On the Boltzmann equation without angular cut-off.

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Non-equilibrium Dynamics and Random Matrices Seminar,
IAS, School of Mathematics,
Tuesday, March 18, 2014

L. Boltzmann (1844-1906), J. Maxwell (1831-1879)

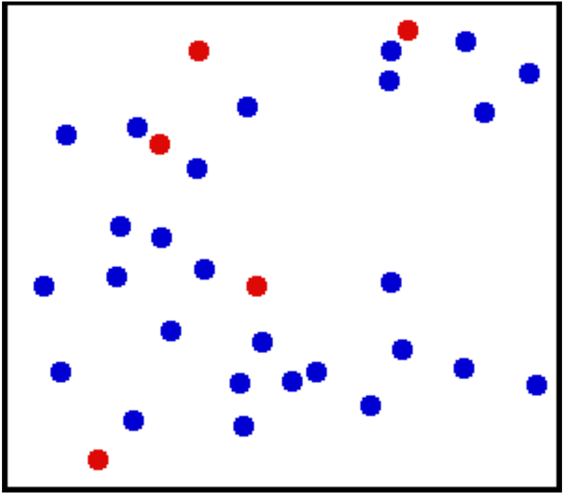
Overall theme: The full Boltzmann equation is a physical non-local geometric fractionally diffusive (somewhat degenerate) PDE.

The Boltzmann Equation: modeling assumptions

The *Boltzmann Equation*, derived by Boltzmann in 1872 (and Maxwell 1866), models the behavior of a dilute gas.

- Gas is modeled as a large ($\gtrsim 10^{23}$) number of particles obeying the laws of Newtonian mechanics.
- Each particle has its own position and velocity. It travels in a straight line until it happens to collide with another particle.
- Only binary collisions are allowed.
- Collisions obey the physical conservation laws of momentum and energy.
- The Boltzmann equation is derived by taking an appropriate continuum (Boltzmann-Grad) limit.
- The Boltzmann equation is probabilistic, describing the evolution in time for an arbitrary particle in the ensemble with a given initial position and momentum.

Particles moving around in a box



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The Boltzmann equation (1872)

- The Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \quad x \in \Omega,\ v \in \mathbb{R}^3,\ t \geq 0$$

- $F = F(t, x, v)$: probability density in (position,velocity)
- $\Omega \subset \mathbb{R}^3$: domain in space
- $v \cdot \nabla_x F$: free transport term
- $Q(F, F)$: collision operator, local in (t, x), quadratic integral operator
- Derived from **rarefied gas dynamics**: *Maxwell 1860'*, *Boltzmann 1872*.

Physical conservation laws of energy and momentum

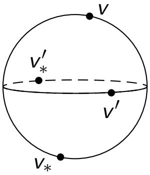
Given two particles, with velocities $v, v_* \in \mathbb{R}^3$, after a binary collision they have outgoing velocities $v', v'_* \in \mathbb{R}^3$.

- These obey the conservation of momentum and energy:

$$\begin{aligned} v + v_* &= v' + v'_*, \\ |v|^2 + |v_*|^2 &= |v'|^2 + |v'_*|^2. \end{aligned}$$

- Since there are six unknowns, (v', v'_*), and four equations, the set of solutions can be parametrized on the sphere $\sigma \in \mathbb{S}^2$:

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$



- Describes the aftermath of a binary collision probabilistically.

Boltzmann equation (1872) for $(x, v) \in \Omega \times \mathbb{R}^3$ is

$$\frac{\partial F}{\partial t} + v \cdot \nabla_x F = \mathcal{Q}(F, F)$$

$F(t, \cdot, \cdot)$ is the density function of particles in the phase space.

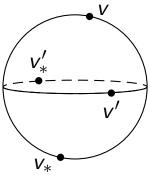
- The *Boltzmann collision operator* is local in (t, x) as:

$$\mathcal{Q}(F, G)(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma \; B(v - v_*, \sigma) \left[F'_* G' - F_* G \right].$$

Shorthand $G = G(v)$, $F_* = F(v_*)$, $G' = G(v')$, $F'_* = F(v'_*)$.

- the pre-post collisional velocities (v, v_*) and (v', v'_*) satisfy

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma,$$
$$v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$



- Lanford (1975). Rigorous derivation for Hard-Spheres.

