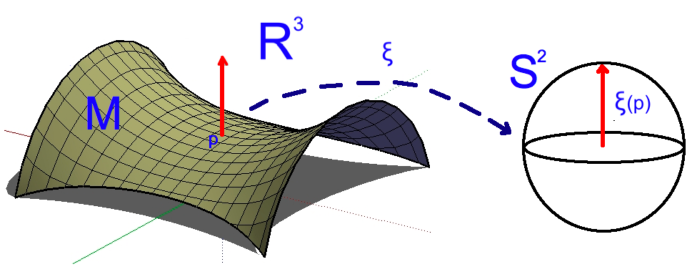
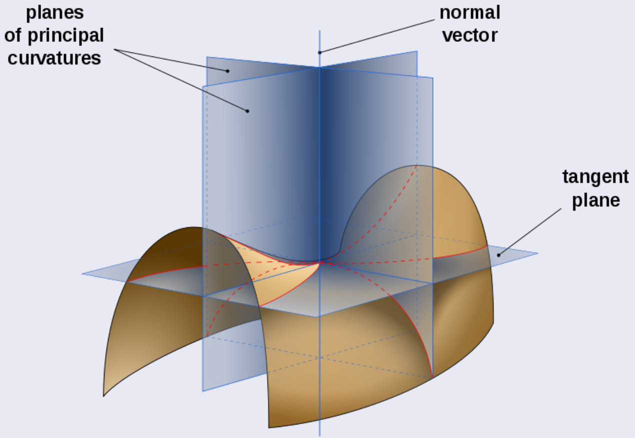


- Let **M** be an oriented surface in $\mathbf{R}^3$, let ξ be the unit vector field normal to **M**:

$$\mathbf{A_p} = -\mathbf{d}\xi_{\mathbf{p}}\colon T_{\mathbf{p}}\mathbf{M} \to T_{\xi(\mathbf{p})}\mathbf{S}^2 \simeq T_p\mathbf{M}$$

  is the **shape operator** of **M**. Here $\mathbf{d}\xi_{\mathbf{p}}$ is the derivative map.

- The map $\xi\colon \mathbf{M} \to \mathbf{S}^2$ is called the **Gauss map** of **M** at **p**.



- The geometry of eigenvalues of the symmetric matrix for

$$\mathbf{A}_\mathbf{p} = -\mathbf{d}\xi_\mathbf{p} \colon T_\mathbf{p}\mathbf{M} \to T_{\xi(\mathbf{p})}\mathbf{S}^2 \simeq T_p\mathbf{M}$$

- **Principal curvatures** $k_1(\mathbf{p}), k_2(\mathbf{p})$ at $\mathbf{p}$ are the eigenvalues of $\mathbf{A}_\mathbf{p}$.

Definition

- $\mathbf{K} = \det(\mathbf{A}) = k_1 k_2 =$ **Gauss curvature** function.
- $\mathbf{H} = \frac{1}{2}\mathrm{tr}(\mathbf{A}) = \frac{k_1+k_2}{2} =$ **mean curvature** function.
- $|\mathbf{A}| = \sqrt{k_1^2 + k_2^2} =$ **norm of 2nd fundamental form (shape op).**

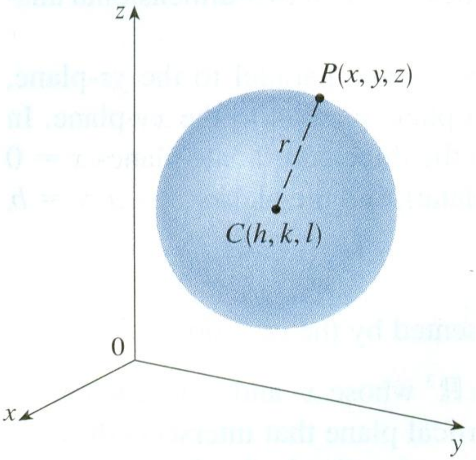
Gauss equation

$$4\mathbf{H}^2 = |\mathbf{A}|^2 + 2\mathbf{K} \quad (\mathbf{K} = \text{Gaussian curvature})$$

So, when $\mathbf{H}(\mathbf{p}) = 0$, then $\mathbf{K}(\mathbf{p}) \leq 0$.

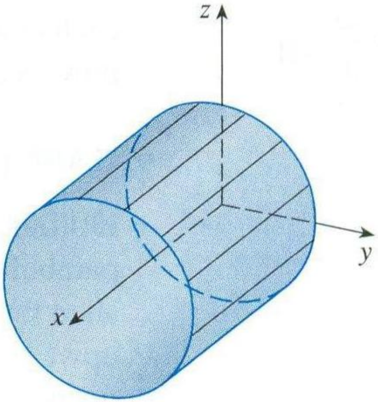
Example

The sphere **S** in $\mathbf{R}^3$ with center $C(h, k, l)$ and radius **r** has constant mean curvature $\mathbf{H} = \dfrac{\mathbf{1}}{\mathbf{r}}$.



Example

- The infinite cylinder $\mathbf{C}$ in $\mathbf{R}^3$ in the image below of radius $\frac{1}{2}$ is an example of a complete, properly embedded planar domain of finite topology and mean curvature $\mathbf{H} = 1$.



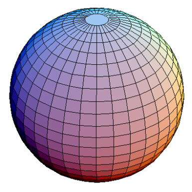
Introduction to the theory of CMC surfaces.

Definition 1

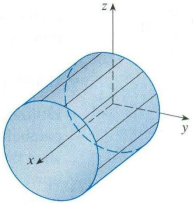
M is an **H-surface** means that it has constant mean curvature **H**.

