# Applications of the trace of Frobenius
# past, present, and future

Tony Feng

September 25, 2020

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$\implies$ there is a **symmetry** $(a, b) \mapsto (a^p, b^p)$ on the space of solutions, called the Frobenius endomorphism.

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**Fundamental identity** (Grothendieck)

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Enables geometric and algebraic tools:

- nearby cycles, monodromy, Lefschetz pencils, spectral sequences, etc.

# The past

$$\boxed{\underbrace{\#\{\text{solutions to } Z \text{ in } \mathbb{F}_{p^n}\}}_{\text{number}} = \mathrm{Tr}(\mathrm{Frob}_p^n,\ \ \underbrace{H^*(Z)}_{\text{vector space}}\ \ )}$$

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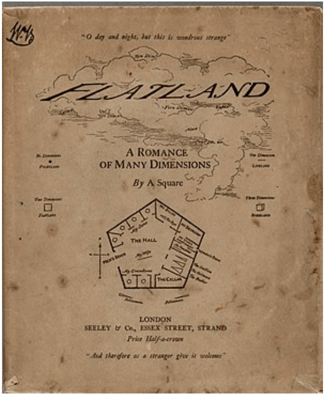
What's next?



"O day and night, but this is wondrous strange"

FLATLAND

A ROMANCE
OF MANY DIMENSIONS

*By* A Square

LONDON
SEELEY & Co., ESSEX STREET, STRAND
*Price Half-a-crown*

"And therefore as a stranger give it welcome"

# Hierarchy of traces

$$\begin{tikzcd} \text{Numbers} & \text{Vector spaces} \arrow[l, bend right, "\text{Trace}"'] & ??? \arrow[l, bend right, "\text{Trace}"'] \end{tikzcd}$$

(Rings, complexes, etc.)

2

3

$4 = 2^2$

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$\mathbb{R}^2 \to \mathbb{R}^3$

$\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^n}$

$$\begin{tikzcd} \mathbb{C}[x] \arrow[d] \\ \mathbb{C}[x]/x^2 \end{tikzcd}$$

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- Equality between objects.

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$\mathbb{C}[x] \twoheadrightarrow \mathbb{C}[x]/x^2$

- Maps between objects.
- Equality between maps.

# Hierarchy of traces

$$\begin{tikzcd} \text{Numbers} & \text{Vector spaces} \arrow[l, bend right, "\text{Trace}"] & & \text{Categories} \arrow[ll, bend right, "\text{Trace}"] \end{tikzcd}$$

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$$\begin{tikzcd} \mathrm{Sheaves}(X) \arrow[d, bend right, "f_!"'] \arrow[d, bend left, "f_*"] \arrow[d, phantom, "\Longrightarrow"] \\ \mathrm{Sheaves}(Y) \end{tikzcd}$$

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# Categorical traces

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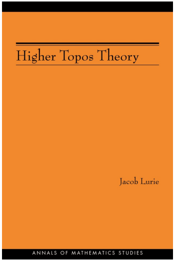

Higher Topos Theory

Jacob Lurie

ANNALS OF MATHEMATICS STUDIES

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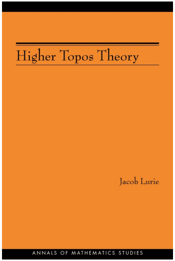

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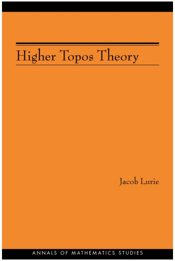

Applications:

- Categorical proof of Hirzebruch-Riemann-Roch
- (Xiao-Zhu) Tate conjecture on products of Shimura varieties

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**Conjecture**

*The commuting variety* $\{X, Y \in \mathfrak{g} \colon [X, Y] = 0\}$ *is reduced, i.e.* $f^2 = 0 \implies f = 0$.

# The commuting variety conjecture

Example: $\mathfrak{gl}_2$

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$$

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Reduced means if $P^2 = 0 \in \mathbb{C}[a, b, c, d, e, f, g, h]/(ae + bg = ae + fc, \ldots)$, then $P = 0 \in \mathbb{C}[a, b, c, d, e, f, g, h]/(ae + bg = ae + fc, \ldots)$.

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**Conjecture**

*The variety $\{X, Y \in \mathfrak{g} \colon [X, Y] = 0\}/$conjugation by $G$ is reduced.*

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$$\begin{tikzcd}
\text{(modular) coherent} \\ \text{Satake category for } G \arrow[d, dashed, "\text{trace of Frob}"'] \\
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*If this picture is correct, then the derived Hecke algebra for* $\widehat{G}$ *is commutative (except possibly in small characteristics).*

For $T\colon V \to V$, $\mathrm{Tr}(T)$ is

$$k \to V \otimes V^\vee \xrightarrow{T \otimes \mathrm{Id}} V \otimes V^\vee \to k.$$

$$(\text{Derived commuting variety of } \mathfrak{g})^{\widehat{G}} \xrightarrow{\sim} (\text{Derived commuting variety of } \mathfrak{t})^{W}$$