.

W^P: minimal representatives for W/W_P in Bruhat order. w_P^{-1}: longest element of W^P.

B_-: opposite Borel $\psi: U \longrightarrow \mathbb{A}^1$ non degenerate additive character.

<u>Berenstein-Kazhdon geometric crystal</u>: $U\,Z(L_P)\,w_P\,U \cap B_-$

$a \in Z(L_P)$ $\quad f_a(u_1 w_P u_2) := \psi(u_1) + \psi(u_2)$ potential.

$= $ sum of ratios of generalized minors on G

$\eta(u)\, w_P u^{-1} \longleftarrow\!\shortmid\; u$

$$X_a \xleftarrow{\sim} B_- w_P B_- \cap U \xrightarrow{\sim} \mathcal{R}^{w_0}_{w_P w_0} \subset G/B$$

Fomin-Zelevinsky twist map η

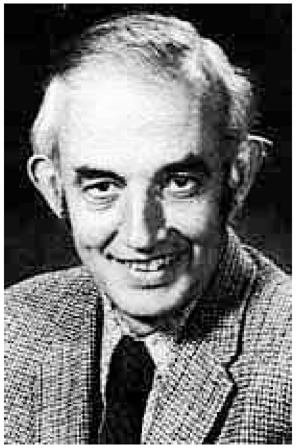
$$\mathcal{R}^{w_0}_{w_P w_0} \overset{\wr}{\downarrow} \mathring{G}/P \subset G/P \qquad G/B \downarrow G/P$$

projected Richardson variety (cluster, affine Calabi-Yau) : $\mathring{G}/P \subset G/P$

(Knutson-Lam-Speyer)

$$\mathbb{A}^1 \xleftarrow{f} U\,Z(L_P)\,w_P\,U \cap B_- \xrightarrow{\pi} Z(L_P)$$

$\pi_!\, f^* \mathcal{D}/\mathcal{D}(\partial - 1)$ crystal $\mathcal{D}$-module



Geometric summary

homogeneous projective G/P

$\Rightarrow$ Fano

• $\exists$ anticanonical $\partial_{G/P}$ multiplicity free union of Schubert divisors.

$\Longrightarrow \exists\, vol_{G/P}$ volume form with simple pole $\partial_{G/P}$ (Knutson-Lam-Speyer '09)

$\Longrightarrow \mathring{G/P} :=$ complement of $\partial_{G/P}$ is log CY.

also $\mathring{G/P} \overset{\sim}{\leftarrow} \mathcal{R}^{w_0}_{w_0^P}$ open Richardson

$\mathcal{R}^{v}_{u} := \underbrace{BvB}_{\text{Bruhat}} \cap \underbrace{B_-uB}_{\text{opposite Bruhat}}$ $u \leq v$.

• $f_t : \mathring{G/P} \longrightarrow \mathbb{A}^1$ regular function $\forall t \in Z(L_P)$

(Berenstein-Kazhdan '00, Rietsch '02)

upper cluster algebra (Berenstein-Fomin-Zelevinsky '03) wild character variety.

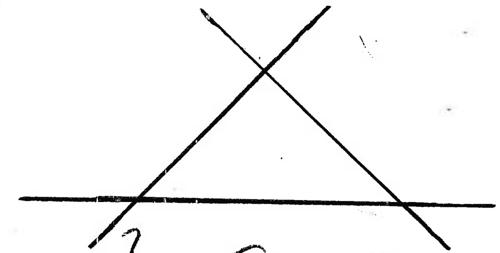
We can reformulate Rietsch conjecture '08 in a way compatible with recent work of Gross-Hacking-Keel (log CY), Katzarkov-Kontsevich-Pantev (compactified Fano) and Seidel (Lefschetz pencils):

$$G/P,\ \partial_{G/P},\ vol_{\mathring{G/P}},\ f_{t\in Z(L_P)} \quad \text{mirror to} \quad G^\vee/P^\vee,\ \partial_{G^\vee/P^\vee},\ vol_{\mathring{G^\vee/P^\vee}},\ f_{t\in Z(L_{P^\vee})}$$

Example $G/P = \mathbb{P}^2$. anticanonical $\simeq \mathcal{O}(3)$ $\quad \partial_{G/P} := \bigcup$ 3 coordinate lines

$[x_0 : x_1 : x_2]$
projective coordinates

section $vol^{-1} := x_0 x_1 x_2$ → divisor



complement of $\partial_{G/P} =: \mathring{G/P} = \{ x_0 \neq 0,\ x_1 \neq 0,\ x_2 \neq 0 \} = \mathbb{G}_m \times \mathbb{G}_m$

$[1 : x_1 : x_2]$

$$\frac{x_1^2 x_2 + x_1 x_2^2 + a x_0^3}{x_0 x_1 x_2} =: f_a(x_1, x_2) = x_1 + x_2 + \frac{a}{x_1 x_2}.$$ affine coordinates

intersection of two cubics : there are 9 indeterminacy points for the potential f_a.

Fibers of f_a are elliptic curves.

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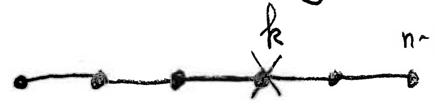
- $\mathbb{P}^n$ and Grassmannian $Gr(k,n)$ 

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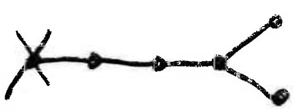
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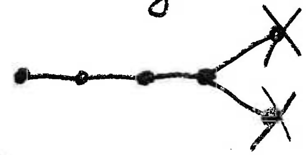
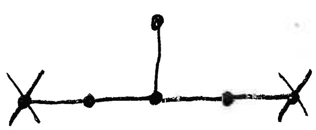

- Cayley plane = projective Octonions (dim = 16) 
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Fundamental example: projective space

$$\mathbb{P}^n \xrightarrow{\text{MIRROR}} (\mathbb{C}^\times)^n, \quad f_a(z) = z_1 + z_2 + \dots + z_n + \frac{a}{z_1 z_2 \cdots z_n}$$

ex (n=2) Kontsevich's formula for # of rational curves of deg d through 3d-1 generic points in the plane.

d	1	2	3	4	5	6
#	1	1	12	620	87 304	26 312 976

How to generalize it?