The aftermath of a binary collision

A complete description of the aftermath of collisions requires more precise physics. There are two major physical approaches:

- Hard spheres: Here the particles are assumed to be literal billiard balls of vanishingly small diameter in the continuum limit. This assumption leads to collision physics in which there is most likely a large deviation in angle after collisions.
- Inverse power-law interactions: Here the particles are assumed to interact pairwise through inverse power law potentials, $\phi(r) = r^{-p+1}$ for some $p > 2$. In this case, the preferred consequence of a collision is a glancing or grazing collision, meaning that the velocities change only a small amount.
- The Coulomb potential $\phi(r) = r^{-1}$: this requires some additional limiting arguments and leads to what is called the Landau equation. It is a reasonably good physical model for the dynamics of plasma.

Boltzmann collision kernel: $B(v - v_*, \sigma)$

$$\mathcal{Q}(F,G)(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma \ B(v - v_*, \sigma) \left[F'_* G' - F_* G \right].$$

J. C. Maxwell in 1866 computed $B(v - v_*, \sigma)$ from the potential:

$$\phi(r) = r^{-(p-1)}, \quad p \in (2, +\infty).$$

This kernel takes product form in its arguments as

$$B(v - v_*, \sigma) = |v - v_*|^\gamma \, b(\cos\theta), \quad \cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma.$$

The angular singularity in σ is not locally integrable:

$$\sin\theta b(\cos\theta) \approx \theta^{-1-2s}, \quad s = \frac{1}{p-1} \in (0,1), \quad \forall\, \theta \in \left(0, \frac{\pi}{2}\right].$$

- The kinetic factor $|v - v_*|^\gamma$ can be singular: $\gamma = \frac{p-5}{p-1} > -3.$
- Maxwell 1866: $\left| \langle \mathcal{Q}(F,G), \phi \rangle_{L^2(\mathbb{R}^n_v)} \right| < \infty.$

Harold Grad angular cut-off assumption 1963

$$b(\cos\theta) \in L^{\infty}(\mathbb{S}^2), \quad \text{or} \quad b(\cos\theta) \in L^{1}(\mathbb{S}^2)$$

Vast majority of research on the Boltzmann equation from the 1960's to say the year 2000 made these assumptions. These assumptions are still in widespread usage today.

- Cut-off versions of the Boltzmann equation are known to propagate singularities: see Boudin-Desvillettes (2000), Duan-Li-Yang (2008), Bernis-Desvillettes (2009).
- The full Boltzmann equation without cut-off experiences smoothing effects: see Lions (1994), Desvillettes (1995), ..., Chen-Desvillettes-He (2009), Alexandre-Morimoto-Ukai-Xu-Yang (2010)
- Full Boltzmann equation without cut-off (near Maxwellian) decays to equilibrium much faster - "diffusion improves the H-theorem" - Gressman-S, AMUXY

Old conjecture regarding the angular singularity

The Boltzmann collision operator operator without cut-off has been widely conjectured to "essentially behave" as

$$F \mapsto Q(g, F) \sim (-\Delta_v)^s F + \text{lower order terms}$$

Linear intuition due to Carlo Cercignani in 1969

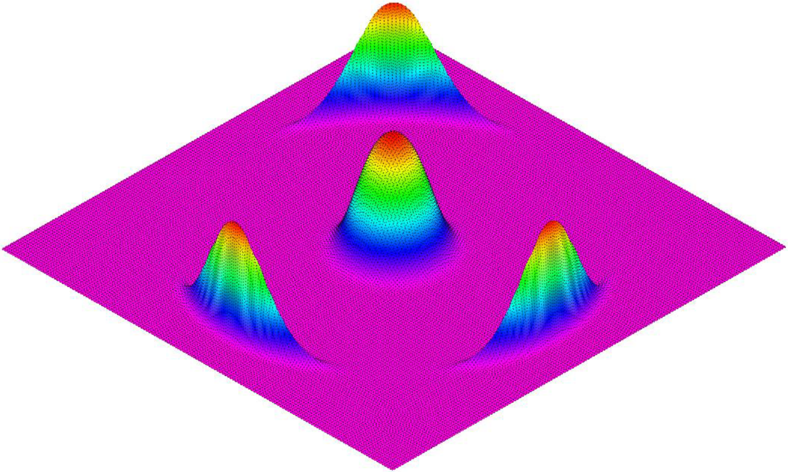
- Conjecture does not take collisional geometry into account. This is an isotropic "flat" Laplacian.
- This is correct to the order s, however we can prove that

$$F \mapsto Q(g, F) \sim \langle v \rangle^{\gamma+2s} (-\Delta_P)^s F + \text{lower order terms.}$$

- If $\mathbb{R}^3$ is identified with a paraboloid in $\mathbb{R}^4$ by means of the mapping $v \mapsto (v, \frac{1}{2}|v|^2)$, then Δ_P is the metric Laplacian on the paraboloid induced by the Euclidean metric on $\mathbb{R}^4$.

