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Notation and language:

- $\mathbf{Y}$ denotes a simply connected $\mathbf{3}$-dimensional homogeneous manifold.
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- In particular, $\mathbf{H}(\mathbf{Y}) = 1$ if $\mathbf{Y} = \mathbb{H}^3$ and $\mathbf{H}(\mathbf{Y}) = 1/2$ if $\mathbf{Y} = \mathbb{H}^2 \times \mathbb{R}$.

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Theorem (Geometry of **H**-spheres, 2017 Meeks-Mira-Pérez-Ros)

Let **S** be an **H**-sphere in **M**.

1. If $\mathbf{H} = 0$ and **X** is a product $\mathbf{S^2} \times \mathbb{R}$, where $\mathbf{S^2}$ is a sphere of constant curvature, then **S** is totally geodesic, stable and has nullity **1** for its Jacobi operator.

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3. There is a point $\mathbf{p_S} \in \mathbf{M}$, called the **center of symmetry** of $\mathbf{S}$, such that every isometry of $\mathbf{M}$ that fixes $\mathbf{p_S}$ also leaves invariant $\mathbf{S}$.

Previous influential results on the **Hopf Uniqueness Problem**:

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Theorem (Daniel-Mira (2013), Meeks (2013))

- If **X** is the Lie group $\mathbf{Sol_3}$ with the left invariant metric

$$e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

  then **H**-spheres in **X** are unique, embedded and have index 1.
- After left translation, these spheres have ambient symmetry group generated by reflections in the (x, z) and (y, z)-planes and rotations by π around the two lines $y = \pm x$ in the (x, y)-plane.

Theorem (Classification Theorem for $\mathbf{H}$-spheres, Meeks-Mira-Pérez-Ros)

Suppose $\mathbf{X}$ is a simply connected $\mathbf{3}$-dimensional homogeneous manifold different from $\mathbf{S}^2(\kappa) \times \mathbb{R}$, where $\mathbf{S}^2(\kappa)$ is a sphere of curvature κ.

- $\mathbf{X}$ diffeomorphic to $\mathbf{S}^3 \implies$ the moduli space of $\mathbf{H}$-spheres in $\mathbf{X}$ is parameterized by the mean curvature values $\mathbf{H} \in \mathbb{R}$.
- $\mathbf{X}$ diffeomorphic to $\mathbf{R}^3 \implies$ moduli space of $\mathbf{H}$-spheres in $\mathbf{X}$ is parameterized by the $\mathbf{H} \in \mathbb{R}$ values, where $|\mathbf{H}| \in (\mathbf{H}(\mathbf{X}), \infty)$.
- $\mathbf{X}$ diffeomorphic to $\mathbf{S}^3 \implies$ the areas of all $\mathbf{H}$-spheres form a half-open interval $(0, \mathbf{A}(\mathbf{X})]$.
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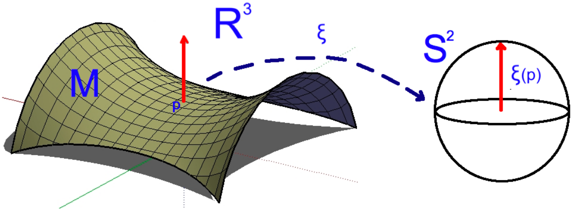
Remark

In the following proof, choose a metric Lie group structure on **X**.



Definition (Left invariant Gauss map)

- Let $\mathbf{X}$ be a $\mathbf{3}$-dimensional metric Lie group.
- Given an oriented immersed surface $f\colon \mathbf{M} \to \mathbf{X}$ with unit normal vector field ξ, the **left invariant Gauss map** of $\mathbf{M}$ is the map $\mathbf{G}\colon \mathbf{M} \to \mathbf{S}^2 \subset T_{\mathbf{e}}\mathbf{X}$ that assigns to each $\mathbf{p} \in \mathbf{M}$, the unit tangent vector to $\mathbf{X}$ at the identity element $\mathbf{e}$ given by left translation:

$$(dl_{f(\mathbf{p})})_{\mathbf{e}}(\mathbf{G}(\mathbf{p})) = \xi_{\mathbf{p}}.$$

Steps of the proof of the Classification Theorem for **H**-spheres.

