Edge behavior of deformed Wigner matrices

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Sept. 25, 2013

Wigner matrix

Definition

A Hermitian Wigner matrix of size N is a Hermitian random matrix

$$W = (w_{ij}) = \frac{1}{\sqrt{N}}(x_{ij}) \,, \qquad (1 \le i, j \le N) \,,$$

whose entries (x_{ij}) are complex random variables, independent up to the constraint $x_{ij} = \overline{x}_{ji}$, such that $\mathbb{E}\, w_{ij} = 0$, and

$$\mathbb{E}|w_{ij}|^2 = \frac{1}{N} \,, \qquad \mathbb{E}\, w_{ij}^2 = \begin{cases} \frac{1}{N} \,, & i = j \,, \\ 0 \,, & i \neq j \,. \end{cases}$$

For simplicity, assume subexponential decay of the matrix entries,

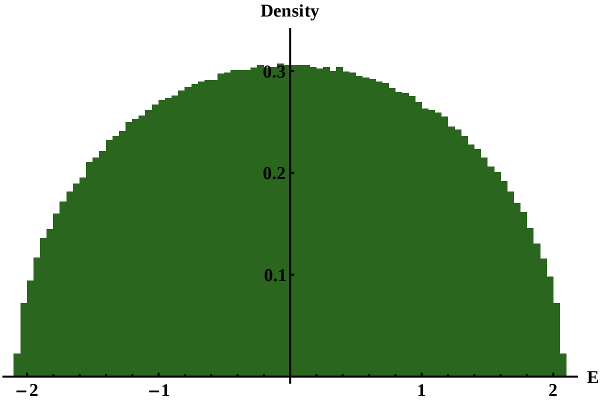
$$\mathbb{P}\left(|x_{ij}| > s\right) \le C_0 \mathrm{e}^{-s^{\vartheta}} \,,$$

for some constants C_0 and $\vartheta > 0$.

Special case: Gaussian unitary ensemble, c.f., Bourgade's talk.

Then the eigenvalues of W follow the semicircle law, Wigner [1955]:

Wigner's Semicircle law



Histogram of the eigenvalues of a $N = 5000$ Hermitian Wigner matrix with complex Gaussian entries

Deformed Wigner matrix

- Let $V = \operatorname{diag}(v_i)$ be an $N \times N$ diagonal random matrix whose entries are real, centered, i.i.d. random variables.
- Assume that the distribution of (v_i) is

$$\mu(v) = Z^{-1}(1+v)^{\mathrm{b}}(1-v)^{\mathrm{b}} f(v)\chi_{[-1,1]}(v)\,,$$

where

$$-1 < \mathrm{b} < \infty\,, \qquad f \in C^1\,, \quad \text{with } f(v) > 0\,, \qquad Z \text{ is a normalization}\,.$$

In particular: $\mathbb{E}v_i = 0$ and $\mathbb{E}v_i^2 = O(1)$.

Deformed Wigner matrix / Wigner matrix with random potential

For $\lambda \in \mathbb{R}^+$, set

$$H = (h_{ij}) := \lambda V + W\,,$$

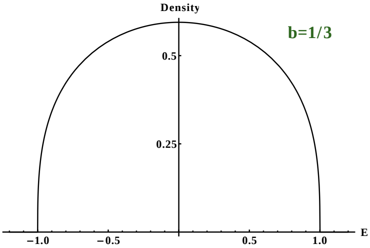
and assume that V and W are independent.

- Note $\lambda = O(N^0)$, so that the eigenvalues of λV and W are of the same order.
- Then the eigenvalues follow the deformed semicircle law, μ_{fc}, Pastur [1972]:

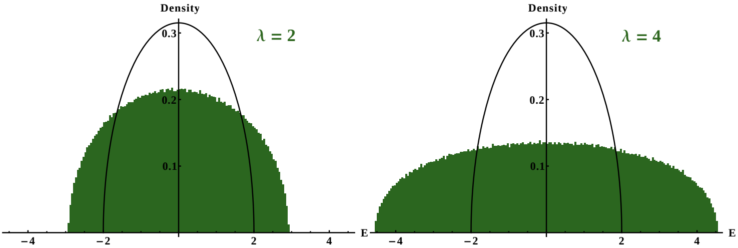
Deformed semicircle law I

Example 1: (v_i) are distributed according to

$$\mu(v) = Z^{-1}(1-v)^{\frac{1}{3}}(1+v)^{\frac{1}{3}}\chi_{[-1,1]}(v) \ :$$



Distribution of (v_i)



Histogram of eigenvalues of a $N = 5000$ deformed Wigner matrix with $\lambda = 2$, respectively $\lambda = 4$.