Definition 2

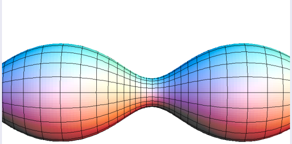
M is an **H-surface** $\iff$ **M** is a critical point for the area functional under compactly supported variations **preserving the volume**.



- Sphere



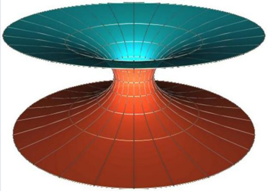
- Cylinder



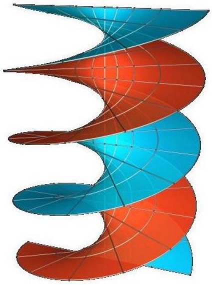
- Delaunay surfaces

Definition 3

An **H**-surface **M** is a **minimal surface** $\iff$ **H** $\equiv 0$ $\iff$ **M** is a critical point for the area functional under compactly supported variations.



- Catenoid



- Helicoid

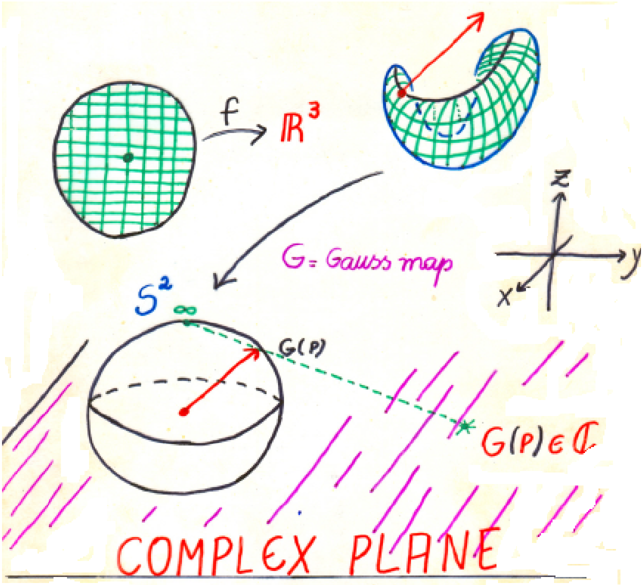
Definition of minimal surface

Definition (Minimal Surface)

A surface $\mathbf{f} : \mathbf{M} \to \mathbf{R}^3$ is **minimal** if:

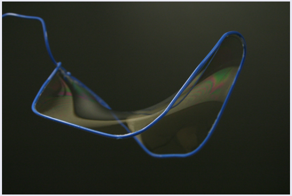
- $\mathbf{M}$ has **MEAN CURVATURE** $\mathbf{H} = \mathbf{0}$.
- Small pieces have **LEAST AREA.**
- Small pieces have **LEAST ENERGY.**
- Small pieces occur as **SOAP FILMS.**
- Coordinate functions are **HARMONIC.**
- Conformal Gauss map $\mathbf{G} : \mathbf{M} \to \mathbf{S}^2 = \mathbf{C} \cup \{\infty\}$.
  (MEROMORPHIC GAUSS MAP)

Meromorphic Gauss map



CMC surfaces in nature.

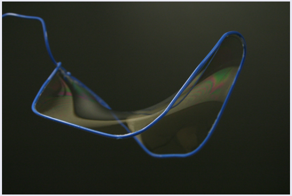
Soap films are minimal surfaces.



Soap bubbles are nonzero **H**-surfaces.

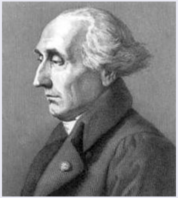
From Soap Films to Minimal Surfaces
– The theory of minimal surfaces

Soap films are minimal surfaces

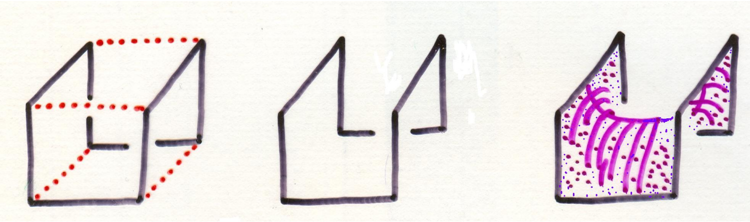


Joseph Plateau (1801 - 1883)

Joseph Lagrange (1736 - 1813)



- **Plateau** proved that a soap film minimizes area **among nearby surfaces**. (Surface tension is at work.)



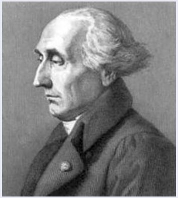
Example (Plateau's soap film characterization of minimal surfaces, $\mathbf{H} = \mathbf{0}$)

- Soap film surfaces bounding a closed wire shape **Γ** tend to a stable physical surface **M** which **locally minimizes energy**.
- At a point **p** of the stable soap film **M**, the **surface tension** breaks up into 2 balanced forces pushing the surface with equal magnitudes in opposite normal directions.
- These 2 force vectors are proportional to the curvature vectors of the trace curves in the 2 principal planes P_1, P_2 orthogonal to **M** at **p**.
- Thus, the **mean curvature H of M vanishes!**

From Soap Films to Minimal Surfaces
– Plateau Problem

Joseph Plateau (1801 - 1883)

Joseph Lagrange (1736 - 1813)



- **Given a simple closed curve in $\mathbb{R}^3$, does there exist a minimal surface with the topology of a disk of least area spanning it?**
- This became known as the **Plateau Problem**.