- $\mathbb{P}^n$ is an example of toric variety $\rightsquigarrow$ mirror symmetry for toric varieties

(Auroux's talk yesterday)

- $\mathbb{P}^n$ is an example of projective homogeneous variety $\longrightarrow$ this talk

$$\mathbb{P}^{n-1} = GL(n) \Big/ \begin{pmatrix} * & * \\ 0 \cdots 0 & * \end{pmatrix}$$

and today talks by Williams and Pech

- $\mathbb{P}^n$ is an example of Fano variety $\longrightarrow$ del Pezzo surfaces, Fano 3-folds, Fano 4-folds

$$\partial = a\frac{d}{da} - \begin{pmatrix} 0&0&0&0&a&0 \\ 1&0&0&0&0&a \\ 0&1&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&1&0&0 \\ 0&0&0&0&1&0 \end{pmatrix} \qquad f_a(z) := = z_1 + \frac{z_2}{z_1} + \frac{z_3}{z_2} + \frac{z_4}{z_3} + \frac{z_4}{z_2} + \frac{a}{z_4}$$

The mirror theorem implies that $I(a) := \oint e^{f_a(z)} \frac{dz}{z}$ is the last entry of a solution

<u>elementary proof</u> $I(a) = \sum_{r \geq 0} \frac{(2r)!}{(r!)^6} a^r$ by Cauchy's residue thm.

The series is annihilated by $\partial^5 - 2a(2\partial + 1)$.

(a binomial identity)

The last entry of any solution of the quantum connection is also annihilated by the same operator. (pleasant exercise!)

1^{st} order vector ODE $\longleftrightarrow$ high order scalar ODE $\square$

The mirror theorem also implies that we obtain an integral representation of all six independent solutions.

Example $\bullet$ — $\times$ — $\bullet$ A_3

$$Gr(2,4) = GL(4) \Big/ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

$= 4$-dim quadric

quantum connection is given by

$$\underbrace{a \frac{d}{da}}_{\partial} - \begin{pmatrix} 0 & 0 & 0 & 0 & a & 0 \\ 1 & 0 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

$\mathring{Gr}(2,4) \supset (\mathbb{C}^\times)^4$ Gelfand-Tsetlin coordinates.

Landau-Ginzburg model given by the regular function

$$\begin{tikzcd} z_1 \arrow[r] \arrow[d] & z_3 \arrow[d] & \\ z_2 \arrow[r] & z_4 \arrow[r] & a \end{tikzcd}$$

$$f_a(z) := \sum_{\text{arrows}} \frac{\text{head}}{\text{tail}} = z_1 + \frac{z_2}{z_1} + \frac{z_3}{z_2} + \frac{z_4}{z_3} + \frac{z_4}{z_2} + \frac{a}{z_4}$$

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$\mathcal{D}$-modules on $\mathbb{G}_m$: $\mathcal{O} := \mathbb{C}[a, a^{-1}]$ structure sheaf

free $\mathcal{O}$-module = alg. vector bundle V on $\mathbb{G}_m$.

Throughout this talk, $\boxed{\text{a will always denote the coordinate on } \mathbb{G}_m}$

$\mathcal{D} := \mathbb{C}[a, a^{-1}]\langle \frac{d}{da} \rangle$ Weyl algebra. $\left[a, \frac{d}{da}\right] = 1$.

$$\begin{tikzcd}
\text{scalar linear ODE} \arrow[r, leftrightarrow, "\text{cyclic vector}"] \arrow[d, leftrightarrow] & \text{matrix ODE} \arrow[d, leftrightarrow, "\text{connection 1-form}"] \\
\mathcal{D}\text{-module that is } \mathcal{O}\text{-free} \arrow[r, leftrightarrow] & \text{alg. vector bundle with connection}
\end{tikzcd}$$

$Lu = 0$

$a \frac{d}{da} \psi = A \psi$

(V, ∇)

$\mathcal{D}/\mathcal{D}L = \mathcal{M}$

$\mathcal{H}om(\mathcal{M}^{an}, \mathcal{O}^{an})$

local system

flat sections $(V^{an})^{\nabla}$

example: $L = \left(a \frac{d}{da}\right)^3 - a$ matrix form $A = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

${}_0F_2$ - hypergeometric

<u>Example</u>: Hypergeometric equation $L := \frac{d}{da}\prod_{j=1}^{q}\left(a\frac{d}{da}+\beta_j-1\right) - \prod_{i=1}^{p}\left(a\frac{d}{da}+\alpha_i\right)$

Hypergeometric $\mathcal{D}$-module $= \mathcal{D}/\mathcal{D}L$

Power series solution at the regular singular point $a=0$

$$ {}_pF_q\left(\begin{matrix}\alpha_1 \cdots \alpha_p\\ \beta_1 \cdots \beta_q\end{matrix}\middle| a\right) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k}\,\frac{a^k}{k!} $$

<u>Special cases</u>:

$\frac{d}{da} - 1$ ${}_0F_0(a) = e^a$

$\frac{d}{da} - \left(a\frac{d}{da}+\alpha\right)$ ${}_1F_0\left(\begin{matrix}\alpha\\ -\end{matrix}\middle| a\right) = (1-a)^{-\alpha}$

$\frac{d}{da}\left(a\frac{d}{da}+\beta-1\right) - 1$ ${}_0F_1\left(\begin{matrix}-\\ \beta\end{matrix}\middle| a\right) = I_0(2\sqrt{a})$ Bessel function

$\frac{d}{da}\left(a\frac{d}{da}+\beta-1\right) - \left(a\frac{d}{da}+\alpha\right)$ ${}_1F_1\left(\begin{matrix}\alpha\\ \beta\end{matrix}\middle| a\right)$ is Kummer confluent hypergeometric

$\frac{d}{da}\left(a\frac{d}{da}+\gamma-1\right) - \left(a\frac{d}{da}+\alpha\right)\left(a\frac{d}{da}+\beta\right)$ ${}_2F_1\left(\begin{matrix}\alpha\ \beta\\ \gamma\end{matrix}\middle| a\right)$ is Gauss hypergeometric function.

