IAS 4/4/2014

Nonlinear Fluctuating Hydrodynamics in One Dimension

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IAS and TUM

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outline

1. randomly forced Burgers equation — scalar field
2. two conserved fields — "phase diagram"
3. one-dimensional fluid — hard shoulder potential
4. molecular dynamics (simulation)

1. randomly forced Burgers

conserved field $\phi(x,t)$, $x \in \mathbb{R}$, $t \geq 0$

$$\partial_t \phi + \partial_x \left(G \phi^2 - \tfrac{1}{2} \partial_x \phi + W \right) = 0$$

coupling nonlinear dissipation space-time white noise

Gaussian

$\langle W(x,t) W(x',t') \rangle = \delta(x-x') \delta(t-t')$

interest: large x, t

finite volume, short distances : E, Khanin, Mazel, Sinai (2000)

mean zero, stationary process

- stationary measure $x \mapsto \phi(x)$ white noise

Hairer 2012, Funaki, Quastel 2013

- stationary covariance

$$S(x,t) = \langle \phi(x,t)\,\phi(0,0) \rangle \qquad S(x,0) = \delta(x)$$

$$\int S(x,t)\,dx = 1\,, \qquad S(x,t) = S(-x,t)$$

$$S \geq 0$$

- scaling limit $x = \mathcal{O}(t^{2/3})$

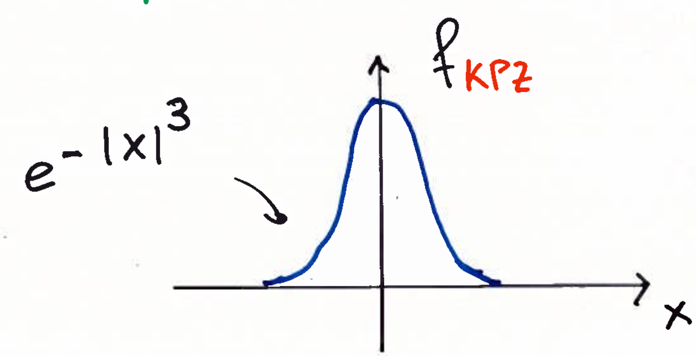
$$S(x,t) = \frac{1}{(\lambda_B t)^{2/3}}\, f_{KPZ}\left(\frac{x}{(\lambda_B t)^{2/3}}\right)$$

proved for PNG, TASEP

non universal

$$\lambda_B = 2\sqrt{2}\,|G|$$

exact solution / scaling



$$\int f_{KPZ}(x)\,x^2\,dx = 0.51\ldots$$

2. two components

$\phi_\alpha(x,t)$, $\alpha = 1,2$

- velocities c_α $\quad c_1 \neq c_2$
- couplings 2×2 matrix G^α
- diffusion $D > 0$, 2×2 matrix

$$\partial_t \phi_\alpha + \partial_x \left(c_\alpha \phi_\alpha + \langle \phi, G^\alpha \phi \rangle - \tfrac{1}{2} \partial_x D \phi_\alpha + \sqrt{D}\, W_\alpha \right) = 0 , \quad \alpha = 1,2$$

W_1, W_2 independent

- normal mode representation

$\partial_x A \phi$, A is diagonal , noise strength $\sqrt{D}$

stationary covariance $\quad S_{\alpha\beta}(x,t) = \langle \phi_\alpha(x,t)\, \phi_\beta(0,0) \rangle$

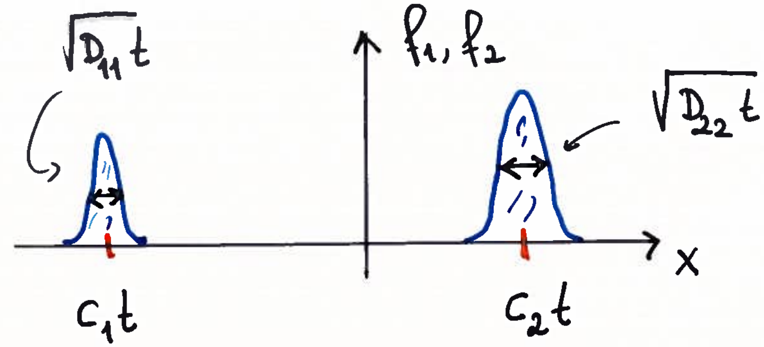
ONLY diagonal

$$S_{\alpha\beta}(x,t) = \delta_{\alpha\beta} f_\alpha(x,t) , \qquad f_\alpha(x,0) = \delta(x)$$