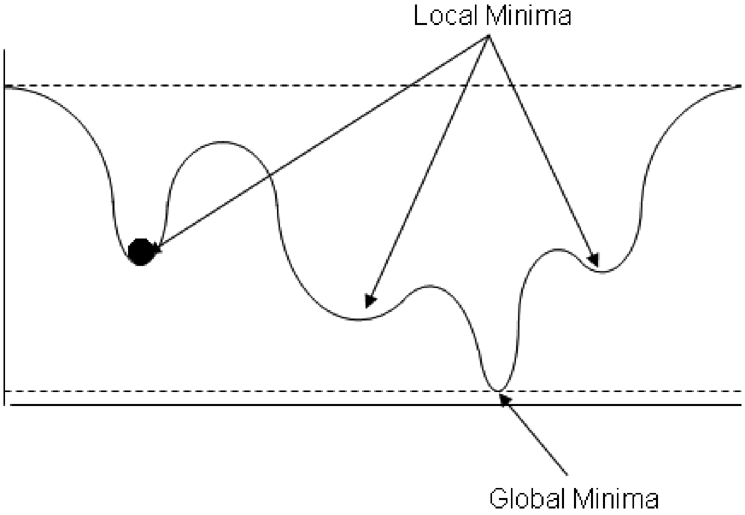
From Soap Films to Minimal Surfaces
– Plateau Problem

In 1930, it was solved independently by

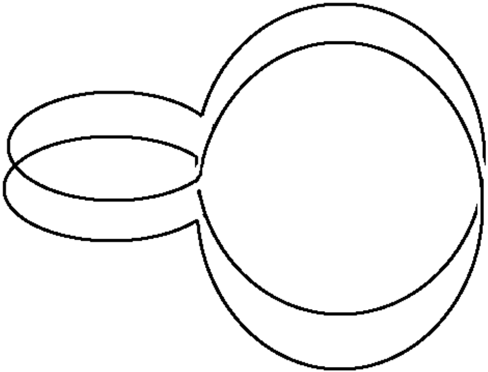
Jesse Douglas
(1897 - 1965)
Fields Medal in 1936

Tibor Rado
(1895 - 1965)

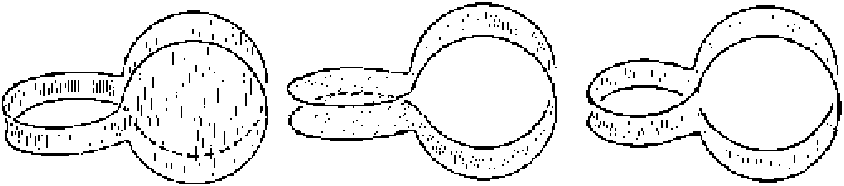
From Soap Films to Minimal Surfaces
– Non-uniqueness of minimal surfaces



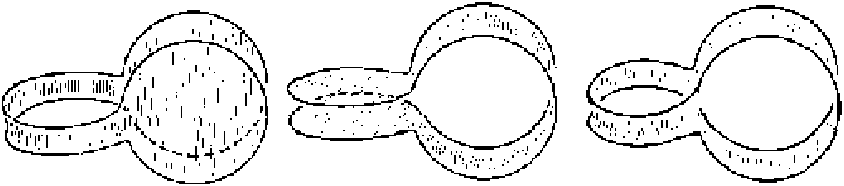
Here is a wire that bounds more than one soap film.



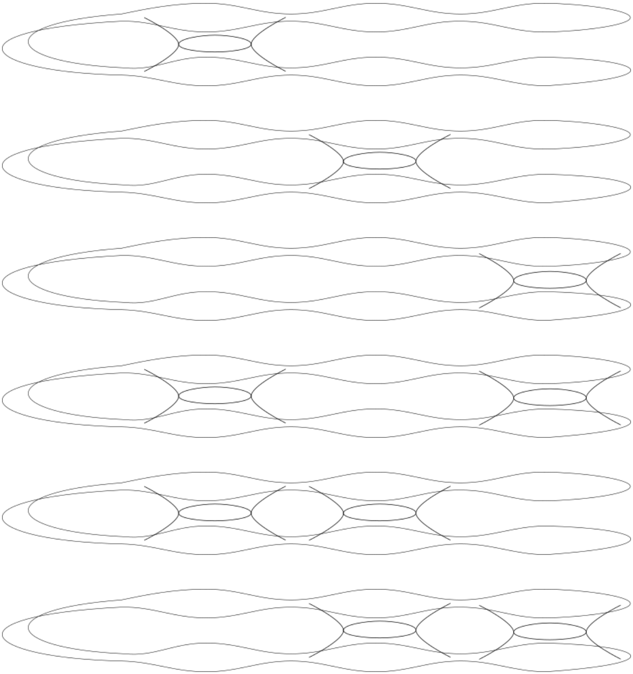
From Soap Films to Minimal Surfaces
– Non-uniqueness of minimal surfaces



From Soap Films to Minimal Surfaces
– Non-uniqueness of minimal surfaces



- Thurston described similar **extremal** simple closed curves $\Gamma_{\mathbf{n}}$ with total curvature $4\pi + 1/\mathbf{n}$ that are the boundary of at least $\mathbf{n}$ embedded soap films of genus $\mathbf{k}$ for every $1 < \mathbf{k} < \mathbf{n}$.
- For $\mathbf{n} = 3$ see the following figures.



The uniqueness and embeddedness of certain minimal surfaces

Theorem (Meeks-Yau 1980)

Let $\Gamma \subset \mathbf{R}^3$ be a simple closed extremal curve. Then:

- If the total curvature of Γ is $\leq 4\pi$, then Γ bounds a **unique** compact (branched) minimal surface.
- Every least-area disk with boundary Γ is **embedded**. Furthermore any two such least-area disks are **disjoint** in their interiors.