(Note that $\langle v \rangle = \sqrt{1 + |v|^2}$)

Heat diffusion on the paraboloid



Sharp geometric fractional diffusive behavior: $N^{s,\gamma}$

We introduce a weighted geometric fractional semi-norm: $\dot{N}^{s,\gamma}$

$$|f|^2_{\dot{N}^{s,\gamma}} \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dv \ \langle v\rangle^{\gamma+2s+1} \int_{d(v,v')\leq 1} dv' \ \frac{(f(v')-f(v))^2}{d(v,v')^{3+2s}}.$$

- We use the metric on the "lifted" paraboloid

$$d(v,v') = \sqrt{|v-v'|^2 + (|v|^2-|v'|^2)^2/4},$$

- The precise statement: $(|f|^2_{L^2_\gamma} \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dv \ \langle v\rangle^{\gamma} \ |f(v)|^2.)$

Theorem (Gressman-S, Adv. Math., 2011)

$$\langle Q(g,F),F\rangle_{L^2(\mathbb{R}^n_v)} \approx |F|^2_{\dot{N}^{s,\gamma}} + |F|^2_{L^2_\gamma} + \textit{lower order terms}.$$

We think of the function g as a parameter; derivative lower bound above holds only assuming $g \in L\log L \cap L^1(B_R)$ locally.

(Recall $\langle v\rangle = \sqrt{1+|v|^2}$)

Intuition from relativistic physics - "on a mass shell"

To "flatten" the diffusive effects of the collision operator we define $\underline{v'} = (v'_1, v'_2, v'_3, v'_4) \in \mathbb{R}^4$, $\underline{v'_*}$, $\underline{v_*}$, with $|v'|^2 = (v'_1)^2 + (v'_2)^2 + (v'_3)^2$:

$$\int_{\mathbb{R}^3} dv' = \int_{\mathbb{R}^4} d\underline{v'}\ \delta\left(v'_4 - |v'|^2/2\right)$$

- We further consider the measure ($\underline{v} = (v, |v|^2/2)$)

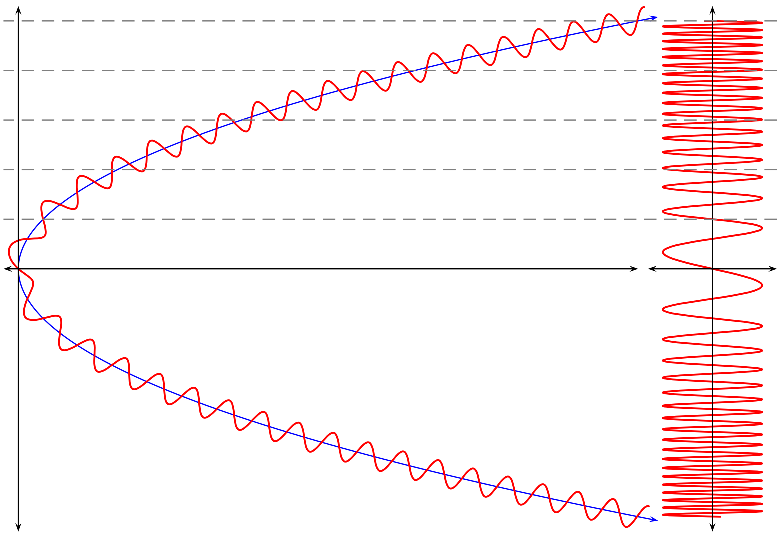
$$\int d\mu = \int_{\mathbb{R}^4} d\underline{v'} \int_{\mathbb{R}^4} d\underline{v_*} \int_{\mathbb{R}^4} d\underline{v'_*}\ \delta^{(4)}\left(\underline{v} + \underline{v_*} - \underline{v'} - \underline{v'_*}\right)$$
$$\times\, \delta\left(v'_4 - |v'|^2/2\right)\ \delta\left((v_*)_4 - |v_*|^2/2\right)\ \delta\left((v'_*)_4 - |v'_*|^2/2\right)$$

- Equivalent expression for the Boltzmann collision operator

$$\mathcal{Q}(F,F)(\underline{v}) = \int d\mu\ B(v - v_*, \sigma)\ \left[F(\underline{v'_*})F(\underline{v'}) - F(\underline{v_*})F(\underline{v})\right].$$

- Insight: Singularity not only $v' \to v$, but also $|v'|^2 \to |v|^2$ (!!)

