Lagrangian and Legendrian skeleta

Zack Sylvan

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- A Legendrian subvariety $L \subset \partial M$ is an n-dimensional C^∞ subvariety such that $\alpha|_{L_{smooth}} = 0$.

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- A Legendrian $L \subset \partial M$ is a finite union of boundary points.

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- All of these Lagrangians have Legendrian boundary.

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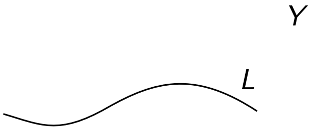

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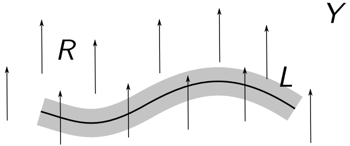

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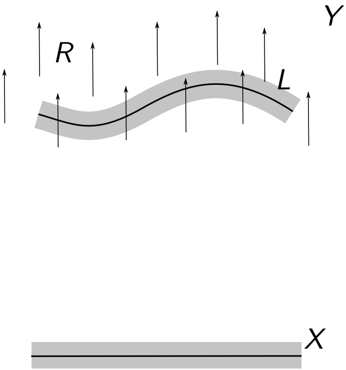

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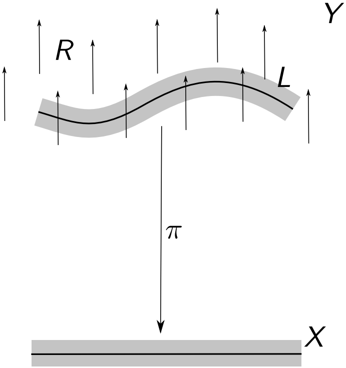

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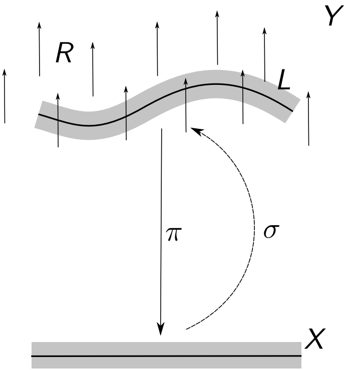

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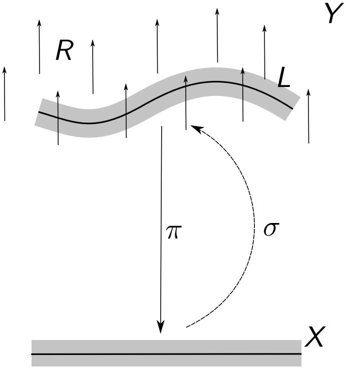
- This σ is the "thickening" of L.



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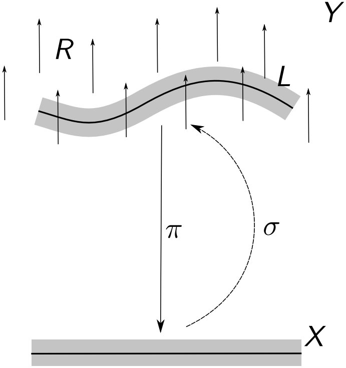

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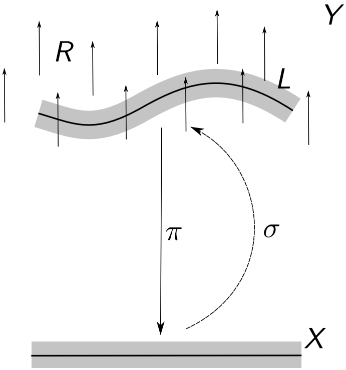

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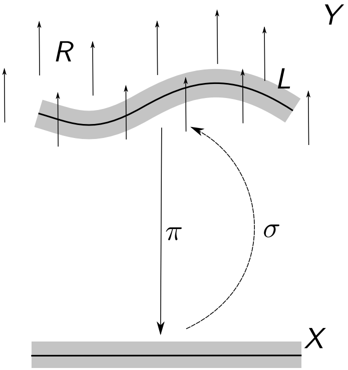

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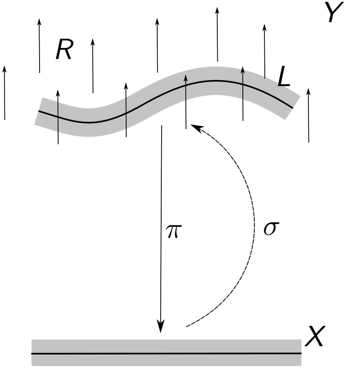

Question

Can the previous thickening procedure be carried out for singular Legendrians? In other words, can we produce a lift σ such that $\sigma \circ \pi(L) = L$ and such that $(X, \sigma^*\alpha)$ is a Liouville domain?

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Looking at the effect on 1-forms, we obtain

$$(\sigma')^*\alpha = \sigma^*\alpha + df.$$

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I don't know any good theorems in this direction. The strongest existing statements are technical lemmas in a book of Cieliebak–Eliashberg.

Thank you!