<u>Theorem.</u> (Lam - T '16)

If $P^\vee$ is a minuscule parabolic, then there is an isomorphism

quantum connection for $G^\vee/P^\vee$ $\simeq$ crystal $\mathcal{D}$-module for $(\overset{\circ}{G}/P, f_a)$

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Mirror theorem

$\underbrace{\text{Something you do not understand}}_{\text{B-side (complex geometry)}}$ is equal to $\underbrace{\text{something you cannot compute}}_{\text{A-side (symplectic geometry)}}$.

Mirror theorem

Something $\underbrace{\text{you do not understand}}_{\text{B-side (complex geometry)}}$ is equal to some$\underbrace{\text{thing you cannot compute}}_{\text{A-side (symplectic geometry)}}$.

Our proof is via a complicated $\underbrace{\text{thing that you can half understand / compute}}_{\text{automorphic form (number theory)}}$

Gross automorphic form A_G

Let G be a complex reductive group. Gross constructed an automorphic form A_G which one can think as the *simplest automorphic form*.

Theorem (Gross) There exists a unique automorphic form A_G over P^1 which is Steinberg at zero, *simple supercuspidal* at infinity and unramified otherwise.

It is rigid, similarly as Riemann’s theory of Gauss hypergeometric function. The proof is via the *simple trace formula*.

“Everything should be made as simple as possible, but not simpler”

On the Galois side it coincides with some of the local systems found by Katz.

Heinloth-Ngo-Yun constructed A_G by writing down a newvector inside as the trace function of an l-adic sheaf.

I like to think of their construction as a *far-reaching generalization* of Poincaré q-expansion of Poincaré series. Nowadays known as Petersson trace formula, see also Bump-Friedberg-Goldfeld.

We are going to use the construction of Heinloth-Ngo-Yun which also works over the complex numbers in the sense of geometric Langlands.

Historical note: This was Poincaré's <u>last paper</u> written in 1912 a few days before he died. Whereas Poincaré series was the <u>first</u> major work of Poincaré, during the years 1880-1882, when he discovered automorphic forms, the theory of Fuchsian and Kleinian groups, the uniformization theorem, monodromy, etc.

idea of proof: via automorphic forms

quantum $G^\vee/P^\vee$ —— MIRROR THM —— $(\mathring{G}/P, f_a)$

Langlands reciprocity

Kloosterman sheaves as Hecke eigenvalues

rigid automorphic form A_G on $\mathbb{P}^1$

Heinloth-Ngô-Yun '13: construct A_G
- tame unipotent at $\{0\}$
- wild simple supercuspidal at $\{\infty\}$

Zhu '16: quantization Hitchin system in the ramified case.

we identify the crystal $\mathcal{D}$-module as the automorphic side. (technically our main result).

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remark. Witten "gauge theory and wild ramification" '07 relates Langlands reciprocity and T-duality of the Hitchin systems for G and $G^\vee$.
See also Hausel-Thaddeus, Kapustin-Witten, Gukov-Witten, Boalch, Donagi-Pantev, ...

Solving linear ODEs as a Goal

$$\int_{\mathbb{G}_m} e^{z+\frac{q}{z}} \quad \text{solves} \quad q\frac{d}{dq} - \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} \quad \text{(Gauss–Manin)}$$

I want to make the following two key observations:

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I want to make the following two key observations:

We have seen that <u>Mirror Symmetry</u> relates to integral representations of linear ODE. Indeed a key expectation is that the Landau–Ginzburg model provides a solution of the quantum differential equation.

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We have seen that Mirror Symmetry relates to integral representations of linear ODE. Indeed a key expectation is that the Landau–Ginzburg model provides a solution of the quantum differential equation.

Geometric Langlands reciprocity also relates to ODE on the Galois side. The reciprocity conjecture says that the Hecke integral solves the ODE.

Mirror symmetry implies that a q-generating series of an enumerative problem is equal to a contour integral of a potential:

$$\oint e^{f_q} = \sum_{\text{degree } d} c_d q^d.$$

Gauss reciprocity says that a prime number q is a square modulo 5 if and only if 5 is a square modulo q. For example, the largest known prime $q = 2^{74207281} - 1$ is a square modulo 5, because $74207281 \equiv 81 \equiv 1 \pmod 4$ and $2^4 \equiv 1 \pmod 5$, so the last digit of q is 1. Therefore 5 is a square modulo q, which is hard to check directly.

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You should think of quantum q and prime q as analogous!

$$\sum_{n=1}^{5} e^{2i\pi q n^2/5} = \begin{cases} \sqrt{5} & \text{if 5 is a square mod } q, \\ -\sqrt{5} & \text{if 5 is not a square mod } q. \end{cases}$$

On the LHS is a Gauss sum, finite field analogue to the Gamma function.

<u>idea of proof</u>: via automorphic forms

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we identify the crystal $\mathcal{D}$-module as the automorphic side. (technically our main result).

Mirror symmetry relates to integral representations of special functions.

I cherish integral representations because of:

- number theory (used in virtually all applications!)
- Gelfand program (asymptotics of special functions, integrable hierarchies)
- representation theory (Jacquet integral)
- exponential sums (l-adic sheaves are built out of integrals)

Solving linear ODEs in the History

- Abel, Gauss: elliptic, hypergeometric functions.
- Riemann, Poincaré: monodromy. In 1880, automorphic forms are discovered.
- Lie groups: arised because of Sophus Lie thinking of differential Galois theory.
- Lefschetz: topological methods in algebraic geometry. Picard–Fuchs equation on de Rham cohomology (Gauss–Manin connection).
- Grothendieck, Deligne, Katz: ℓ-adic theory.
- Kashiwara, Bernstein: $\mathcal{D}$-modules.
- Dwork, Faltings, Scholze: p-adic theory.
- National Institute of Standards and Technology: `dlmf.nist.gov`
- Mirror symmetry and Langlands program, too!