Graphing the Chirp on the Parabola



Conservation Laws for the Boltzmann equation

- Conservation of mass, momentum and energy hold (formally):

$$\boxed{\int_{\Omega\times\mathbb{R}^3} dxdv \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} F(t,x,v) = \int_{\Omega\times\mathbb{R}^3} dxdv \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} F_0(x,v)}$$

- Equilibrium states are the Maxwellians:

$$\boxed{\mu(\rho,u,T)(v) = \frac{\rho}{(2\pi T)^{3/2}} \exp\left(-\frac{|v-\mathbf{u}|^2}{2T}\right)}$$

Above the density $=\rho$, the bulk velocity $=\mathbf{u}$, and the temperature $=T$ are the fluid variables.

- The five conservation laws and the five parameter Maxwellian $\Longrightarrow \exists\,!$ equilibrium state for any given initial condition

$\sim$ 140 year old Boltzmann H-theorem for solutions

Boltzmann's celebrated H-theorem: Entropy is increasing:

$$-\frac{d}{dt}\int_{\Omega} dx \int_{\mathbb{R}^3} dv \; F \log F \geq 0.$$

This is a manifestation of the second law of thermodynamics.

- Entropy is maximized by the Maxwellian equilibria:

$$\mu(\rho, u, T)(v) = \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|v-\mathbf{u}|^2}{2T}}.$$

  $\rho, \mathbf{u}, T$: the density, bulk velocity and temperature of the gas

- Boltzmann himself predicted convergence in large time to these equilibria because of the H-theorem. The "proof" was however held back by "slight analytical difficulties".
- Big open problem for mathematicians to prove!
- We have a rigorous proof in the close to equilibrium context.

Lots of previous results (this is a short incomplete list)

- Hilbert (approx 1900)
- $\cdots$
- Ukai (1974 near equilibrium for Hard spheres)
- Ellis-Pinsky (1975, fluid approximation...)
- Illner-Shinbrot (1980's near Vaccum)
- Caflish (1980, in $\mathbb{T}^3_x$),
  Ukai-Asano (1982, in $\mathbb{R}^n_x$)
- DiPerna - Lions (1992)
- $\cdots$
- Guo (2002, 2003), Liu-Yang-Yu (2004)
- Desvillettes-Villani (2004), Guo-S (2005)
- Mouhot-S (2006), Gressman-S, AMUXY, ...
- Duan-S (2011, 2012), Guo-Wang (2012), Sohinger-S ...

Linearization and reformulation

We consider the time evolution of perturbations:

$$F(t,x,v) = \mu(v) + \sqrt{\mu(v)}f(t,x,v)$$

The global Maxwellian equilibrium is $\mu(v) = (2\pi)^{-3/2}e^{-|v|^2/2}$.

- The perturbation $f = f(t,x,v)$ evolves via the equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + Lf = \Gamma(f,f)$$

$$\Gamma(f,g) = \mu^{-\frac{1}{2}}\mathcal{Q}(\sqrt{\mu}f, \sqrt{\mu}g), \quad Lf = -\Gamma(f,\sqrt{\mu}) - \Gamma(\sqrt{\mu}, f)$$

- Boltzmann's H-Theorem: $F(t,x,v) \to \mu(v)$ as $t \to \infty$, or equivalently $f(t,x,v) \to 0$ as $t \to \infty$.
- The linear operator has a large null space: $\mathbf{P}L = L\mathbf{P} = 0$. $\mathbf{P}f = a^f(t,x)\sqrt{\mu} + \sum_{i=1}^3 b_i^f(t,x)v_i\sqrt{\mu} + c^f(t,x)(|v|^2 - 3)\sqrt{\mu}$.

Definition of various Sobolev spaces

We first define a unified weight function

$$w(v) \overset{\text{def}}{=} \begin{cases} \langle v \rangle, & \gamma + 2s \geq 0, \quad \text{"hard potentials"} \\ \langle v \rangle^{-\gamma-2s}, & \gamma + 2s < 0, \quad \text{"soft potentials"}. \end{cases}$$

Also spatial and velocity derivatives: $\partial^\alpha_\beta = \partial^{\alpha^1}_{x_1} \partial^{\alpha^2}_{x_2} \partial^{\alpha^3}_{x_3} \partial^{\beta^1}_{v_1} \partial^{\beta^2}_{v_2} \partial^{\beta^3}_{v_3}$.