Throughout $\boldsymbol{\Sigma}$ denotes a fixed $\mathbf{H}_0$-sphere in $\mathbf{X}$ of **index 1**.

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- **Step 0:** **Σ** has **nullity 3**: **Cheng's theorem**.
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  (A) If **X** is isomorphic to $SU(2)$, areas of spheres in $\mathcal{M}(X)$ are uniformly bounded.

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- **Conclusions:**
  - The space of non-congruent $\mathbf{H}$-spheres in $\mathbf{X}$ equals $\mathcal{M}(\mathbf{X})$ which is an interval parameterized by the mean curvature values in $[0,\infty)$ if $\mathbf{X}$ is isomorphic to $\mathbf{SU(2)}$ and otherwise, in the interval $(\mathbf{H}(\mathbf{X}),\infty)$.
  - Each $\mathbf{H}$-sphere in $\mathbf{X}$ has **index 1** and **nullity 3**.
  - Each $\mathbf{H}$-sphere in $\mathbf{X}$ is the boundary of an immersed $\mathbf{3}$-ball $\mathbf{F}\colon \mathbf{B}\to\mathbf{X}$ on its mean convex side (Alexandrov embedded).
  - If $\mathbf{X}$ is isomorphic to $\mathbf{SU(2)}$, then the areas of $\mathbf{H}$-spheres in $\mathbf{X}$ form a half-open interval $(0,\mathbf{A}(\mathbf{X})]$. □

Theorem (Curvature Estimates for H-Disks, Meeks-Tinaglia 2018)

Fix $\varepsilon, \mathbf{H}_0 > 0$ and a complete locally homogenous **3**-manifold $\mathbf{X}$. $\exists\ \mathbf{C} > 0$ s.t. for all embedded $(\mathbf{H} \geq \mathbf{H_0})$-disks $\mathbf{D}$ in $\mathbf{X}$:

$$|\mathbf{A_D}|(p) \leq \mathbf{C} \quad \text{for all} \quad p \in \mathbf{D} \text{ s.t. } \mathbf{dist_D}(p, \partial\mathbf{D}) \geq \varepsilon,$$

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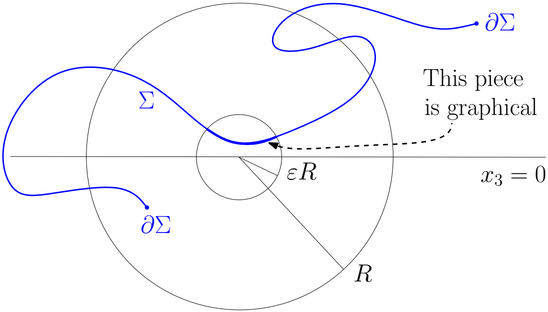
- **Step 4:** One extends the double multigraph in the forming helicoid near $p_n \in \mathbf{D}(n)$ a definite distance for n large, a contradiction.

Theorem (One-sided curvature estimate for H-disks, Meeks-Tinaglia)

$\exists \mathbf{C}, \varepsilon > 0$ s.t. for any embedded **H**-disk $\mathbf{\Sigma} \subset \mathbf{R}^3$ as in the figure below:

$$|\mathbf{A}_{\mathbf{\Sigma}}| \leq \frac{\mathbf{C}}{R} \quad \text{in} \quad \mathbf{\Sigma} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\}.$$



This result generalizes the one-sided curvature estimates for minimal disks by Colding-Minicozzi, and uses their work in its proof.