Corollary (Peterson isomorphism)

Jacobian ring

quantum cohomology ring $\quad QH(G^\vee/P^\vee) \overset{\text{MIRROR}}{\simeq} \mathrm{Jac}(\mathring{G}/P, f_a)$

$:= \mathbb{C}[\mathrm{Critical}(f_a)]$

Peterson (announced '97) $\quad \wr\!\wr \qquad \quad \wr\!\wr$ Rietsch '08

$\mathbb{C}[Y_{(G,P)}]$

Peterson variety

The idea of proof is to take the semiclassical limit $\hbar \to 0$ of the main theorem

$$\mathcal{Q}_T^{G^\vee/P^\vee}(\hbar) \simeq \mathcal{C}r_{(G,P,T)}(\hbar)$$

We need to prove the T-equivariant version because the non-equivariant $QH^*(G^\vee/P^\vee)$ is not generated by H^2 in general.

Compare Givental equivariant GW (toric variety), Knutson-Tao puzzles $(Gr(k,n))$

Our mirror theorem in particular yields an isomorphism

$$\bigoplus_{i=0}^{d} H^{2i}(G^\vee/P^\vee) \overset{\text{MIRROR}}{\simeq} H^d_{dR}(\overset{\circ}{G/P}, e^f)$$

The RHS is $\left\{ e^f \omega,\ \omega \in \Omega^d(\overset{\circ}{G/P}) \right\} \Big/ \left\{ \begin{array}{c}\text{exact differentials} \\ d(e^f \eta)\end{array},\ \eta \in \Omega^{d-1}(\overset{\circ}{G/P}) \right\}$

"Twisted" de Rham complex because
$f=0$ is the usual de Rham cohomology: $\quad d + df \wedge$

<u>Example</u> $G/P = \mathbb{P}^1 \quad \overset{\circ}{G/P} = \mathbb{G}_m \quad f(x) = x + \frac{1}{x}$.

$$\left\{ e^{f(x)} P(x)\, dx,\ P \in \mathbb{C}[x, x^{-1}] \right\} \Big/ \left\{ d(e^{f(x)} Q(x)),\ Q = \sum_m a_m x^m \in \mathbb{C}[x, x^{-1}] \right\}$$

$$= e^{f(x)} \sum_{\mathbb{Z}} a_m (m x^{m-1} + x^m - x^{m-2})\, dx.$$

$$//$$

$$H^1_{dR}(\mathbb{G}_m, e^{x + \frac{1}{x}}) = \mathbb{C} \frac{dx}{x} \oplus \mathbb{C}\, dx.$$

<u>ex</u>: Note that $\frac{dx}{x^2} = dx$ because
$e^{f(x)}\left(1 - \frac{1}{x^2}\right) dx = d\left(e^{f(x)}\right)$.

$$\bigoplus_{i=0}^{d} H^{2i}(G^\vee/P^\vee) \overset{\text{MIRROR}}{\simeq} H^d_{dR}(\overset{\circ}{G}/P, e^f)$$

<u>Corollary</u> (Deligne-Yu filtration)

In particular $F^d H^d_{dR}(\overset{\circ}{G}/P, e^f) = \mathbb{C}\cdot vol$ ← unique non-vanishing form with log poles

(Knutson-Lam-Speyer '09)

Deligne '84, '07 : irregular Hodge filtration that is not a Hodge structure

Esnault-Sabbah-Yu '15

Kontsevich complex '12 : f-adapted log forms $\Omega_f^\bullet$

<u>example</u> $G/P = \mathbb{P}^2$ $\quad \overset{\circ}{G}/P = \mathbb{G}_m^2$ $\quad f = x_1 + x_2 + \frac{1}{x_1 x_2}$

$$\begin{array}{ccccc} & & 1 \; h^{2,2} & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 \; h^{0,0} & & \end{array} \qquad\qquad \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 1 & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{array}$$

$H^0(\mathbb{G}_m^2, \Omega_f^2) = \mathbb{C}\cdot vol = \mathbb{C}\,\frac{dx_1 dx_2}{x_1 x_2}$

← $p+q = 2$ (purity)

Hodge diamond of $\mathbb{P}^2$ $\qquad\qquad$ $\dim H^p(\mathbb{G}_m^2, \Omega_f^q)$

Recall: $$Kl_n(a) = \sum_{x_1 \cdots x_{n-1} \in \mathbb{F}_p^\times} e_p\left(x_1 + \cdots + x_{n-1} + \frac{a}{x_1 \cdots x_{n-1}}\right)$$

<u>THM</u> (Deligne SGA $4\frac{1}{2}$, Sperber 77) ($n=2$: Weil 48, Dwork 74)

for every $a \in \mathbb{F}_p^\times$ (i) $Kl_n(a) = \alpha_1 + \cdots + \alpha_n$ is a sum of Weil numbers of wt $n-1$

(ii) $v_p(\alpha_1) = 0$, $v_p(\alpha_2) = 1$ $\cdots$ $v_p(\alpha_n) = n-1$.

Towards the general case: Families

In our work, we exploited the rigidity, namely we have a single ODE, which admit no deformation. Beyond the rigid case, I propose to work with families.

Families in number theory: Selberg, Bombieri, Iwaniec, Taylor–Wiles, Katz–Sarnak. Langlands functoriality conjecture says that all automorphic forms can be pushed to $\mathrm{GL}(N)$ over $\mathbb{P}^1$. So a family is a certain spectral set of automorphic forms with the same monodromy (Sato–Tate group).

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In Mirror Symmetry: Witten–Dijkgraaf–Verlinde–Verlinde (WDVV), Dubrovin, Givental (Frobenius manifolds), Katzarkov–Kontsevich–Pantev (nc Hodge structure), Abouzaid (family Floer). The key is to construct an isomonodromic deformation, parametrized by $H^*(X)$, of $\mathbb{C}((\hbar))$-connections, plus additional Hodge data.

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One wants to relate the two worlds, e.g. by examining more families of meromorphic connections on $\mathbb{P}^1$.

summary

Q. What number theory brings to mirror symmetry?

Purity, weight-monodromy, Ramanujan conj, are at the heart of number theory and automorphic forms. There are statements inside mirror symmetry that do involve purity. In this talk I focus on those statements.