- For $\ell, K \geq 0$, we define the $H^K_\ell(\mathbb{T}^3 \times \mathbb{R}^3)$ space by

$$\|h\|^2_{H^K_\ell(\mathbb{T}^3 \times \mathbb{R}^3)} \overset{\text{def}}{=} \sum_{|\alpha|+|\beta| \leq K} \|w^{\ell-|\beta|} \partial^\alpha_\beta h\|^2_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)}.$$

- Weighted non-isotropic derivative space $|h|^2_{N^{s,\gamma}_\ell}$ is

$$|w^\ell h|^2_{L^2_{\gamma+2s}} + \int_{\mathbb{R}^3} dv \, \langle v \rangle^{\gamma+2s+1} w^{2\ell}(v) \int_{\mathbb{R}^3} dv' \frac{(h'-h)^2}{d(v,v')^{3+2s}} \mathbf{1}_{d(v,v') \leq 1}$$

- $N^{s,\gamma}_{\ell,K}$: $\|h\|^2_{N^{s,\gamma}_{\ell,K}(\mathbb{T}^3 \times \mathbb{R}^3)} \overset{\text{def}}{=} \sum_{|\alpha|+|\beta| \leq K} \|\partial^\alpha_\beta h\|^2_{N^{s,\gamma}_{\ell-|\beta|}(\mathbb{T}^3 \times \mathbb{R}^3)}$.
- $L^2(\mathbb{R}^3)$-norms in the velocity variables are denoted $|\cdot|_{L^2}$.

Global classical solutions of the Boltzmann equation

Theorem (Gressman-S, J. Amer. Math. Soc., 2011)

Fix $K \geq 4$ the number of derivatives, and $\ell \geq 0$ the velocity weight. Choose $f_0(x,v) \in H^K_\ell(\mathbb{T}^3 \times \mathbb{R}^3)$ which satisfies the conservation laws. $\exists \eta_0 > 0$ such that if $\|f_0\|_{H^K_\ell} \leq \eta_0$, then there exists a unique global classical solution to the Boltzmann equation which satisfies

$$f(t,x,v) \in L^\infty_t H^K_\ell \cap L^2_t N^{s,\gamma}_{\ell,K}((0,\infty) \times \mathbb{T}^3 \times \mathbb{R}^3).$$

When $\gamma + 2s \geq 0$, for a fixed $\lambda > 0$, we have exponential decay

$$\|f(t)\|_{H^K_\ell(\mathbb{T}^3\times\mathbb{R}^3)} \lesssim e^{-\lambda t}\|f_0\|_{H^K_\ell(\mathbb{T}^3\times\mathbb{R}^3)}.$$

When $\gamma + 2s < 0$, for any $m > 0$ we have polynomial decay

$$\|f(t)\|_{H^K_\ell(\mathbb{T}^3\times\mathbb{R}^3)} \leq C_m(1+t)^{-m}\|f_0\|_{H^K_{\ell+m}(\mathbb{T}^3\times\mathbb{R}^3)}.$$

Also positivity, i.e. $F = \mu + \sqrt{\mu} f \geq 0$ if $F_0 = \mu + \sqrt{\mu} f_0 \geq 0$.

Other global results without cut-off

Alexandre-Villani (CPAM, 2002) - large data renormalized weak solutions (ala Diperna-Lions) with defect measure $\mu \geq 0$:

$$\frac{\partial \beta(F)}{\partial t} + v \cdot \nabla_x \beta(F) = \beta'(F)\mathcal{Q}(F,F) + \mu.$$

Typically renormalize as for instance: $\beta(F) = \frac{F}{1+F}$.

Alexandre-Morimoto-Ukai-Xu-Yang (AMUXY) (several publications 2011-2012) - global existence for small data in the whole space

- Very different methods. Their norm is, $|||g|||^2$, given by

$$\int_{\mathbb{R}^6} dv dv_* \int_{\mathbb{S}^2} d\sigma \, B(v-v_*,\sigma) \left\{ \mu_* \left(g' - g\right)^2 + g^2 \left(\sqrt{\mu'} - \sqrt{\mu}\right)^2 \right\}.$$

- $K \geq 6$ derivatives.

Identification of the space $X = N^{s,\gamma}$

Our proof is built on the foundation laid by Guo and S for the Landau equation (corresponding to a limiting case $p \to 2$):

- Macroscopic space-time estimates and nonlinear energy methods due to Guo (CMP, 2002), (Invent. Math., 2003)
- Asymptotic time decay rates due to S-Guo (CPDE, 2006), (ARMA, 2008)

The Landau machinery breaks down at the following stage:

Missing piece: Find a Hilbert space X (only in $\mathbb{R}^3_v$) such that

$$|\langle \Gamma(g,h), f\rangle| \lesssim |g|_{L^2} |h|_X |f|_X + \text{l.o.t.}$$

$$\langle Lf, f\rangle \gtrsim |f|_X^2 - \text{l.o.t.}$$

If such a Hilbert space X exists, it is essentially unique.