Old Question

Is the round sphere the only complete simply connected surface **embedded** in $\mathbf{R}^3$ with **non-zero** constant mean curvature?

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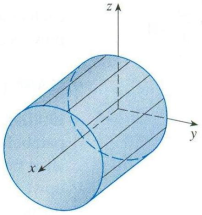

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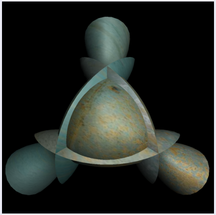

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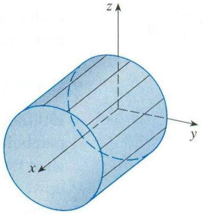

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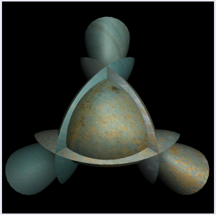

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Answer (Meeks-Tinaglia)

Yes!

Corollary (Radius Estimates for H-Disks, Meeks-Tinaglia 2017)

$\exists\ \mathbf{R_0} \geq \pi$ such that every embedded **1**-disk in $\mathbf{R}^3$ has radius $< \mathbf{R_0}$.

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Corollary (Meeks-Tinaglia 2017)

A complete simply connected $\mathbf{H}$-surface embedded in $\mathbf{R^3}$ with $\mathbf{H} > 0$ is a round sphere.

Theorem (Calabi-Yau Holds for Embedded Finite Topology **H**-surfaces, Meeks-Tinaglia 2017)

Let $\mathbf{M} \subset \mathbf{R}^3$ be a complete, connected embedded **H**-surface.

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When $\mathbf{H} = \mathbf{0}$, items **1** and **2** were proved by Meeks-Rosenberg, based on:

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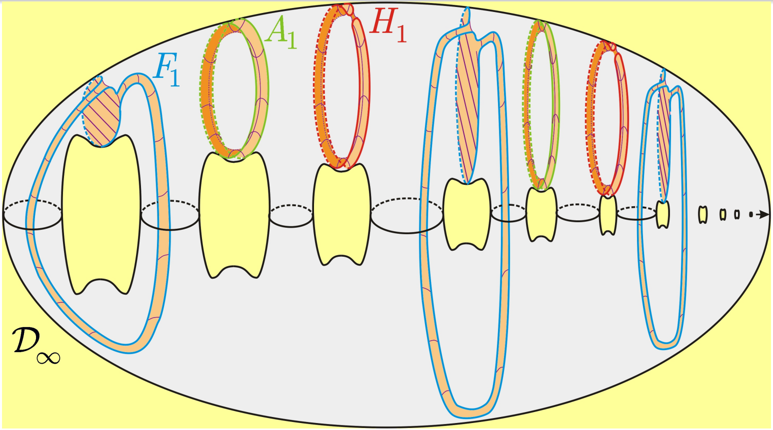
Item 3 in the above theorem holds for 3-manifolds which have bounded absolute sectional curvature; in particular it holds in closed Riemannian 3-manifolds.



- $\mathcal{D}_\infty$ = above **bounded domain, smooth except at $\mathbf{p}_\infty$** on right.
- **Ferrer, Martin** and **Meeks** conjecture: An open surface **properly embeds as a complete minimal surface in $\mathcal{D}_\infty$** $\iff$ every end has **infinite genus** $\iff$ it admits a complete bounded minimal embedding in $\mathbb{R}^3$.