Q. What mirror symmetry brings to number theory?

Hodge structures, which could be transported to congruences via p-adic Hodge theory. *Asymptotics, Purity,* which could be exploited directly.

$$\bigoplus_{i=0}^{d} H^{2i}(G^\vee/P^\vee) \overset{\text{MIRROR}}{\simeq} H^d_{dR}(\mathring{G}/P, e^f)$$

Corollary (purity) $\dim H^i_{dR}(\mathring{G}/P, e^f) = \begin{cases} |W^P| & \text{if } i=d \\ 0 & \text{o/w} \end{cases}$.

example: $G/P = \mathbb{P}^n$, $\mathring{G}/P = \mathbb{G}_m^n$ $\quad f = x_1 + \dots + x_n + \dfrac{a}{x_1 \cdots x_n}$.

This is Deligne purity theorem (SGA $4\frac{1}{2}$) for hyperKloosterman sums:

$\mathcal{L}_\psi$: Artin-Schreier sheaf $\quad H^i_{\acute{e}t}(\mathbb{G}_m^n, f^* \mathcal{L}_\psi)$.

its trace function is $Kl_n(a) = \sum_{x \in (\mathbb{F}_p^\times)^n} e^{f(x)}$ $\quad |Kl_n(a)| \leq \dim(H^d_{\acute{e}t})\, p^{\frac{d}{2}}$

$= (n+1)\, p^{\frac{n}{2}}$.

M. Hien thm: $H_i^{\text{rapid decay}}(U, \nabla) \times H^i_{dR}(U, \nabla^\vee) \longrightarrow \mathbb{C}$ is a perfect pairing. $\forall i$

Inv. '09

Definition of rapid decay cycles in general follows from monumental work of T. Mochizuki: "resolution of singularities for $\mathcal{D}$-modules".

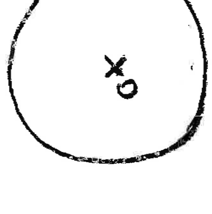
Here for $\nabla = e^f$ we can use Hironaka + Deligne, Bloch-Esnault.

example:
$$G^\circ/P = \mathbb{G}_m \qquad f(x) = x + \frac{1}{x}.$$

$$\mathrm{Re}(f) \longrightarrow -\infty \iff \mathrm{Re}(x) \longrightarrow -\infty \text{ or } \begin{array}{l} x \to 0 \\ \arg(x) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \end{array}$$

these two cycles are basis of

basis of $H_1^{\text{rapid decay}}(\mathbb{C}^\times, e^{-f})$.



Corollary $\dim H_i^{\text{rapid decay}}(G^\circ/P, e^f) = \begin{cases} |W^P| & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$

Compare with $H_*(G/P)$, very different!

Corollary (combinatorial formula for GW invariants)

The hypergeometric series of $G^\vee/P^\vee$ has the integral representation:

$$I_{G^\vee/P^\vee}(a) = \oint e^{f_a} \cdot \mathrm{vol}$$

half-dimensional compact cycle $\in H_d(\overset{\circ}{G/P}) \subset H_d^{\text{rapid decay}}(\overset{\circ}{G/P}, e^{-f_a})$

Beilinson-Bernstein $\Longrightarrow$ $\dim H_d(\overset{\circ}{G/P}) = \dim \mathrm{Ext}^0(M_{w_0}, M_{w_0^P}) = 1$

Verma modules

example $G^\vee/P^\vee = \mathbb{P}^n$

$$I_{\mathbb{P}^n}(a) = \sum_{k=0}^{\infty} \frac{a^k}{(k!)^{n+1}} = \oint e^{x_1 + \cdots + x_n + \frac{a}{x_1 \cdots x_n}} \frac{dx_1 \cdots dx_n}{(2i\pi)^n \, x_1 \cdots x_n}$$

Erdelyi integral for ${}_0F_n(\overline{1 \cdots 1}; a)$.

example $Gr(k,n)$ conjecture of Batyrev-Ciocan-Fontanine-Kim-van Straten, Acta Math '00

proved by Marsh-Rietsch '13.

$$\left(\mathbb{G}_m^2 ,\ x_1 + x_2 + \frac{a}{x_1 x_2} \right) \xleftrightarrow{\text{MIRROR}} \mathbb{P}^2$$

"<u>Hodge numbers for $Kl_3(a)$</u>"

$$\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 1 & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{array}$$

$$h^{pq} = \begin{cases} 1 & \text{if } p+q = 2 \\ 0 & \text{o/w} \end{cases}$$

<u>Hodge diamond of $\mathbb{P}^2$</u>

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

generated by the hyperplane class σ (purely algebraic)

$$H^{p,q}(\mathbb{P}^2) = \begin{cases} \mathbb{C}\cdot\sigma^p & \text{if } p = q \\ 0 & \text{o/w} \end{cases}$$

purity $\xleftrightarrow{\text{MIRROR}}$ Hodge-Tate type

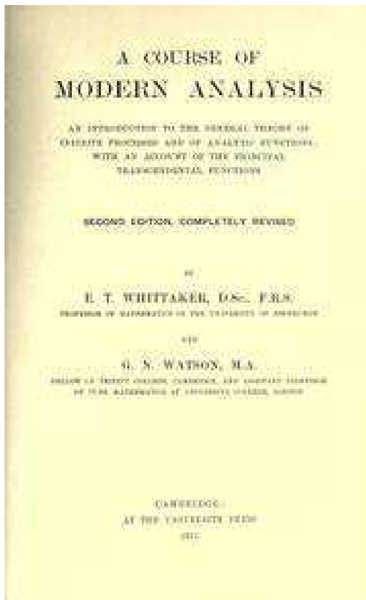
slopes $\xleftrightarrow{\text{MIRROR}}$ degrees

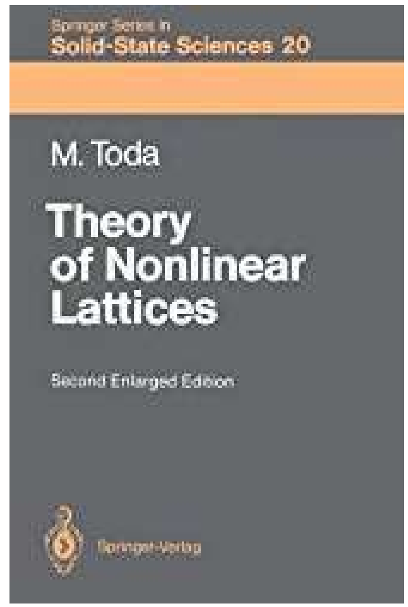
Kim-Givental '95: complete flag variety G/B, dual $G^\vee/B^\vee$. B = Borel subgp.