See the blackboard for a description of the **free boundary value problem**. The solid torus $\mathbf{M}$ in the drawing is assumed to have mean curvature function $\mathbf{H_M} \geq 0$.

The embedded solutions to the free boundary value problem

Theorem (Geometric Loop Theorem, Meeks-Yau 1980)

Let $\mathbf{M}$ be a Riemannian 3-manifold with mean convex boundary and suppose that there exists an nontrivial loop in $\partial\mathbf{M}$ that is trivial in $\mathbf{M}$. Then:

- There exists an immersed disk $f\colon (D, \partial D) \to (\mathbf{M}, \partial\mathbf{M})$ of least-area s.t. ∂D is nontrivial in $\partial\mathbf{M}$.
- Solutions to this free boundary value problem form a **pairwise disjoint** collection $\mathcal{D}$ of **embedded** disks.

The embedded solutions to the free boundary value problem

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- Solutions to this free boundary value problem form a **pairwise disjoint** collection $\mathcal{D}$ of **embedded** disks.

Note that if $g\colon \mathbf{M} \to \mathbf{M}$ is an isometry, then $\mathcal{D}$ is invariant under g.

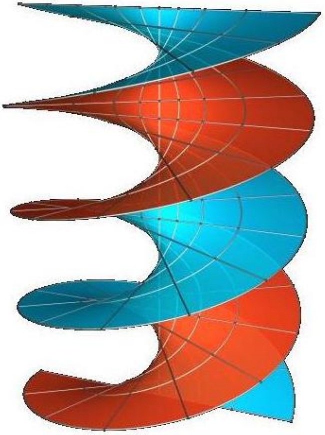
Application: Classification of finite subgroups of Diff($\mathbb{R}^3$)

- Around 1976 **Thurston** reduced the Smith Conjecture to the "Equivariant Loop Theorem", which is an immediate consequence of the Geometric Loop Theorem.
- In 1983 **Meeks-Yau** used the solution to Smith Conjecture together with their Geometric Sphere Theorem to reduce the generalized Smith Conjecture to understanding group actions on a compact ball.

Theorem (Generalized Smith Conjecture: Meeks-Yau, Thurston)

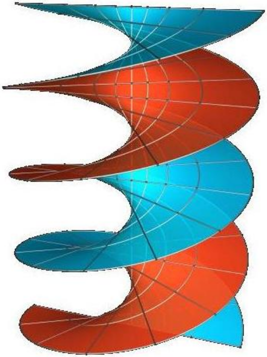
A finite subgroup of Diff($\mathbf{R}^3$) is conjugate to a subgroup of the orthogonal group O(3) in Diff($\mathbf{R}^3$).

Helicoid - 2 double infinite staircases glued together



From Soap Films to Minimal Surfaces
– The Helicoid and its applications

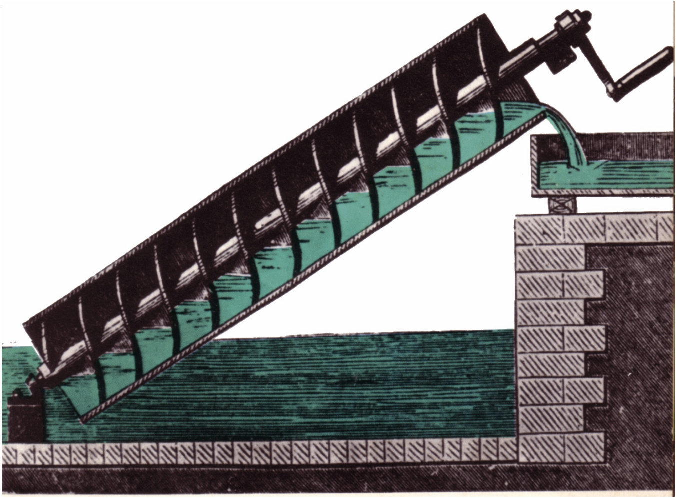
Other uses of the helicoid:



The Chateau of Chambord in the Valley of the Loire.

From Soap Films to Minimal Surfaces
– The Helicoid and its applications

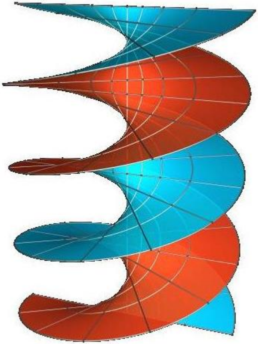
Shape used by **Archimedes** to pump water in 250 BC.



From Soap Films to Minimal Surfaces
– The Helicoid and its applications

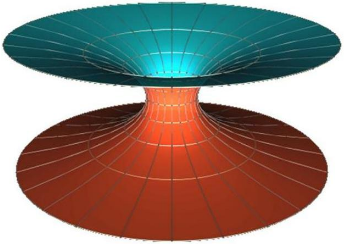
Proof that the Helicoid is a minimal surface

From Soap Films to Minimal Surfaces

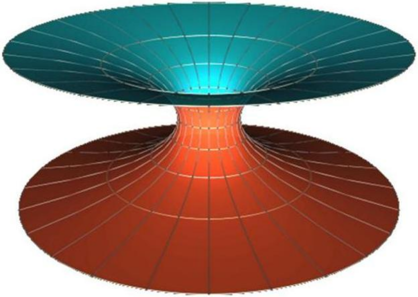


- Proved to be a minimal surface by **Meusnier** in 1776.
- Together with the plane, the helicoid is the only ruled minimal surface (proved by **Catalan** in 1842).
- The plane and the helicoid are the only complete simply connected minimal surfaces embedded in $\mathbf{R}^3$(**2005 - Meeks-Rosenberg, Colding-Minicozzi**).