General very high level strategy of proof

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + Lf = \Gamma(f,f).$$

Consider the high order norm

$$\mathcal{G}(f)(t) \overset{\text{def}}{=} \|f(t)\|^2_{H^K(\mathbb{T}^3\times\mathbb{R}^3)} + \int_0^t d\tau \sum_{|\alpha|\leq K} \|\partial^\alpha f(\tau)\|^2_{N^{s,\gamma}}.$$

We have the energy inequality

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2(\mathbb{T}^3\times\mathbb{R}^3)} + (f, Lf)_{L^2(\mathbb{T}^3\times\mathbb{R}^3)} = (f, \Gamma(f,f))_{L^2(\mathbb{T}^3\times\mathbb{R}^3)}$$

Key steps to prove non-linear stability: $\mathcal{G}(f)(t) \lesssim \|f_0\|^2_{H^K(\mathbb{T}^3\times\mathbb{R}^3)}$.

- Coercive estimate: $(f, Lf)_{L^2(\mathbb{T}^3\times\mathbb{R}^3)} \gtrsim \|f\|^2_{N^{s,\gamma}}$. This is false as as an operator inequality, only true for solutions.
- Non-linear estimate: $|(f, \Gamma(f,f))_{L^2}| \lesssim \|f\|_{L^2(\mathbb{T}^3\times\mathbb{R}^3)}\|f\|^2_{N^{s,\gamma}}$.

Main estimates

Split the linearized collision operator $L = N + K$, where N is essentially the "norm" and K is a "compact perturbation":

Lemma (Norm estimate)

$$\langle Ng, g\rangle \approx |g|^2_{N^{s,\gamma}}.$$

Lemma (Compact estimates, no derivatives)

$$|\langle Kg, g\rangle| \leq \eta |g|^2_{L^2_{\gamma+2s}(\mathbb{R}^3)} + C_\eta |g|^2_{L^2(B_{C_\eta})}, \quad \forall \eta > 0, \quad \exists C_\eta > 0.$$

Theorem (Non-linear estimate)

$$|\langle \Gamma(g,h), f\rangle| \lesssim \left(|g|_{L^2}|h|_{N^{s,\gamma}} + |g|_{N^{s,\gamma}}|h|_{L^2}\right)|f|_{N^{s,\gamma}}$$

One idea for the estimates: Littlewood-Paley-Stein theory

Advantage of our concrete definition $N^{s,\gamma}$: we can develop a corresponding Littlewood-Paley theory to prove the upper bounds:

$$P_j f\left(v, \frac{|v|^2}{2}\right) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dv' 2^{3j} \varphi\left(2^j \left(v - v', \frac{|v|^2}{2} - \frac{|v'|^2}{2}\right)\right) \langle v' \rangle f(v')$$

$$Q_j f \stackrel{\text{def}}{=} P_j f - P_{j-1} f, \qquad j \geq 1$$

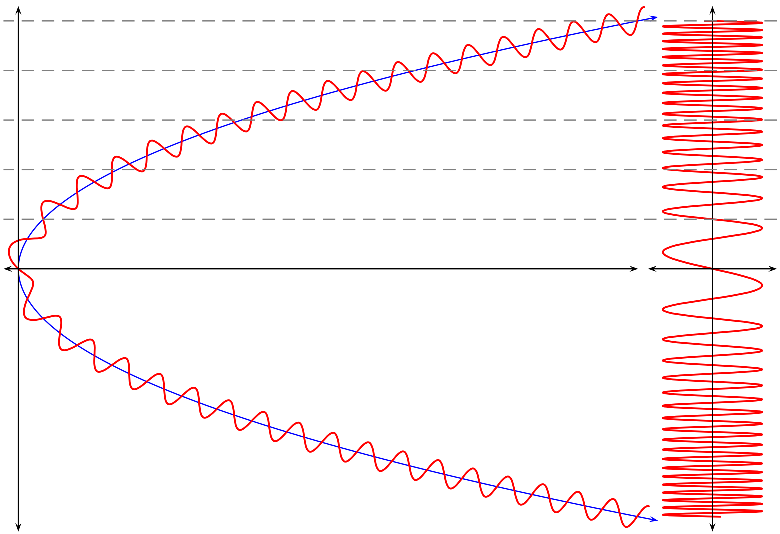
This is a renormalized *four-dimensional Euclidean* Littlewood-Paley decomposition of the measure on the paraboloid in $\mathbb{R}^4$.

- We may phrase our results in terms of $P_j g$, $Q_j g$, and Euclidean derivatives of these functions in $\mathbb{R}^4$ instead of $\mathbb{R}^3$.
- We obtain the sharp characterization of the space $N^{s,\gamma}$:

$$\sum_{j=0}^{\infty} 2^{2(s-|\alpha|)j} \int_{\mathbb{R}^3} dv \ |\tilde{\nabla}^\alpha Q_j f(v)|^2 \langle v \rangle^{\gamma+2s} \lesssim |f|^2_{N^{s,\gamma}}$$

For any multi-index α of Euclidean derivatives $\tilde{\nabla}$ on $\mathbb{R}^4$.