Joe-Kim '03

$G^\vee/B^\vee$ — MIRROR — $(\mathring{G/B}, f)$

quantum connection — Toda lattice — Whittaker function — integral representation





Zuckerman conjecture (unpublished from '79) First appears in To's PhD thesis '95.

T. in progress: study it using ideas from mirror symmetry.

- Lefschetz thimbles of f in $\mathring{G/B}$.
- Dubrovin conjecture: exceptional collection on $G^\vee/B^\vee$.

(see also Gamma conjecture of Golyshev-Iritani-Galkin '13)

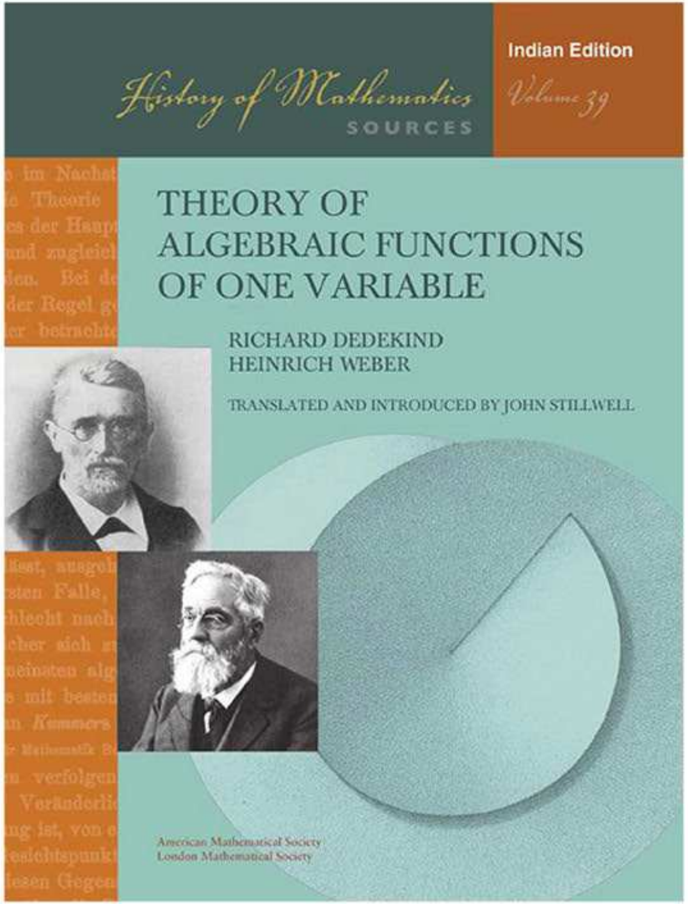
<u>Global fields</u>

Number fields	Smooth projective curve X	Riemann surface X	
$\mathbb{Q}$	$\mathbb{F}_p(X)$	$\mathbb{C}(X)$	physics

Dedekind-Weber, *Theory of algebraic functions of one variable* (1882).

~~Birkhoff-Grothendieck~~

~~Weil's letter to his sister~~



$$Kl^{Std}_{GL(3)}(a) := \sum_{z_1, z_2 \in \mathbb{F}_p^\times} e^{\frac{2i\pi}{p}\left(z_1 + z_2 + \frac{a}{z_1 z_2}\right)}$$

hyper-Kloosterman

Deligne, SGA $4\frac{1}{2}$: pure lisse $\left| Kl^{Std}_{GL(3)}(a) \right| \leq 3p$ $\quad \forall a \in \mathbb{F}_p^*$

Katz book '96: rigid local system

Frenkel-Gross, Annals '09: construct rigid connection ∇_G^V on $\mathbb{P}^1$

for any representation (G, V)

Heinloth-Ngô-Yun, Annals '13: construct $Kl_G^V(a) = \sum_{z \in X(\mathbb{F}_p)} e^{\frac{2i\pi}{p} f_a(z)}$

Hecke eigensheaf

$$\begin{tikzcd} & Hk_V \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Bun_{G^\vee} & & \mathbb{P}^1 \times Bun_{G^\vee} \end{tikzcd} \qquad p_{2!}\, p_1^*$$

Ramanujan bound over function field. Compare Ramanujan $\tau(p)$.

Lam-T '16: X is identified with $\mathring{G}/P$ and $f_a(z)$ is the potential function.

<u>example above</u>: $z \in X = \mathbb{G}_m \times \mathbb{G}_m = \mathring{\mathbb{P}}^2 = \mathring{\overline{GL(3)/\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}}}$

Suppose you want to study an exponential sum (e.g. square-root cancellation).

Step 1. Apply Deligne theorem “Weil I”. It is equivalent to prove purity of cohomology:

$$H^i_{\text{ét}}(Y, f^*\mathcal{L}_\psi) = 0, \text{ if } i \neq \dim Y$$

Step 2. Apply Fourier transform, see Katz, “differential equations and exponential sums.”

Step 3. Apply complex to l-adic comparison theorem.

Step 4. Identify the mirror Fano variety X.

Step 5. Prove that X has cohomology of Hodge-Tate type:

$$H^{(p,q)}(X, \mathbb{C}) = 0, \text{ if } p \neq q$$

Step 6. Prove mirror symmetry for X and (Y,f).

The same outline could work for p-adic slopes (replace etale by Dwork cohomology) and for asymptotics (replace steps 1-3 by symplectic Lefschetz thimbles).

cell in Beilinson-Drinfeld affine Grassmannian.