Catenoid - the case of 2 concentric circles in parallel planes

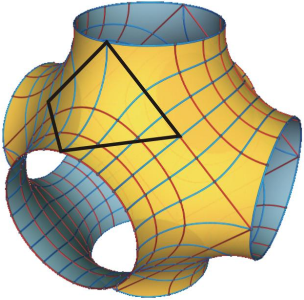


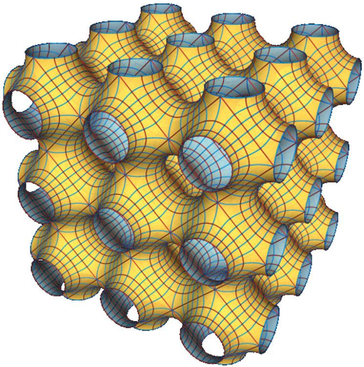
- In 1741, **Euler** discovered that when a small arc on the catenary $x_1 = \cosh x_3$ is rotated around the x_3-axis, then one obtains a surface which minimizes area among surfaces of revolution after prescribing boundary values for the generating curves.



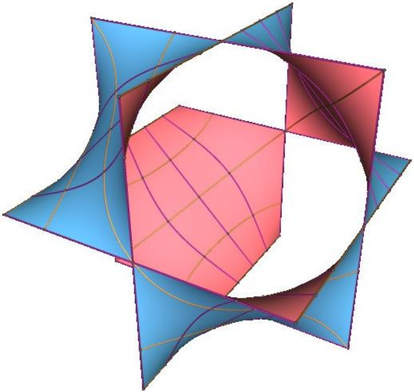
- The catenoid is the **unique** complete embedded minimal surface with finite topology and two ends (**Schoen 1983**) or of finite topology and genus zero (**Lopez-Ros 1991**).
- These characterizations also depend on more recent work of **Colding-Minicozzi 2008 and Collin 1997**.

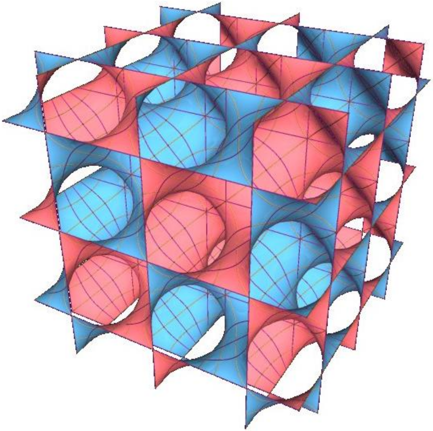
From Soap Films to Minimal Surfaces





- Discovered by **Schwarz** in the 1880's, it is also called the **P**-surface.
- It contains many infinite straight lines and by Schwartz reflection these are symmetry lines of the **P**-surface.



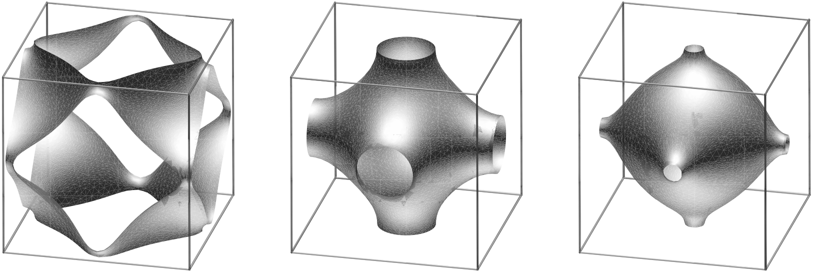


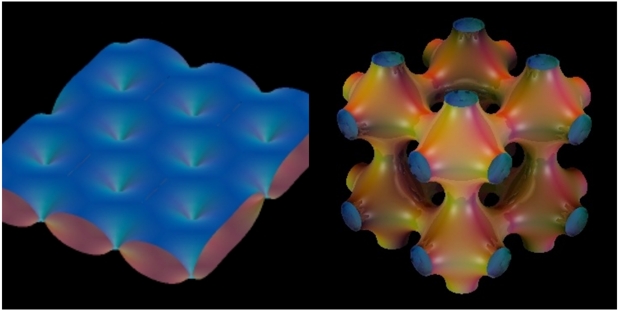
Discovered by **Schwarz**, it is the conjugate surface to the **P**-surface, and is another famous example of an embedded **TPMS**.

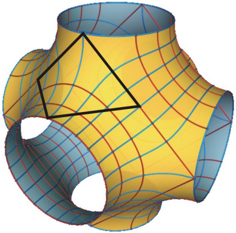
1970 - Alan Schoen's Gyroid surface.

- Discovered by **Schoen** while working for NASA, it is an associate surface to the **P**-surface, and is another famous example of an embedded **TPMS**.
- **Ross 1992** proved that the P, Diamond and Gyroid surfaces are stable with respect to volume preserving variations.

Closed H-surfaces in a flat 3-torus. By K. Grosse-Brauckmann (top) and N. Schmitt (bottom)







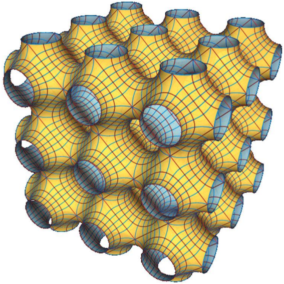


Figure: A body-centered cubic interface or Fermi surface in salt crystal.