I) $G^\alpha = 0$, Gaussian theory

modes decouple for $t \to \infty$

diffusive



couplings

$$\begin{array}{cc} \alpha = 1 & \alpha = 2 \\ \begin{pmatrix} \blacksquare & 0 \\ 0 & \bullet \end{pmatrix}, & \begin{pmatrix} \bullet & 0 \\ 0 & \blacksquare \end{pmatrix} \end{array}$$

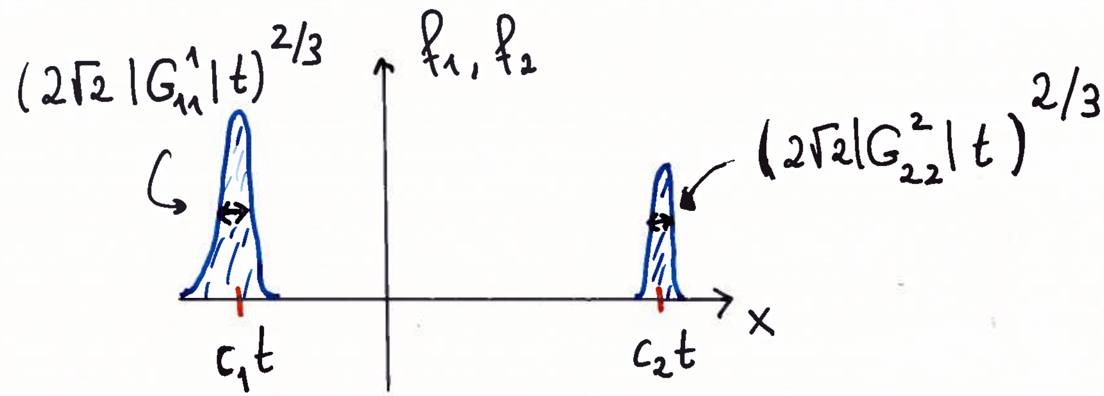
$\blacksquare$ self-coupling leading

$\bullet$ cross-coupling subleading

II) $G^1_{11} \neq 0$, $G^2_{22} \neq 0$,

decoupling

two KPZ peaks



III) $G^1_{11} = 0, \; G^2_{22} = 0$ $\qquad \begin{pmatrix} 0 & 0 \\ 0 & \bullet \end{pmatrix}, \; \begin{pmatrix} \bullet & 0 \\ 0 & 0 \end{pmatrix}$

III a)

$$\partial_t \phi_1 + \partial_x \left(c_1 \phi_1 - \tfrac{1}{2} D \partial_x \phi_1 + \sqrt{D}\, W_1 \right) = 0$$

$$\partial_t \phi_2 + \partial_x \left(c_2 \phi_2 + G \phi_1^2 + W_2 \right) = 0$$