$p_{2!}\, p_1^* =$ Hecke correspondence

Heinloth-Ngô-Yun rigid automorphic form A_G is a D-module on Bun_G

$$\begin{tikzcd} & Gr^\circ \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Bun_G & & \mathbb{P}^1 - \{0,\infty\} \times Bun_G \\ U \arrow[u, phantom, "\cup" description] & & \{pt\} \arrow[u, phantom, "\cup" description] \end{tikzcd}$$

$$\mathbb{A}^1 \xleftarrow{\ \psi\ } U/[U,U] \times \mathbb{A}^1$$

restrict this diagram to this point and compare with previous page $\Longrightarrow$

$$p_{2!}\, p_1^*(A_G) = Kl^\vee_{G^\vee} \boxtimes A_G$$

generalized Kloosterman D-module on $\mathbb{P}^1 - \{0,\infty\}$ as Hecke eigenvalue of A_G.

(compare $T_a(f) = \lambda(a) f$ for a classical modular form f on $SL_2\mathbb{Z}$)

Thm. (Lam-T '16) If $P^\vee$ is minuscule, then $Kl^\vee_{G^\vee}$ coincides with $\int_{\mathring{G}/P} e^{f_a}$

Consequences of the mirror theorem

- we establish the Peterson isomorphism (announced '97) for minuscule flag varieties $G^\vee/P^\vee$. This is the semi-classical limit ($\hbar \to 0$) of the mirror theorem

$$QH^*(G^\vee/P^\vee) \simeq \mathcal{O}(Y_P)$$

small quantum cohomology ring

ring of regular functions on the Peterson variety Y_P

$$Y := \{ g \in G/B \,,\ \mathrm{Ad}(g^{-1}) f \in [u,u]^\perp \}$$

$$Y_P := Y \cap B_- w_P B$$

principal nilpotent in $\mathfrak{b}_-$

- the conjecture of Batyrev – Ciocan-Fontanine – Kim – Van Straten (Acta Math '00) for Grassmannians $Gr(k,n)$ using Gelfand-Tsetlin coordinates as a cluster chart.
- a conjecture of Marsh-Rietsch '13 for $Gr(k,n)$ and Pech-Rietsch-Williams '15 for quadrics

Euler-Poincaré characteristic calculation + purity.

Smooth projective Fano.		Landau-Ginzburg model = (quasi-projective Calabi-Yau, potential).
symplectic invariants : A-side		complex invariants : B-side
Fukaya category	$\overset{?}{\simeq}$ HMS	Matrix factorization category
quantum cohomology = enumerating rational curves		singularity theory.
Frobenius manifold		Saito mixed Hodge modules
isomonodromic deformations		miniversal deformations
small quantum differential equation (linear ODE)	$\overset{?}{\simeq}$ MIRROR	pushforward D-module

Reconstruction theorems: e.g. quantum product is associative (WDVV equation)

e.g. from small to big when cohomology is generated in degree 2.

early works that launched the program:

Givental ICM '94, Kontsevich ICM '94, Dubrovin ICM '98

Example: $Gr(2,4) = GL(4) \Big/ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} = 4$-dimensional quadric

$\circ\!-\!\!\!-\!\times\!-\!\!\!-\!\circ$ A_3: Dynkin.

quantum connection is

$$a\frac{d}{da} - \begin{pmatrix} 0 & 0 & 0 & 0 & a & 0 \\ 1 & 0 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Gelfand-Tsetlin coordinates

$$\begin{tikzcd} {} \arrow[d] & & \\ z_1 \arrow[r] \arrow[d] & z_3 \arrow[d] & \\ z_2 \arrow[r] & z_4 \arrow[r] & a \end{tikzcd}$$

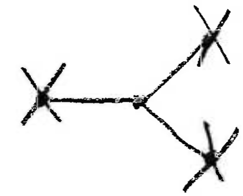
$$f_a(z) = \sum_{\text{arrows}} \frac{\text{head}}{\text{tail}} = z_1 + \frac{z_2}{z_1} + \frac{z_3}{z_2} + \frac{z_4}{z_3} + \frac{z_4}{z_2} + \frac{a}{z_4}$$

Corollary $\Longrightarrow$ $\oint e^{f_a(z)} \frac{dz}{z}$ is in the kernel of the connection.

Which can be verified directly: $\displaystyle\sum_{r=0}^{\infty} \frac{(2r)!}{r!^6} a^r$

is in the kernel of $\partial^5 - 2a(2\partial + 1)$ $\qquad \partial = a\dfrac{d}{da}$

<u>Example:</u> 6-dimensional quadric $= SO(8)/P$



D_4-triality

Hasse diagram

middle cohomology is 2-dim.

quiver Pech-Rietsch-Williams '15

$$\begin{tikzcd} 1 \arrow[d] & \\ z_1 \arrow[d] & \\ z_2 \arrow[d] & \\ z_3 \arrow[r] \arrow[d] & z_5 \arrow[d] \\ z_4 \arrow[r] & a \end{tikzcd}$$

$$f_a(z) = z_1 + \frac{z_2}{z_1} + \frac{z_3}{z_2} + \frac{z_4}{z_3} + \frac{z_5}{z_3} + \frac{a}{z_4} + \frac{a}{z_5}.$$

quantum connection is

$$a\frac{d}{da} - \begin{pmatrix} 0 & \cdots & \cdots & a & 0 \\ 1 & & & 0 & a \\ & 1 & 0 & & \vdots \\ & 1 & & & \vdots \\ & 1 & 1 & & \vdots \\ 0 & & 1 & & \\ & & & 1 & 0 \end{pmatrix}$$

7-dim stable subspace generated by σ.

$= \mathcal{D}/\mathcal{D}(\partial^7 - 2a(2\partial+1))$ where $\mathcal{D} = \mathbb{C}[a, a^{-1}]\langle \partial \rangle$.

<u>Thm</u> (Katz, Frenkel-Gross) The monodromy group is G_2.

because of S_3-symmetry of D_4

because it is the $(1,7)$-hypergeometric ${}_1F_7\left(\begin{matrix} 1/2 \\ 1111111 \end{matrix}; a\right)$

thm 4.1.5 in "Exponential sums and diff. equations", Annals of Math Studies.