Graphing the Chirp on the Parabola (again)



Soft potential decay rates in the whole space: $\mathbb{R}^n_x$

Theorem (Ukai-Asano 1982, with angular cut-off, $-1 < \gamma \le 0$)

$$\| \langle v \rangle^b f(t,\cdot,\cdot)\|_{L^\infty_v(H^m_x)} \lesssim (1+t)^{-a}.$$

Here $a \overset{def}{=} \min\left\{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{2}\right),1\right\}$ *for* $p \in [1,2)$. *With* $\| \langle v \rangle^{b+a\gamma} f_0(\cdot,\cdot)\|_{L^\infty_v(H^m_x)} + \|f_0\|_{L^2_v(L^p_x)}$ *sufficiently small.*

Above $a = \frac{n}{4}$ when $p=1$ is optimal in dimensions $n = 2$, 3, 4 but the rate of $a=1$ is not optimal for $n \ge 5$.

Theorem (AMUXY, 2011, non cut-off)

Optimal L^2_x *decay rates for the hard potentials* $\gamma + 2s > 0$. *For the soft potentials* $\max\{-2s-3/2,-3\} < \gamma < -2s$ *they have*

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^3_x} |f(t,x)|_{H^{K-3}(\mathbb{R}^3_v)} \lesssim (1+t)^{-1}, \quad (K \ge 6).$$

This L^∞_x rate is non-optimal.

Optimal decay rates in Besov spaces in $\mathbb{R}^n_x$ $(n \geq 3)$

We use the homogeneous mixed Besov space $\dot{B}_2^{-\varrho,\infty} L^2_v$ with norm

$$\|g\|_{\dot{B}_2^{-\varrho,\infty} L^2_v} \stackrel{\text{def}}{=} \sup_{j \in \mathbb{Z}} \left(2^{-\varrho j} \|\Delta_j g\|_{L^2(\mathbb{R}^n_x \times \mathbb{R}^n_v)} \right), \quad \varrho > 0.$$

Here Δ_j is the standard LP projection onto frequencies $|\xi| \approx 2^j$.

Theorem (Sohinger - S, Adv. Math., accepted, 2014)

Fix $\varrho \in (0, n/2]$ and consider any $k \in \{0, 1, \dots, K-1\}$. Suppose additionally that $\|f_0\|_{\dot{B}_2^{-\varrho,\infty} L^2_v} < \infty$ (not necessarily small). Then

$$\sum_{k \leq |\alpha| \leq K} \|\partial^\alpha f(t)\|^2_{L^2_x L^2_v} \lesssim (1+t)^{-(k+\varrho)}.$$

This holds uniformly for $t \geq 0$.

These decay rates are said to be "optimal" by comparison to the analogous results for the linear decay. Lower bounds unknown.

Optimal decay rates in Besov spaces in $\mathbb{R}^n_x$ $(n \geq 3)$

Theorem (Sohinger-S, same paper, faster decay)

Furthermore for $\varrho \in (n/2, (n+2)/2]$, *if*

$$\|\mathbf{P} f_0\|_{\dot{B}_2^{-\varrho,\infty} L^2_v} + \|\{\mathbf{I} - \mathbf{P}\} f_0\|_{\dot{B}_2^{-\varrho+1,\infty} L^2_v} < \infty,$$

then, in the hard potentials case, the solution uniformly satisfies

$$\sum_{|\alpha|+|\beta| \leq K} \|w^{\ell-|\beta|} \partial^\alpha_\beta f(t)\|^2_{L^2_x L^2_v} \lesssim (1+t)^{-\varrho},$$

Recall the macroscopic projection is
$\mathbf{P} f = a^f(t,x)\sqrt{\mu} + \sum_{i=1}^n b_i^f(t,x) v_i \sqrt{\mu} + c^f(t,x)(|v|^2 - n)\sqrt{\mu}$.

Comparisons...

Most previous results for decay rates in the whole space for these types of equations require that the initial data is small in L^1_x (say).