$$f_1(x + c_1 t, t) = \frac{1}{\sqrt{2\pi D t}} e^{-x^2/2Dt}$$

diffusive

Bernardin, Gonçalves, Jara 2014

$$\frac{d}{dt} Y_j = Y_{j+1} - Y_{j-1} \quad + \text{random exchanges}$$

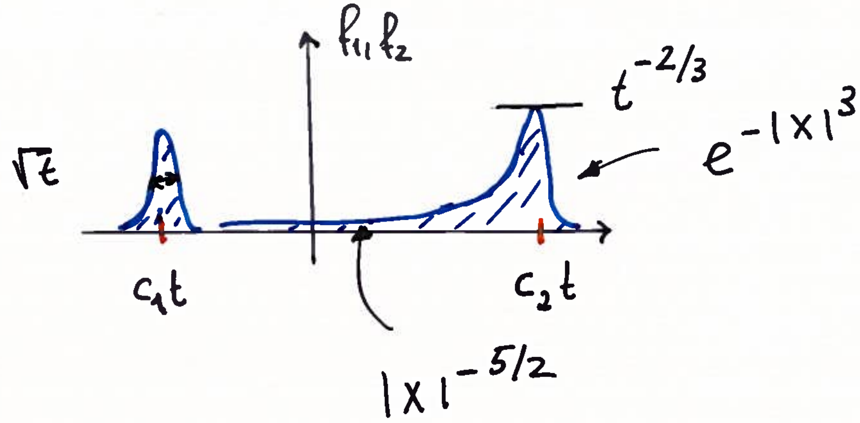
$$Y_j \leftrightarrow Y_{j+1}$$

$$f_2(x + c_2 t, t) = \frac{1}{(\lambda_L t)^{2/3}} f_{L,\frac{3}{2}} \left(\frac{x}{(\lambda_L t)^{2/3}} \right)$$

$$\hat{f}_{L,\frac{3}{2}}(k) = e^{-|k|^{3/2}(1 - i \beta_{max} \operatorname{sgn}(k))}$$

maximally asymmetric Lévy $\frac{3}{2}$

$$\lambda_L = \sqrt{2}\, G^2 \left(|c_2 - c_1|\, D\right)^{-1/2}$$

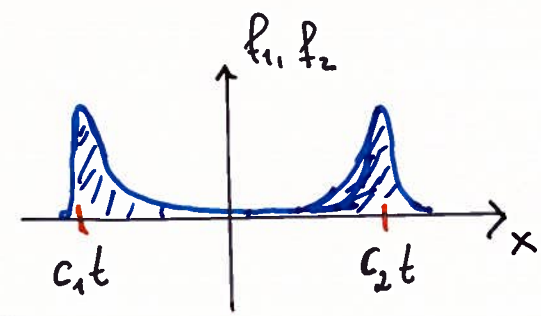


III b) cross coupled

$$\partial_t \phi_1 + \partial_x \left(c_1 \phi_1 - \phi_2^2 - \tfrac{1}{2} \partial_x \phi_1 + W_1 \right) = 0$$

$$\partial_t \phi_2 + \partial_x \left(c_2 \phi_2 - \phi_1^2 - \tfrac{1}{2} \partial_x \phi_2 + W_2 \right) = 0$$

$$f_\alpha(x + c_\alpha t, t) = \frac{1}{(\lambda_k t)^{2/3}}\, f_{L,k}\left(\frac{x}{(\lambda_k t)^{2/3}} \right)$$



maximally asymmetric Lévy, $k = \frac{1}{2}(1 + \sqrt{5})$

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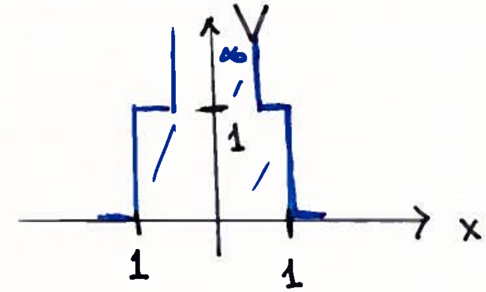
method: mode coupling $\quad \partial_t S(t) = \partial_x^2 \int_0^t ds\, S(t-s) * S(s)^2$