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- $\overline{\mathbf{\Sigma}}$ is an $\mathbf{H}$-lamination in $\mathbf{R}^3$ **iff** $\mathbf{\Sigma}$ has locally bounded genus in $\mathbf{R}^3$.
- $\exists\ \mathbf{A}_{\mathbf{\Sigma}}$ s.t. $\forall \mathbf{r} \geq 1$ and $\mathbf{p} \in \mathbf{R}^3$,

$$\mathbf{Area}(\mathbf{\Sigma} \cap \mathbb{B}(\mathbf{p}, \mathbf{r})) \leq \mathbf{A}_{\mathbf{\Sigma}} \cdot \mathbf{r}^3 \quad \textbf{iff}$$

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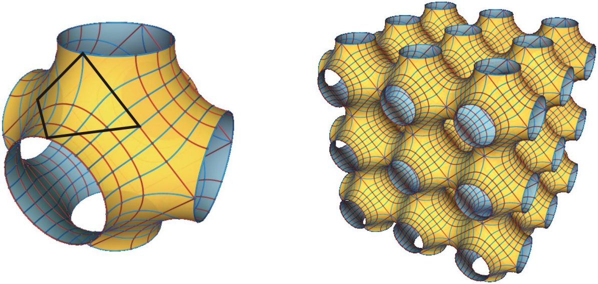
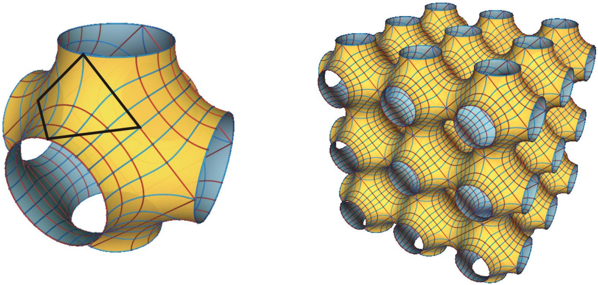


Figure: A body-centered cubic interface or Fermi surface in salt crystal.

Next theorem is motivated by the study of **3**-periodic **H**-surfaces that appear as interfaces in material science or as equipotential surfaces in crystals. This result contrasts with the failure of area estimates for compact minimal surfaces of genus $\mathbf{g} > 2$ in any flat **3**-torus (**Traizet**).



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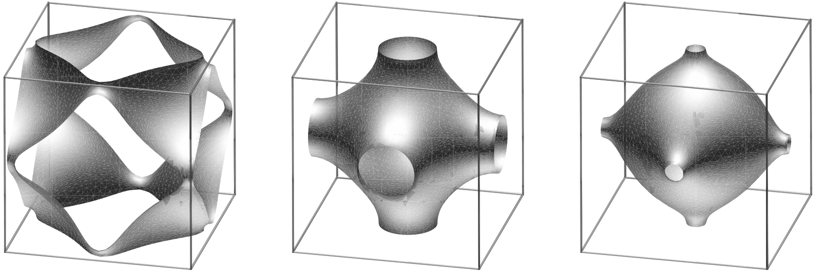
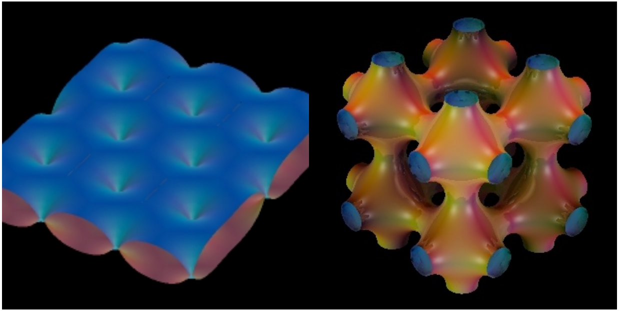
Theorem (Meeks-Tinaglia(2016))

Given a flat **3**-torus $\mathbb{T}^3$ and $\mathbf{H} > 0$, $\forall \mathbf{g} \in \mathbb{N}$, $\exists \mathbf{C}(\mathbf{g}, \mathbf{H})$ s.t. a closed **H**-surface $\boldsymbol{\Sigma}$ embedded in $\mathbb{T}^3$ with genus at most $\mathbf{g}$ satisfies

$$\mathbf{Area}(\boldsymbol{\Sigma}) \leq \mathbf{C}(\mathbf{g}, \mathbf{H}).$$

Closed H-surfaces in a flat 3-torus. By K. Grosse-Brauckmann (top) and N. Schmitt (bottom)





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Let $\mathbf{N}$ = a closed Riemannian $\mathbf{3}$-manifold with Ricci curvature > 0. Then:

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2. For **every** closed Riemannian **3**-manifold **X** and any non-negative integer **g**, the space of strongly Alexandrov embedded closed surfaces in **X** of genus at most **g** and constant mean curvature $\mathbf{H} \in [a, b]$ is **compact**. (Similar compactness result holds for any fixed smooth compact family of metrics on **X**.)