Stieltjes transform & deformed semicircle law I

- Stieltjes transform of a measure ω,

$$m_\omega(z) := \int_{\mathbb{R}} \frac{\mathrm{d}\omega(x)}{x - z}\,, \qquad z = E + \mathrm{i}\eta \in \mathbb{C}^+\,.$$

- Inversion formula (for abs. continuous ω)

$$\omega(E) = \lim_{\eta \searrow 0} \frac{1}{\pi} \mathrm{Im}\, m_\omega(E + \mathrm{i}\eta)\,.$$

For example: $\mathrm{d}\mu_{sc}(x) := \frac{1}{2\pi}\sqrt{4 - x^2}\chi_{[-2,2]}(x)\mathrm{d}x$,

$$m_{\mu_{sc}}(z) = -\frac{1}{z + m_{\mu_{sc}}(z)}\,, \qquad z \in \mathbb{C}^+\,,$$

with $\mathrm{Im}\, m_{\mu_{sc}}(z) \geq 0$, $\eta > 0$.

Stieltjes transform & deformed semicircle law II

$$m_\omega(z) := \int_{\mathbb{R}} \frac{\mathrm{d}\omega(x)}{x-z}\,, \qquad m_{\mu_{sc}}(z) = -\frac{1}{z + m_{\mu_{sc}}(z)}\,, \qquad z \in \mathbb{C}^+\,.$$

- The Stieltjes transform of the deformed semicircle law, $m_{\mu_{fc}}$, satisfies

  $$m_{\mu_{fc}}(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(v)}{\lambda v - z - m_{\mu_{fc}}(z)}\,, \qquad \text{(Pastur relation)}$$

  and $\operatorname{Im} m_{\mu_{fc}}(z) \geq 0$, $\eta > 0$.
- The deformed semicircle law, μ_{fc}, is then obtained through the inversion formula (μ_{fc} is abs. continuous).
- Alternative definition, additive free convolution $\mu_{fc} = \mu \boxplus \mu_{sc}$, c.f., free probability theory, Voiculescu,...[1985-...].
- For the special choice of μ above, μ_{fc} is supported on a single interval, $\operatorname{supp} \mu_{fc} = [L_-, L_+]$.

Eigenvector behavior I

Denote by (λ_α) the eigenvalues (with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$), by (u_α) the associated (ℓ^2)-normalized eigenvectors and by $(u_\alpha(k))$ the components of the eigenvectors of H. Then:

- $\lambda = 0$ (Wigner matrix),

$$|u_\alpha(k)| \lesssim N^{-1/2}, \qquad (1 \leq \alpha \leq N, 1 \leq k \leq N).$$

  All eigenvectors are completely delocalized.
  Erdős-Schlein-Yau,...[2009-2012].

- $\lambda \neq 0$, $\mathbf{b} < 1$,

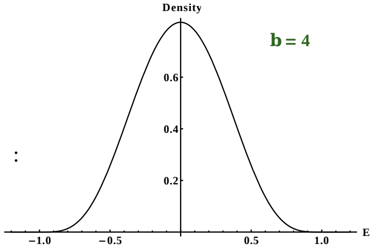
$$|u_\alpha(k)| \lesssim N^{-1/2}, \qquad (1 \leq \alpha \leq N, 1 \leq k \leq N),$$

  for all finite λ. All eigenvectors are completely delocalized.
  Lee-S. [2013]

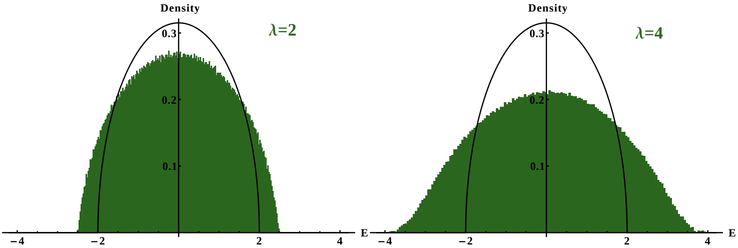
Deformed semicircle law II

Example 2: (v_i) are distributed according to

$$\mu(v) = Z^{-1}(1-v)^4(1+v)^4\chi_{[-1,1]}(v) \ :$$



Distribution of (v_i)



Histograms of eigenvalues of a $N = 5000$ deformed Wigner matrix with $\lambda = 2$, respectively $\lambda = 4$.