3. one-dimensional fluids

particle j, $q_j, p_j \in \mathbb{R}$, mass $= 1$

$$H = \sum_j \tfrac{1}{2} p_j^2 + \tfrac{1}{2} \sum_{i \neq j} V(q_i - q_j)$$

special case: hard shoulder



$q_{j+1} - q_j \geq \frac{1}{2}$ order preserved

nearest neighbor only

lattice field theory $r_j = q_{j+1} - q_j$ stretch

$$\frac{d}{dt} r_j = p_{j+1} - p_j$$

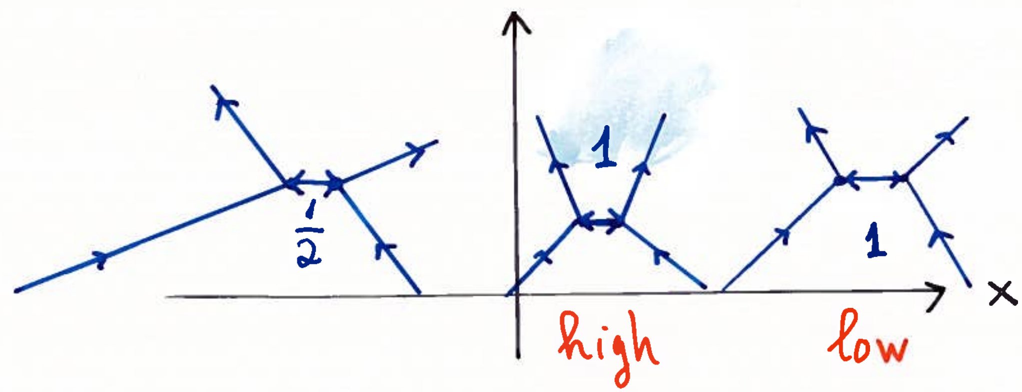
$$\frac{d}{dt} p_j = V'(r_j) - V'(r_{j-1})$$

• 3 conservation laws

r_j

p_j

$e_j = \frac{1}{2} p_j + V(r_j)$



- initial measure : Gibbs $\qquad \{r_j, p_j\}$ are i.i.d.

$$p_j : \frac{1}{Z} e^{-\frac{1}{2}\beta p_j^2}, \quad r_j : \frac{1}{Z} e^{-\beta(V(r_j) + P\, r_j)}$$

pressure

- stationary stochastic process

stretch - stretch

$$S(j,t) = \langle r_j(t)\, r_0(0) \rangle$$

Gibbs

Newton

heat

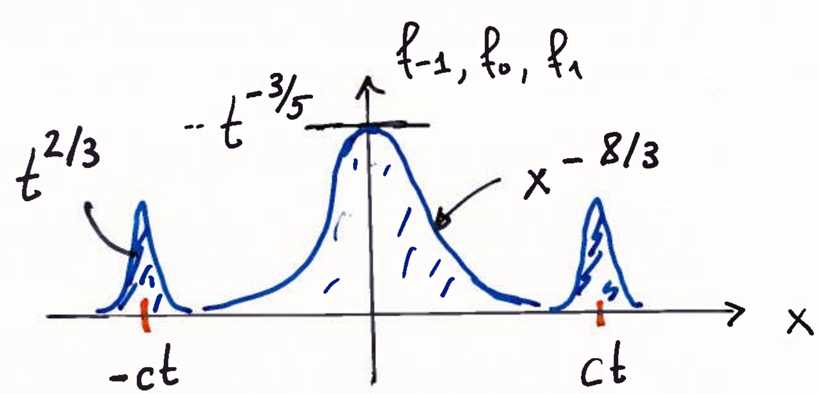
nonlinear hydrodynamics: 3 components $\qquad$ label $-1 \quad 0 \quad 1$

sound

$\vec{c} = (-c, 0, c)$

couplings $\quad G^{1}_{11} = -G^{-1}_{-1-1} \neq 0$

BUT $\quad G^{0}_{00} = 0$ (for all V)