Next theorem is motivated by the study of **3**-periodic **H**-surfaces that appear as interfaces in material science or as equipotential surfaces in crystals. This result contrasts with the failure of area estimates for closed minimal surfaces of genus $\mathbf{g} > 2$ in any flat **3**-torus (**Traizet**).

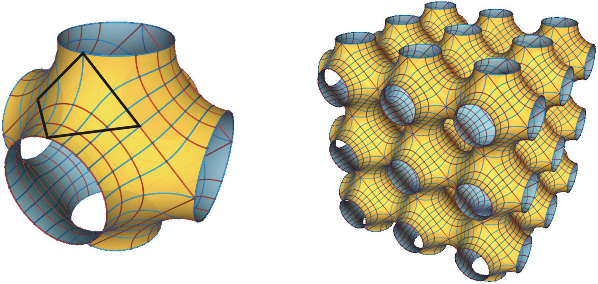


Figure: A body-centered cubic interface or Fermi surface in salt crystal.

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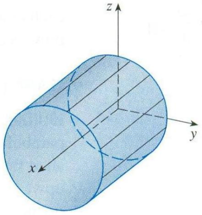
Theorem (2017, Meeks-Tinaglia)

Given a flat **3**-torus $\mathbb{T}^3$, $\mathbf{H} > 0$ and $\mathbf{g} \in \mathbb{N}$, $\exists \mathbf{C_{H,g}}$ s.t., a closed **H**-surface $\mathbf{\Sigma}$ embedded in $\mathbb{T}^3$ with genus at most **g** satisfies $\mathbf{Area}(\mathbf{\Sigma}) \leq \mathbf{C_{H,g}}$.

Question

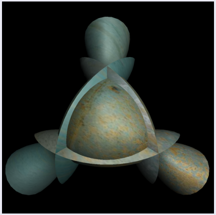
Is the round sphere the only complete simply connected surface **embedded** in $\mathbf{R}^3$ with **non-zero** constant mean curvature?

NOT simply connected



- Cylinder

NOT embedded



- Smyth surface conformally $\mathbb{C}$

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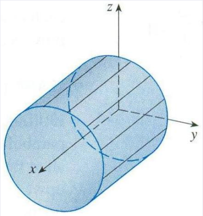

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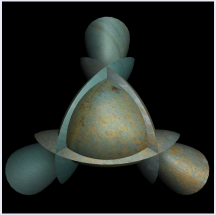

- Smyth surface conformally $\mathbb{C}$

Answer (Meeks-Tinaglia)

Yes!

Theorem (2016 - Meeks-Tinaglia)

Round spheres are the only complete simply connected surfaces **embedded** in $\mathbf{R}^3$ with non-zero constant mean curvature.

- 1950 - **Hopf** proved that an immersed **H**-sphere is a **round** sphere.
- 1986 - Above result proved by **Meeks** for **properly embedded**.

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- Earlier results of **Colding-Minicozzi** and **Meeks-Rosenberg** imply the plane and helicoid are the only complete embedded simply connected minimal surfaces in $\mathbf{R}^3$.

New uniqueness results for CMC surfaces.

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- Earlier results of **Colding-Minicozzi** and **Meeks-Rosenberg** imply the plane and helicoid are the only complete embedded simply connected minimal surfaces in $\mathbf{R}^3$.
- So if **M** is a complete, simply connected **H**-surface **embedded** in $\mathbf{R}^3$, then **M** is either

  a plane, **a helicoid** or **a round sphere**.

Theorem (2016 - Radius Estimates for H-Disks, Meeks-Tinaglia)

$\exists$ $\mathbf{R_0} \geq \pi$ such that every embedded **H**-disk in $\mathbf{R^3}$ has radius $< \mathbf{R_0}/\mathbf{H}$.

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Corollary (2016 - Meeks-Tinaglia)

A complete simply connected **H**-surface embedded in $\mathbf{R^3}$ with $\mathbf{H} > 0$ is a round sphere.

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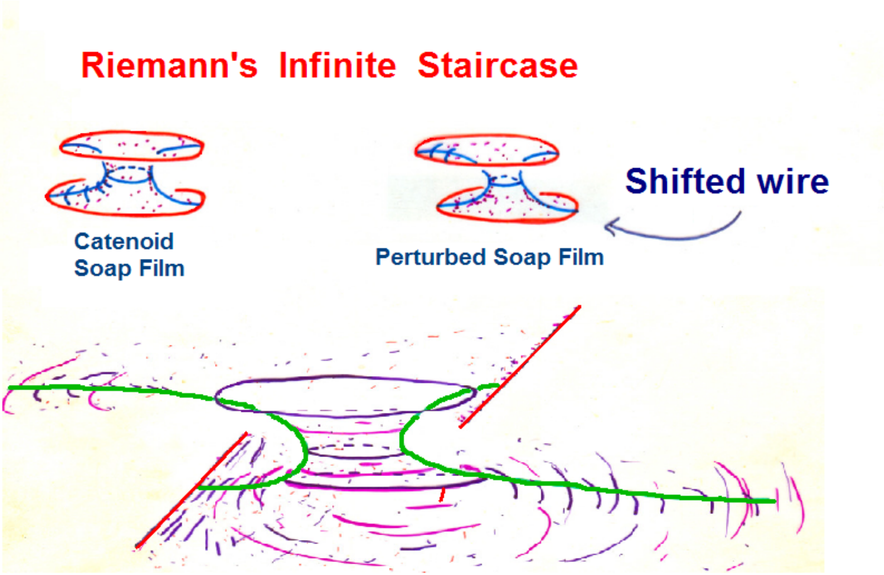
The next classification result depends on previous work of **Colding-Minicozzi**, **Collin** and **Korevaar-Kusner-Solomon**.