Eigenvector behavior II

For $\mathrm{b} > 1$, there is a constant λ_+, such that we have

$$\mu_{fc}(E) \sim \begin{cases} \sqrt{\kappa_E}\,, & \text{if } \lambda < \lambda_+\,, \\ (\kappa_E)^{\mathrm{b}}, & \text{if } \lambda > \lambda_+\,, \end{cases} \qquad E \geq 0\,,$$

where κ_E denotes the distance from E to the upper endpoint of the support of μ_{fc}. (A similar statement holds for $E \leq 0$).

Wlog assume that $v_1 \geq v_2 \geq \ldots \geq v_N$.

- For $\lambda < \lambda_+$, all eigenvectors are completely delocalized:

$$|u_\alpha(k)|^2 \lesssim N^{-1}\,, \qquad (1 \leq \alpha \leq N\,, 1 \leq k \leq N)\,.$$

- For $\lambda > \lambda_+$, the eigenvectors in the bulk are completely delocalized; at the extreme edge they are 'partially localized', i.e.,

$$|u_\alpha(\alpha)|^2 = \frac{\lambda^2 - \lambda_+^2}{\lambda^2} + o(1)\,,$$
$$|u_\alpha(k)|^2 \lesssim \frac{1}{N}\frac{1}{\lambda^2|v_\alpha - v_k|^2}\,, \qquad (\alpha \neq k\,, 1 \leq k \leq N)\,,$$

where $\alpha \leq n_0$, for some fixed n_0. (Similar statement holds for the lower edge)
Lee-S.-Yau

Fluctuations of the largest eigenvalue

The largest eigenvalue λ_1 of H approaches L_+, as $N \to \infty$, where $\operatorname{supp} \mu_{fc} = [L_-, L_+]$.

Fluctuations at the (upper) edge: $\lambda = O(1)$

- In the delocalized regime,

$$\lim_{N\to\infty} \mathbb{P}\left(N^{1/2}(L_+ - \lambda_1) \le s\right) = \Phi_a(s)\,, \qquad s \in \mathbb{R}\,,$$

  where Φ_a is the CDF of centered Gaussian of variance $a = a(\mu, \lambda)$;
- In the 'partially localized' regime,

$$\lim_{N\to\infty} \mathbb{P}\left(N^{1/(\mathrm{b}+1)}(L_+ - \lambda_1) \le s\right) = G_{\mathrm{b}+1}(s)\,, \qquad s \in \mathbb{R}\,,$$

  where

$$G_{\mathrm{b}+1}(s) = \left(1 - \mathrm{e}^{-(\frac{s}{c})^{\mathrm{b}+1}}\right) \chi_{[0,\infty)}(s)$$

  is the CDF of a Weibull distribution with parameters $\mathrm{b}+1$ and $c = c(\mu, \lambda)$.

Lee-S.-Yau

From Tracy-Widom to Gaussian: $\lambda = o(N^0)$

- For $\lambda = 0$ (Wigner matrix),

$$\lim_{N\to\infty} \mathbb{P}\left(N^{2/3}(\lambda_1 - 2) \leq s\right) = \exp\left(-\int_s^\infty (x-s)q(x)^2 \mathrm{d}x\right) =: F_2(s)\,, \tag{1}$$

  where q satisfies

$$q'' = xq + 2q^3\,, \qquad q(x) \sim \mathrm{Ai}(x)\,, \quad \text{as} \quad x \to \infty\,.$$

  GUE: Tracy-Widom [1994-1996], Wigner: Soshnikov [1998], Erdős-Yau-Yin,... [2012], Lee-Yin [2013]

- For $\lambda \neq 0$, in case W is a GUE matrix, it is known that the Tracy-Widom law (1) holds true for $\lambda \ll N^{-1/6}$, Johansson [2007], T. Shcherbina [2011], and that the transition to Gaussian fluctuations occurs at $\lambda \sim N^{-1/6}$, Johansson [2007].