sound : $\quad$ KPZ

heat : $\quad$ symmetric Lévy $\frac{5}{3}$

simulation

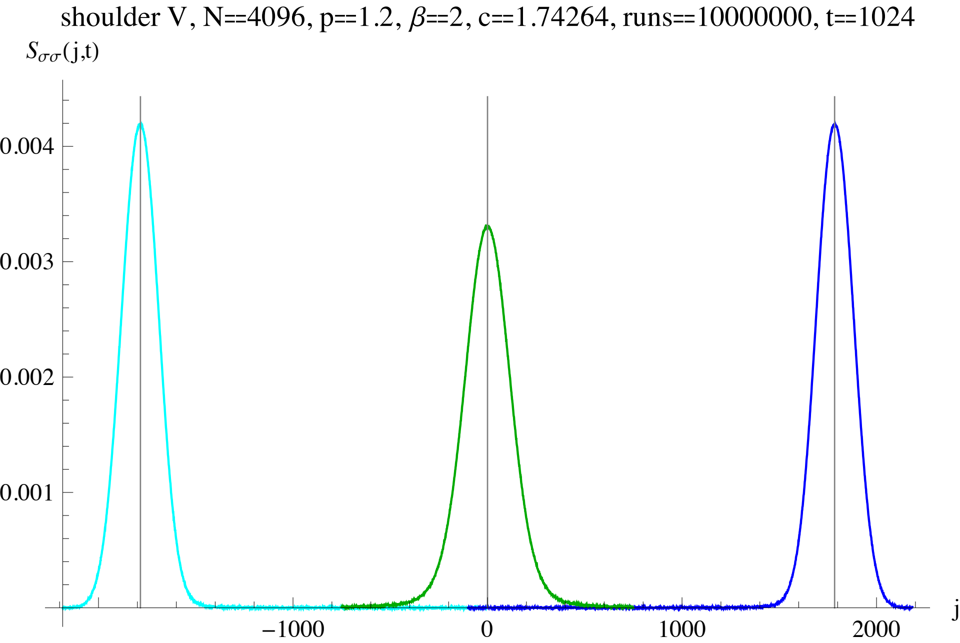
$N = 4096$, $\{r_j, p_j\}$ by random number generator

$t = 1024$ free motion – collision – ....

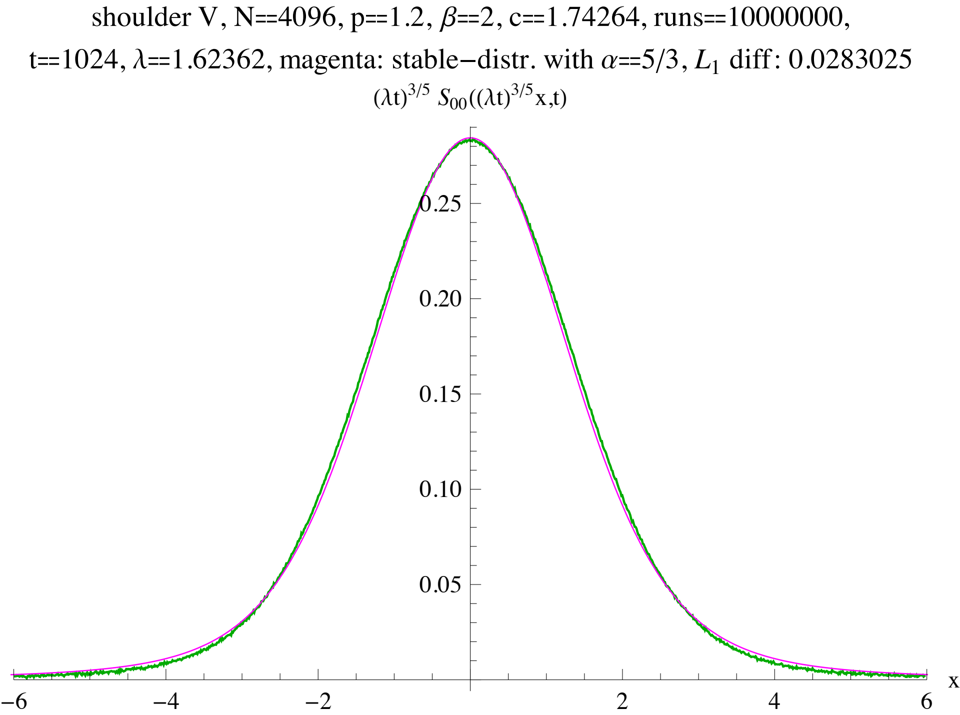
- pressure $p = 1.2$, $\beta = 2$ $\cong$ 1200 collisions / particle
- average over 10^7 initial phase points