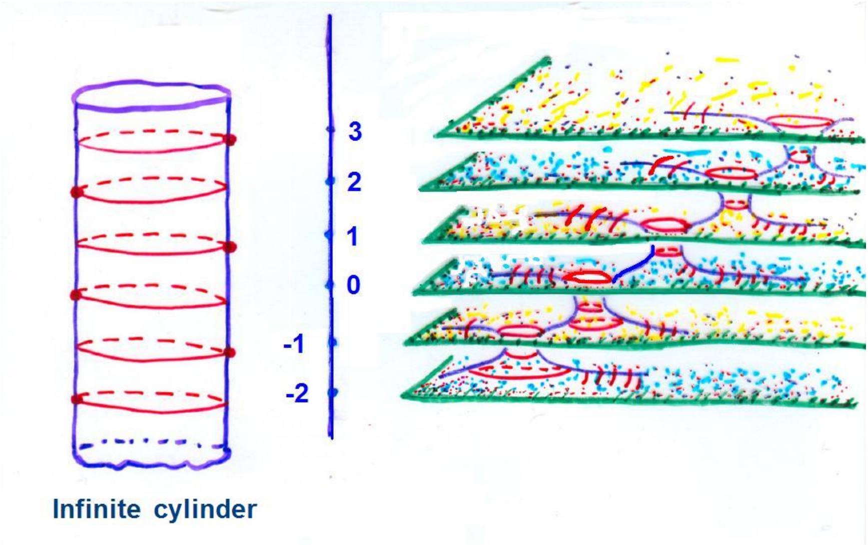
Theorem (2016 - Meeks-Tinaglia)

A complete annulus **embedded** in $\mathbf{R}^3$ with constant mean curvature is a **catenoid** or a surface of revolution discovered by **Delaunay** in 1841.

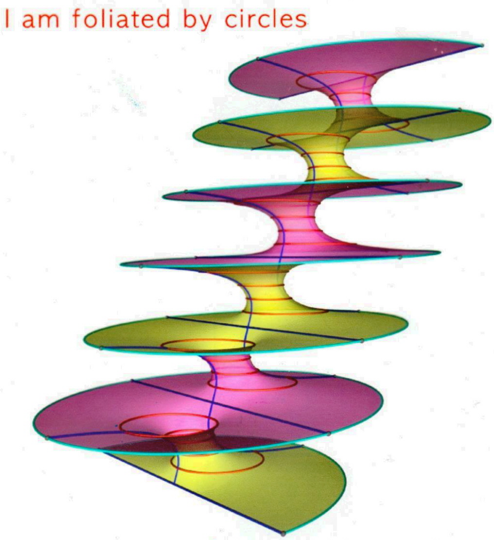
The family $\mathcal{R}_t$ of Riemann minimal examples - 1860



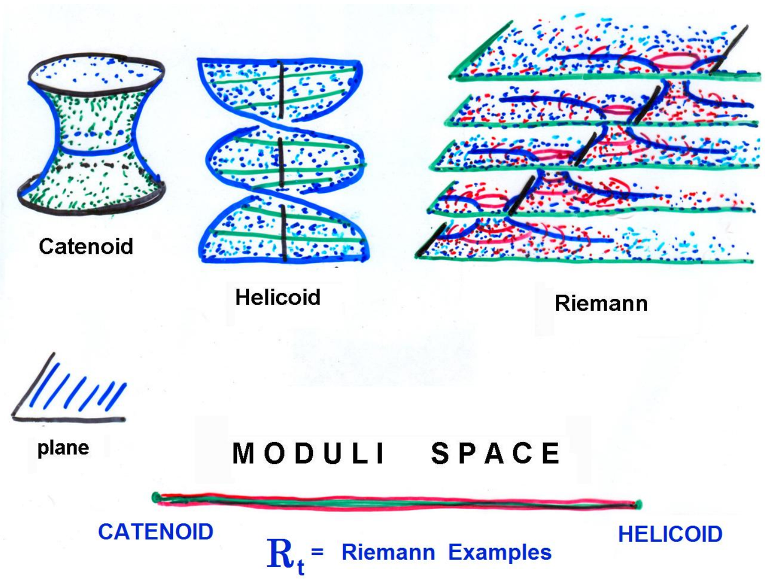
Cylindrical parametrization of a Riemann minimal example



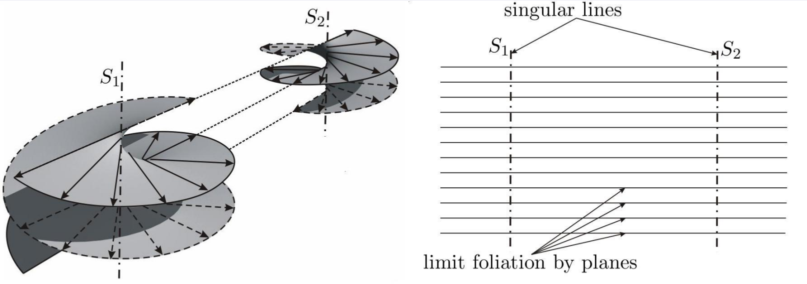
1860 Riemann's discovery! Image by Matthias Weber



