# Hamiltonian Dynamics and Morse theory

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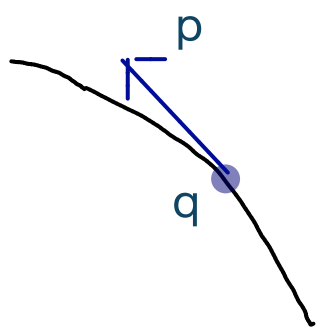
# Classical Hamiltonian mechanics

- Hamiltonian formulation of classical mechanics describes mechanics in terms of position q and momentum p



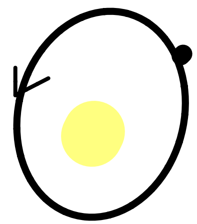

- The Hamiltonian

$$H(t, q, p) = \frac{1}{2m}\|p\|^2 + V(t, q)$$

describes total energy of system, i.e., kinetic plus potential energy.

- Equations of motion are given by

$$\frac{d}{dt}q = \frac{\partial H}{\partial p} \quad \text{and} \quad \frac{d}{dt}p = -\frac{\partial H}{\partial q}.$$



- Typical example: Kepler problem
A planet under influence of gravity of the sun. Here, $V(q) = \frac{1}{\|q\|}$.

# The symplectic setting

- Natural setting for Hamiltonian mechanics: a symplectic manifold.
- In classical mechanics, this is the cotangent bundle of the space of positions (fibers are the momentum coordinates)
- In general, we consider a manifold $M^{2n}$, equipped with a closed, non-degenerate 2-form $\omega$
- Now we can formulate equations of motion:

  For a Hamiltonian $H\colon S^1 \times M \to \mathbb{R}$, define the vector field $X_H$ by

  $$\omega(X_H, \cdot) = -dH.$$

  Then the motion is given by the flow equation

  $$\dot{x}(t) = X_H(x(t)).$$

# Examples

- Classical mechanics on $\mathbb{R}^{2n}$:
  - $\omega$ given by $\omega(v, w) = v^T J w$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
  - The equations of motion are now

$$\dot{x}(t) = X_H(x(t)) = J\nabla H(x(t))$$

  - This reproduces the system of ODEs in classical mechanics

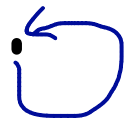
- Geodesic flow on a manifold $B$ is given by the Hamiltonian

$$H(q, p) = \frac{1}{2}\|p\|^2 \text{ on } M = T^*B,$$

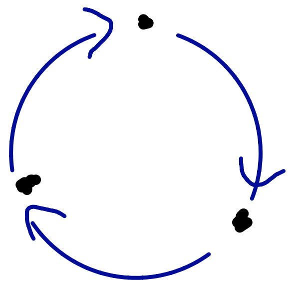
where $q$ position on $B$ and $p$ is the fiber coordinate.

# Number of periodic orbits

- I am mainly interested in periodic orbits of the time-1-map.
- Fixed points
  (one-periodic orbits of the flow)



- Periodic points
  (periodic flow lines with integer period)



- Question: Does the symplectic manifold carry information about the number of periodic orbits?
- Answer: Yes, in many cases even infinitely many periodic orbits are known a priori for all Hamiltonian systems.

# The Conley conjecture

**Theorem**

*On certain symplectic manifolds, every Hamiltonian $H$ has infinitely many periodic orbits.*

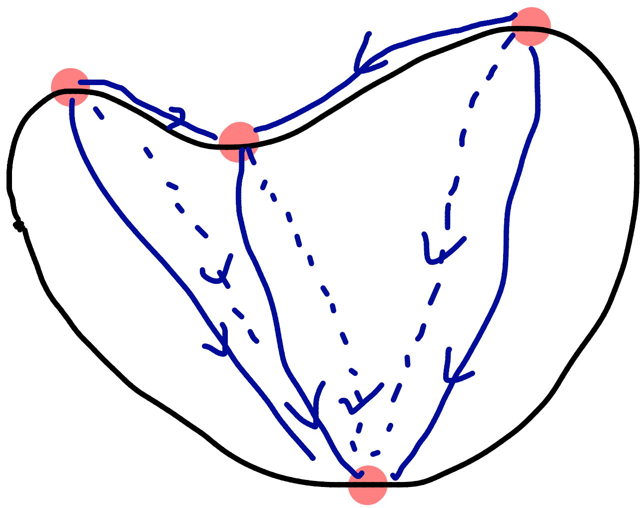
- known for closed manifolds with vanishing first Chern class, on cotangent bundles and $\mathbb{R}^{2n}$ for certain classes of Hamiltonians
- In particular, this theorem is true for classical systems with position $q$ on closed manifolds and momentum $p$ in fibers of the cotangent bundle, i.e. for Hamiltonians of the form $H(t, q, p) = \|p\|^2 + V(t, q)$.

Proofs on different classes of manifolds by Conley-Zehnder, Salamon-Zehnder, Franks-Handel, Hingston, Ginzburg, Ginzburg-Gürel, H., Gürel

- Counter example: rotation on $S^2$ has only two periodic points (the fixed points).

# Morse theory

- Idea: Use gradient flow of a function



Critical points generate chain complex

Gradient flow lines determine the boundary map

- Counting gradient flow lines defines a boundary operator and the resulting homology is the singular homology of the manifold.
- There is also a Morse-theoretic version of the cup product, which can be used to show existence of critical points.

# Critical points of smooth functions

**Definition**

The cuplength of a manifold $Z$ is defined by

$$\mathrm{cuplength}(Z) := \max\{k \in \mathbb{N} \mid \alpha_1 \cup \ldots \cup \alpha_k \neq 0 \text{ for } \alpha_i \in H^{\geq 1}(Z)\}.$$

Then we have the following

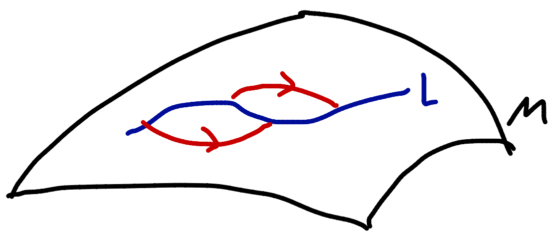
**Theorem (Albers-H.)**

*Fix a function $F$ with a non-degenerate critical submanifold $Z$ and a smooth function $h$ which is close to $F$.*
*Then the function $h$ has at least $\mathrm{cuplength}(Z) + 1$ critical points.*

This method can also be applied to action functionals in Hamiltonian dynamics.

# Applications in Hamiltonian dynamics
# Floer homology

- Floer homology is Morse homology for action functional whose critical points are fixed points of Hamiltonian diffeomorphisms.
- The Morse theory proof for existence of critical points applies here and shows existence of fixed points for small Hamiltonians.
- By the same method, we can also show existence of Hamiltonian chords for certain submanifolds.




Hamiltonian chords

I hope to modify the Morse proof to make it work in other cases of symplectic geometry.

# Contact geometry - Weinstein conjecture

- Contact geometry is in some sense the odd-dimensional version of symplectic geometry. Contact manifolds carry Reeb vector fields, whose flow has similar properties to the Hamiltonian flow.

**Conjecture**

*Weinstein conjecture: Every closed contact manifold has at least one closed Reeb orbit.*

- Known for contact 3-manifolds (Taubes)
- The standard contact structure $S^3$ has at least two periodic orbits. (Christofaro-Gardiner - Hutchings, Ginzburg - H. - Hryniewicz - Macarini, Lui - Long)
- Few things are known in higher dimensions
  $\rightarrow$ Possibly above method applies and give lower bound for number of Reeb orbits.

**Thank you**