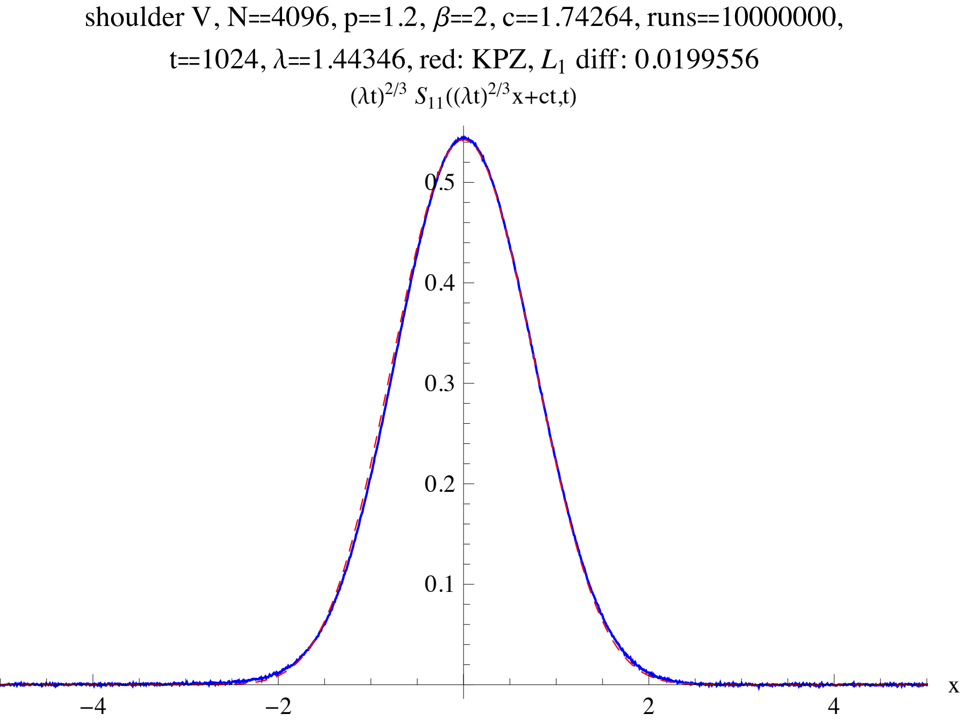
$c = 1.743$

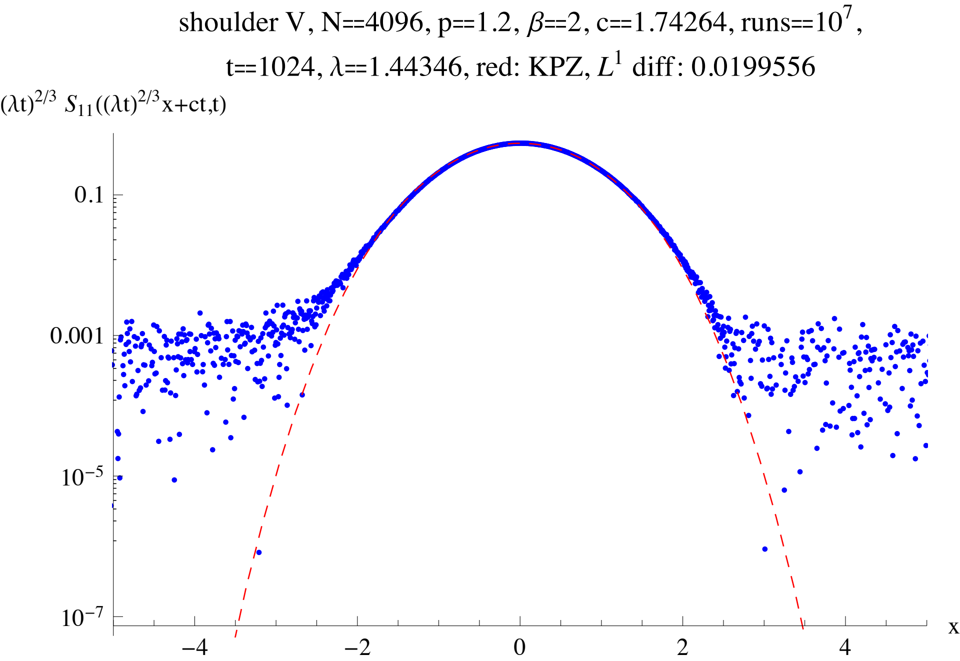
Landau-Placzek 0.082 : 0.065 : 0.082



shoulder V, N==4096, p==1.2, β==2, c==1.74264, runs==10000000, t==1024
$S_{\sigma\sigma}(j,t)$
0.004
0.003
0.002
0.001
−1000
0
1000
2000
j