“Everything should be made as simple as possible, but not simpler.”

$$\mathbb{CP}^2$$

$$Kl_2(a) = \sum_{x_1, x_2 \in \mathbb{F}_p^\times} e\left(x_1 + x_2 + \frac{a}{x_1 x_2}\right)$$

$$x^2 + y^2 + z^2 = 3xyz$$

$$ {}_0F_2\left(\begin{matrix} - \\ 1\ 1 \end{matrix}; a\right) := \sum_{k=0}^{\infty} \frac{a^k}{(k!)^3}$$

$$a\frac{d}{da} - \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$N_6 = 26312976$$

$$\left(a\frac{d}{da}\right)^3 - a$$

$$\phi_{xxy}^2 = \phi_{yyy} + \phi_{xxx}\phi_{xyy}$$

$$\mathrm{Ind}_{\begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}}^{\mathrm{PGL}_2(\mathbb{Q}_p)}(\chi)$$

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

$$\dim S_k(p^3)^{\mathrm{new}} = \frac{k-1}{12}(p-1)^2(p+1)$$

quantum Chevalley formula (Fulton - Woodward '04 Witten '91)

$$H^*(G/P) = \bigoplus_{w \in W/W_P} \mathbb{C}\,\sigma_w$$ Schubert basis.

$$W^P \xrightarrow{\sim} W/W_P \qquad \pi_P : W \longrightarrow W/W_P$$

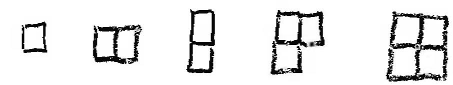
minimal representatives in Bruhat order

Th: If P is minuscule, $\exists!$ root γ such that $\forall w \in W^P$

$$\sigma_1 * \sigma_w = \sum_{\substack{\beta \in R^+ \setminus R_P^+ \\ W^P \ni w s_\beta > w}} \langle \beta^\vee, \omega_1 \rangle\, \sigma_{w s_\beta} + a \langle \gamma^\vee, \omega_1 \rangle\, \sigma_{\pi_P(w s_\gamma)}$$

if $\ell(\pi_P(w s_\gamma)) = \ell(w) + 1 - \langle \gamma^\vee, 2(\rho - \rho_P) \rangle$

Example $Gr(2,4)$ $W = \mathfrak{S}_4$ $W_P = \mathfrak{S}_2 \times \mathfrak{S}_2$ W^P:

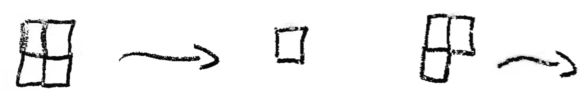


ϕ σ_ϕ σ_1 σ_2 σ_{11} σ_{21} σ_{22}

$\sigma_1 * \sigma_{21} = \sigma_{22} + a$: add a box

σ_{21} σ_{22}

$\sigma_1 * \sigma_{22} = a\,\sigma_1$: remove a rim



ϕ

Berenstein-Kazhdan crystal $\xrightarrow{\text{tropicalize}}$ Lusztig-Kashiwara combinatorial crystal

$$\begin{array}{ccc} \lambda_1 & & \\ \downarrow & & \\ z & \longrightarrow & \lambda_2 \end{array} \qquad f_\lambda(z) = \frac{z}{\lambda_1} + \frac{\lambda_2}{z}$$

$$\begin{array}{ccc} \lambda_1 & & \\ \vee\!\mid & & \\ z & \geq & \lambda_2 \end{array} \qquad \max(z-\lambda_1, \lambda_2 - z) \leq 0$$

Gelfand-Tsetlin pattern $\xleftrightarrow[1:1]{}$ Semi-std Young tableaux of shape λ

Kloosterman sum: $\displaystyle\sum_{z \in \mathbb{F}_p^\times} \chi(z) e^{\frac{2i\pi}{p} f(z)}$ $\quad \chi : \mathbb{F}_p^\times \to S^1$

Schur polynomial $\displaystyle s_\lambda(x) = \sum_{\lambda_2 \leq z \leq \lambda_1} x^z$

Bessel-Whittaker function: $\displaystyle\int \chi(z) e^{f_\lambda(z)} \frac{dz}{z}$

crystal $\mathcal{D}$-module: $\displaystyle\int \mathcal{L}_\chi \otimes f_\lambda^* \mathrm{Exp}$

$\mathcal{D} = \mathbb{C}[z]\langle d \rangle$

$$\mathcal{L}_\chi := \mathcal{D}/\mathcal{D}(d - \chi z) \qquad \mathrm{Exp} := \mathcal{D}/\mathcal{D}(d-1)$$

$$f^* \mathrm{Exp} = \mathcal{D}/\mathcal{D}(d - f')$$

Kim-Givental '95: complete flag variety G/B, dual $G^\vee/B^\vee$. B = Borel subgp."

Joe-Kim '03

$G^\vee/B^\vee$ ——— MIRROR ——— $(\overset{\circ}{G}/B, f)$

quantum connection

Toda lattice

Whittaker function

integral representation

<u>Non-exhaustive list</u> of <u>related works</u>: Jacquet integral '67; Kostant, Goodman-Wallach: rep theory; Jacquet-Piatetski-Shapiro-Shalika: Rankin-Selberg integrals; Casselman-Shalika-Shintani formula; Stade formula; Frenkel-Gaitsgory-Vilonen: geometric Langlands; Peterson, Kostant, Rietsch: Toda; Ginzburg-Jiang-Soudry: automorphic descent; Brubaker-Bump-Chinta-Friedberg: metaplectic; Borodin, Chhaibi, Corwin, O'Connell: probabilistic processes; Gerasimov-Lebedev-Oblezin: integrable systems; Braverman-Maulik-Okounkov: Springer resolution; Brumley-T '14: large values and singularities; Miller-Trinh '16: automorphic growth. Poincaré 1912, Bump-Friedberg-Goldfeld '88: Poincaré series.

<u>Zuckerman conjecture</u> (unpublished from '79) First appears in To's PhD thesis '95.

T. in progress: study it using ideas from mirror symmetry

- Lefschetz thimbles of f in $\overset{\circ}{G}/B$
- Dubrovin conjecture: exceptional collection on $G^\vee/B^\vee$.

(see also Gamma conjecture of Golyshev-Iritani-Galkin '13)