Theorem (Meeks-Tinaglia (2018)

- For $\mathbf{H} \geq 1$, complete embedded finite topology **H**-surfaces $\mathbf{\Sigma}$ in complete hyperbolic **3**-manifolds are **proper**.

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- For every $\mathbf{H} \in [0, 1)$, $\exists$ a complete embedded stable $\mathbf{H}$-plane that is **nonproper** in the hyperbolic $\mathbf{3}$-space $\mathbb{H}^3$.
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- Suppose **X** is a complete hyperbolic **3**-manifold with finite volume, **H** $\in [0,1)$ and **M** is a properly immersed **H**-surface. Then:
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Theorem (Adams-Meeks-Ramos(2018))

- Let **H** ≥ 0 and **M** be a connected noncompact surface of finite topology and negative Euler characteristic.
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- There does **NOT** exist a complete hyperbolic **3**-manifold of finite volume containing a proper embedding of **M** with constant mean curvature **H** ≥ 1.

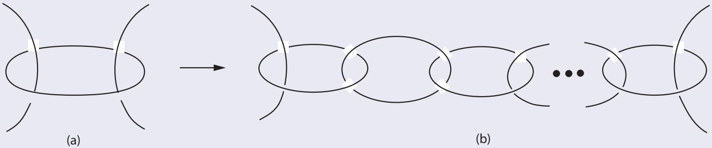

Figure: Replacing (a) with (b) preserves hyperbolicity of the complement.

Theorem (The Chain Lemma)

Let L be a link in a 3-manifold $\mathbf{M}$ such that the link complement $\mathbf{M} \setminus L$ admits a complete hyperbolic metric of finite volume. Suppose that there is a sphere $\mathcal{S}$ in $\mathbf{M}$ bounding a ball $\mathcal{B}$ that intersects L as in Figure 2 (a). Let L' be the resulting link obtained by replacing $L \cap \mathcal{B}$ by the components as appear in Figure 2 (b). Then $\mathbf{M} \setminus L'$ admits a complete hyperbolic metric of finite volume.

Theorem (The Switch Move Lemma)

Let L be a link in a 3-manifold $\mathbf{M}$ such that $\mathbf{M} \setminus L$ admits a complete hyperbolic metric of finite volume. Let $\alpha \subset M$ be the closure in $\mathbf{M}$ of a complete, properly embedded geodesic of $\mathbf{M} \setminus L$ with distinct endpoints on L. Let $\mathcal{B}$ be a closed ball in $\mathbf{M}$ containing α in its interior and such that $\mathcal{B} \cap L$ is composed of two arcs in L, as in Figure 3. Let L_1 be the resulting link in $\mathbf{M}$ obtained by replacing $L \cap \mathcal{B}$ by the components as appearing in Figure 4 (b). Then $\mathbf{M} \setminus L_1$ admits a complete hyperbolic metric of finite volume.

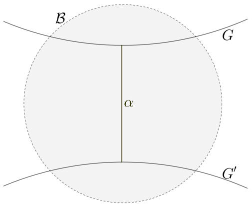

Figure: The trace of a geodesic α of $(\mathbf{M} \setminus L, h)$ joins distinct components G, G' of L, and a neighborhood $\mathcal{B}$ of α intersects L in two arcs $g \subset G$ and $g' \subset G'$.