shoulder V, N==4096, p==1.2, β==2, c==1.74264, runs==10000000,
t==1024, λ==1.62362, magenta: stable−distr. with α==5/3, L_1 diff: 0.0283025
$(\lambda t)^{3/5}\, S_{00}((\lambda t)^{3/5}x,t)$
0.25
0.20
0.15
0.10
0.05
x
−6
−4
−2
2
4
6


shoulder V, N==4096, p==1.2, β==2, c==1.74264, runs==10000000,
t==1024, λ==1.44346, red: KPZ, L_1 diff: 0.0199556
$(\lambda t)^{2/3}\, S_{11}((\lambda t)^{2/3}x+ct,t)$
0.5
0.4
0.3
0.2
0.1
−4
−2
2
4
x

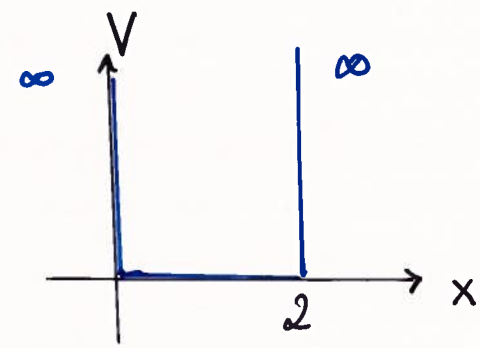

shoulder V, N==4096, p==1.2, β==2, c==1.74264, runs==10^7,
t==1024, λ==1.44346, red: KPZ, L^1 diff: 0.0199556
(λt)^2/3 S11((λt)^2/3 x+ct,t)
0.1
0.001
10^−5
10^−7
−4
−2
0
2
4
x

a further test $\qquad G^{0}_{00} = 0$

special point $\qquad G^{1}_{11} = 0$

$\Rightarrow$ 2 diffusive sound modes + heat mode Lévy $\frac{3}{2}$ symmetric

shoulder NO



$\Rightarrow$ alternating masses , n.n. potential

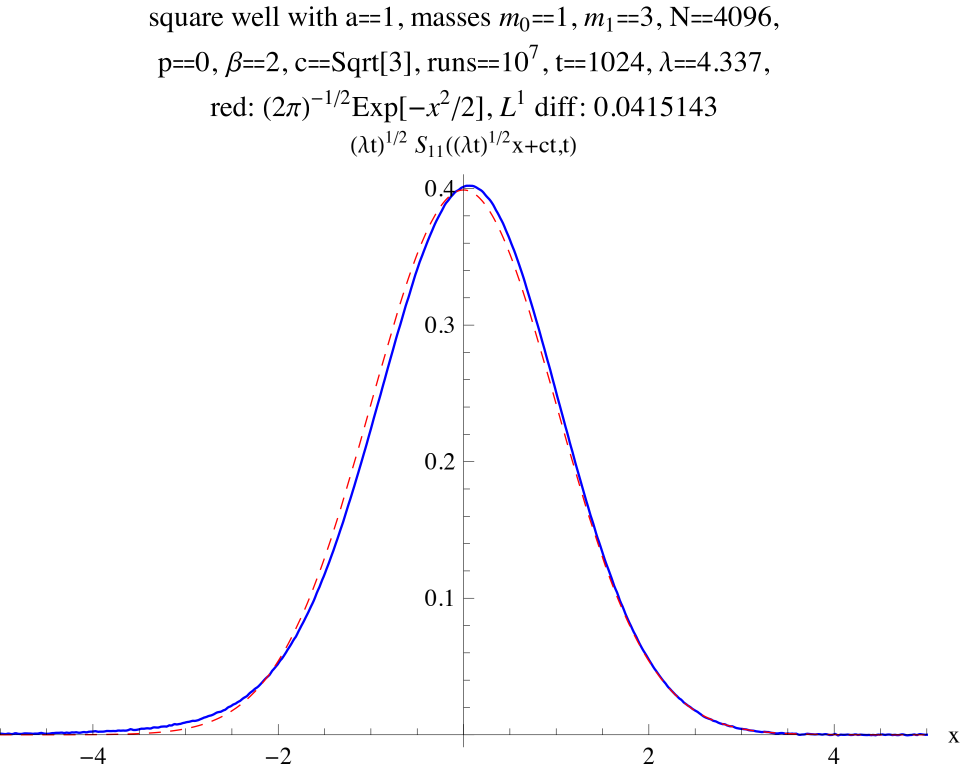
hard collisions, maximal distance for neighbors