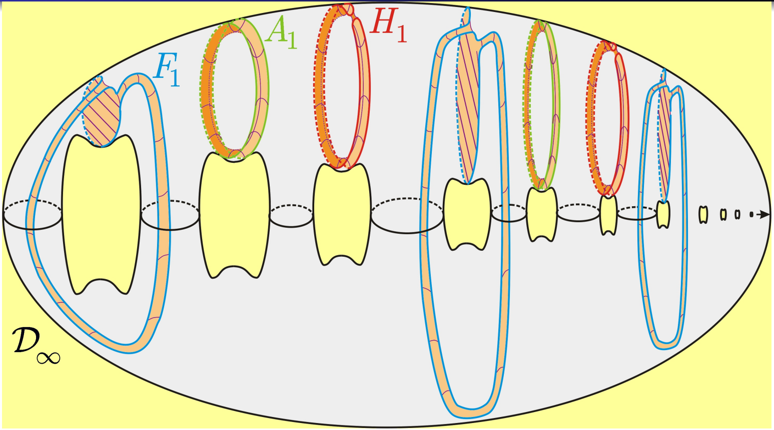
Properly embedded genus-0 examples - Collin-Lopez-Meeks-Perez-Ros-Rosenberg



Riemann minimal examples near helicoid limits



- By appropriately scaling, the Riemann examples $\mathcal{R}_t$ converge as $t \to \infty$ to a foliation $\mathcal{F}$ of $\mathbf{R}^3$ by horizontal planes.
- The set of non-smooth convergence $\mathbf{S}(\mathcal{F})$ to $\mathcal{F}$ consists of **2** vertical lines $\mathbf{S_1}, \mathbf{S_2}$ perpendicular to the planes in $\mathcal{F}$.
- This type of limit is called a minimal parking garage structure on $\mathbf{R}^3$ with columns $\mathbf{S_1}, \mathbf{S_2}$.



- $\mathcal{D}_\infty$ = the above **bounded domain, smooth except at $\mathbf{p}_\infty$**.
- **Ferrer, Martin** and **Meeks** conjecture: An open surface with compact boundary **properly embeds as a complete minimal surface in $\mathcal{D}_\infty$** $\iff$ every end has **infinite genus** $\iff$ it admits a complete bounded minimal embedding in $\mathbb{R}^3$.

The embedded Calabi-Yau problem for finite genus

Theorem (2009, Colding-Minicozzi)

A complete minimal surface embedded in $\mathbf{R}^3$ of finite topology is proper.

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Let $\boldsymbol{\Sigma} \subset \mathbf{R}^3$ be a complete embedded minimal surface of finite genus. Then:

$$\boldsymbol{\Sigma} \text{ is proper} \iff \boldsymbol{\Sigma} \text{ has a } \underline{\text{countable}} \; \# \text{ of ends.}$$

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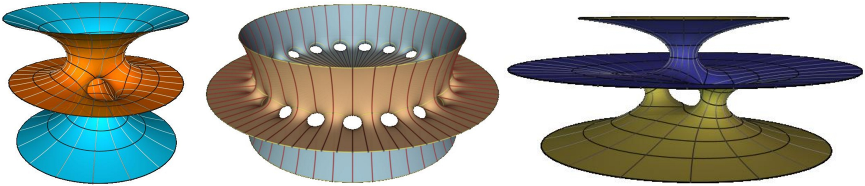
Theorem (2016, Meeks-Tinaglia)

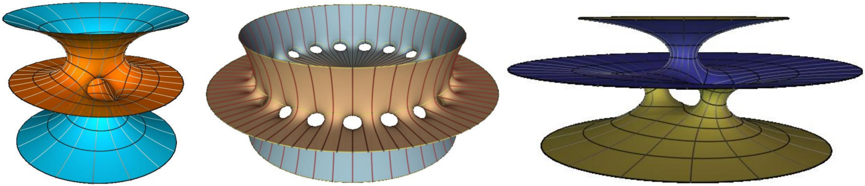
A complete $\mathbf{H}$-surface embedded in $\mathbf{R}^3$ of finite topology is proper.

The Hoffman-Meeks Conjecture Image by Matthias Weber



Conjecture (Hoffman-Meeks Conjecture)

An open surface $\mathbf{\Sigma}$ with finite genus $\mathbf{g}$ and a finite number $\mathbf{e} > 2$ of ends embeds as a complete minimal surface in $\mathbf{R}^{\mathbf{3}}$ $\iff$ $\mathbf{e} \leq \mathbf{g} + 2$.



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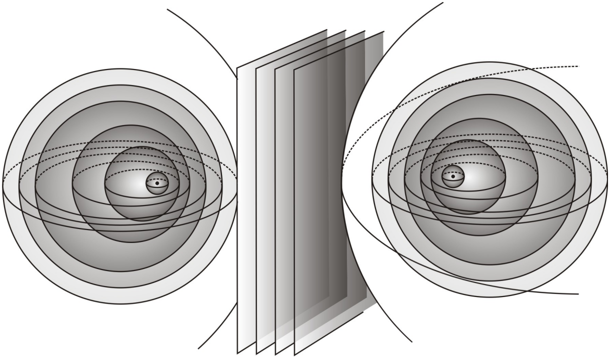
Theorem (2018, Meeks-Perez-Ros)

Given an integer $\mathbf{g} \geq 0$, there exists $\mathbf{C}(\mathbf{g})$ such that the following holds. If $\boldsymbol{\Sigma} \subset \mathbf{R}^3$ be a complete embedded minimal surface of finite genus $\mathbf{g}$ and a finite number $\mathbf{e}$ of ends, then

$$\mathbf{e} \leq \mathbf{C}(\mathbf{g}).$$

CMC foliation of $\mathbf{R}^3$ punctured in two points by spheres and planes



Theorem (Meeks-Perez-Ros)

Suppose $\mathcal{F}$ is a **CMC** foliation of $\mathbf{R}^3 - \mathcal{S}$ where $\mathcal{S}$ is a closed countable set. Then all leaves of $\mathcal{F}$ are contained in planes and round spheres.