- For the heat equation, if $g_0 \in \mathcal{S}'$ and $\|g_0\|_{\dot{B}_2^{-\varrho,\infty}(\mathbb{R}^n_x)} < \infty$ then

$$\|g_0\|_{\dot{B}_2^{-\varrho,\infty}(\mathbb{R}^n_x)} \approx \left\| t^{\varrho/2} \left\| e^{t\Delta} g_0 \right\|_{L^2(\mathbb{R}^n_x)} \right\|_{L^\infty_t((0,\infty))}, \quad \text{for any } \varrho > 0.$$

- For the incompressible Navier-Stokes equations, if $u_0 \in L^2_x \cap \dot{B}_2^{-\varrho,\infty}$ generates a Leray-Hopf weak solution then

$$\sup_{t>0} \left(t^{\varrho/2} \|u(t)\|_{L^2(\mathbb{R}^n_x)} \right) < \infty, \quad \varrho \in \left(0, \frac{n+2}{2}\right], \quad n \in \{2,3\}.$$

See Wiegner 1987. And Lemarié-Rieusset: Recent developments in the Navier-Stokes problem (2002).

Needed to establish vector valued interpolation.

To prove energy inequalities with high spatial derivatives, we need to do interpolation.

- Our solution is to let $\mathcal{H}_v$ be an arbitrary seperable Hilbert space, and re-prove vector valued interpolation inequalities.

$$\|f\|_{\dot{B}_q^{-\varrho,\infty}\mathcal{H}_v} \lesssim \|f\|_{L^p_x\mathcal{H}_v}, \ \frac{1}{p}-\frac{1}{q}=\frac{\varrho}{n}, \ 1\le p\le 2, \ 0<\varrho\le\frac{n}{2}.$$

$$\|f\|_{\dot{B}_p^{k,1}\mathcal{H}_v} \lesssim \|f\|^{1-\theta}_{\dot{B}_r^{m,\infty}\mathcal{H}_v}\|f\|^{\theta}_{\dot{B}_r^{\varrho,\infty}\mathcal{H}_v}, \ 0<\theta<1, \ 1\le r\le p\le\infty,$$

Also $m\neq\varrho$ and $k+\frac{n}{r}-\frac{n}{p}=m(1-\theta)+\varrho\theta$. These are in the "hard order" when the $\mathcal{H}_v$ norm is on the inside. And others...

Coercive Entropy production estimates

Boltzmann's celebrated H-theorem:

$$-\frac{d}{dt}\int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dv \ F\log F = \int_{\mathbb{T}^3} dx \ D(F) \geq 0.$$

- non-negative Entropy production functional:

$$D(F) = -\int_{\mathbb{R}^3} dv \ Q(F,F)\log F$$
$$= \frac{1}{4}\int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma \ B(v-v_*,\sigma)\left(F'F'_* - FF_*\right)\log\frac{F'F'_*}{FF_*}.$$

- Alexandre-Desvillettes-Villani-Wennberg (ARMA, 2000)

$$D(F) \gtrsim C_R \|\sqrt{F}\|^2_{\dot{H}^s(|v|<R)} - \ \text{l.o.t.}, \quad R > 0.$$

Widely used. General Conditions: $F \in L\log L(\mathbb{R}^3) \cap L^1_1(\mathbb{R}^3)$.

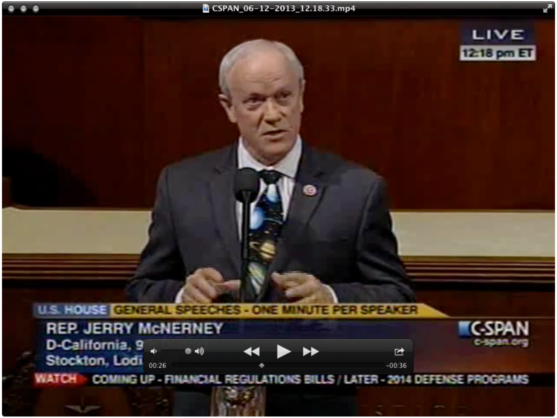
Our Improved Entropy Production Estimate

Our Improved Entropy Production estimate (Gressman-S, 2012, Adv. Math.):

$$D(F) \gtrsim \int_{\mathbb{R}^3} dv \int_{d(v,v')\leq 1} dv' \ \langle v \rangle^{\gamma+2s+1} \ \frac{(\sqrt{F'} - \sqrt{F})^2}{d(v,v')^{3+2s}} - \text{ l.o.t.}$$

- This is the same semi-norm as in the linearized context.
- Stronger global anisotropic semi-norm (in terms of the weight power coupled to the fractional differentiation) than the ADVW (ARMA, 2000) smoothing estimate.
- Derivative estimate holds under general (local) conditions: $F \in L \log L \cap L^1(B_R)$. (Plus other integrability conditions for the error terms which are the same as ADVW).
- Also work by Chen-He (2011), AMUXY, etc, in terms of sharp isotropic lower bound for this Sobolev space.

THANK YOU!



Congressman Jerry McNerney giving a “one-minute speech” on the floor of the US House of Representatives about our NSF-funded research on the Boltzmann equation on June 12, 2013.