special point $\qquad P = 0, \quad \langle r_j \rangle = 1$

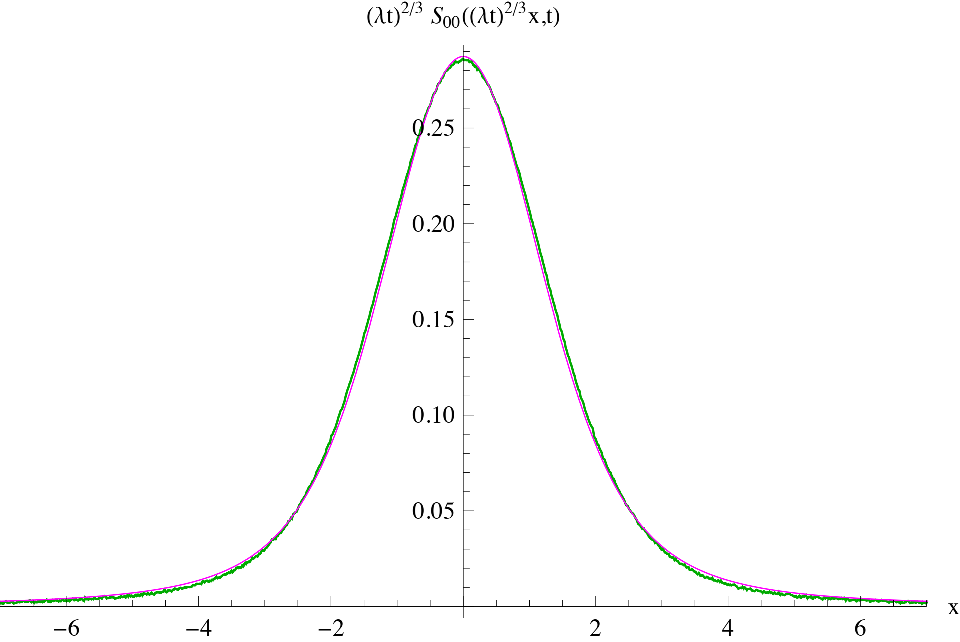
simulation: $\frac{m_1}{m_0} = 3, \quad \beta = 2, \quad P = 0$

Komorowski, Jara, Olla 2014

harmonic chain + random exchanges


square well with a==1, masses m_0==1, m_1==3, N==4096,
p==0, β==2, c==Sqrt[3], runs==10^7, t==1024, λ==4.337,
red: $(2\pi)^{-1/2}$Exp[$-x^2/2$], L^1 diff: 0.0415143
$(\lambda t)^{1/2}$ $S_{11}((\lambda t)^{1/2}$x+ct,t)
0.4
0.3
0.2
0.1
−4
−2
2
4
x

square well with a==1, masses m_0==1, m_1==3, N==4096,
p==0, β==2, c==Sqrt[3], runs==10^7, t==1024, λ==1.32418,
magenta: α−stable−distr. with α==3/2, L^1 diff: 0.025795



Outlook

not integrable

- quantitative predictions for any 1D system with conservation laws

- good agreement with simulations

- HOWEVER
  non-universal coefficients are still changing in time, asymptotics not yet reached!

- exact solutions are welcome