Dynamics of energy critical wave equations

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Part of the talk is based on joint works with Duyckaerts, Kenig, Merle, and with Liu, Schlag, Xu

Introduction

Consider the KdV equations (Diederik Korteweg and Gustav de Vries, 1895):

$$\partial_t u + \partial_{xxx} u + 6u\partial_x u = 0.$$

This equation admit solutions of the form $u = h^2 Q(h(x - h^2 t))$, with $h > 0$ and Q satisfying

$$\partial_{xx} Q - Q + 6Q^2 = 0.$$

All solitons travel to the right; tall solitons are thinner, and travel faster; dispersive wave travels to the left.

Introduction

Zabusky & Kruskal observed that any solution eventually breaks up into the sum of several solitons moving to the right plus a decaying term moving to the left. (Gardner, Greene, Kruskal and Miura, 1967; Eckhaus and Schuur 1983.)

This remarkable "universal behavior" for large times has attracted enormous attentions from physcists and mathematicians.

Now people believe that for general dispersive equations, one has the same "universal behavior". Mathematically, this is called the "soliton resolution conjecture".

Introduction

It turns out that the KdV equation is quite special: it is completely integrable. The method for proving soliton resolution conjecture does not apply for non-integrable equations.

Indeed, the physical phenomenon can also be truly different. (E.g., Soliton collision for quartic KdV, Martel and Merle, 2011.)

The soliton resolution conjecture remains open for most dispersive equations, except in the case of linear equations, or for nonlinear equations which do not permit solitons.

Introduction

For example, one can consider the nonlinear Schrodinger equation

$$i\partial_t u + \Delta u - Vu - |u|^2 u = 0.$$

If the V is attractive ($V < 0$), then there could be solitary waves of the form $e^{iEt}\Psi$, with

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As another example, one can consider the nonlinear Klein Gordon equation

$$\partial_{tt} u - \Delta u + u + Vu + u^3 = 0,$$

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In both cases, the solution are global, and one can ask what are the long time behavior.

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It is still open how to prove such a result in both cases.
There are interesting partial results by Tao for the nonlinear Schrodinger equation, in the radial case and in high dimensions: basically he proved that the solution splits into a linear dispersive part in the far field, and a local part which belongs to a "'compact set" of data. The main remaining question is to classify what are these objects in the local part.

Introduction

Many other models can be considered in connection with the soliton resolution conjecture.
It turns out that for energy critical wave equations, the situation is much better. There are, I think, several reasons:
1. Finite speed of propagation property for wave equations;
2. Speed of propagation does not depend on frequency;
3. Geometric features of wave equations.

In recent years, a fundamental new property of linear wave equations has been found firstly by Duyckaerts-Kenig-Merle, and others, which has had an enormous impact on the understanding of long time dynamics of wave equations.

Channel of energy inequalities

Consider the linear wave equation

$$\partial_{tt} u - \Delta u = 0, \text{ in } R^d \times R, \tag{1}$$

with initial data $\overrightarrow{u}(0) := (u, \partial_t u)|_{t=0} = (u_0, u_1)$.
It is well known that we have the principle of finite speed of propagation for (1):

if (u_0, u_1) vanishes outside B_r, then u vanishes outside B_{r+t} for $t \geq 0$.

nothing travels faster than light

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Duyckaerts, Kenig and Merle made the following simple but fundamental observation:
in many cases, one also has the following quantitative inverse statement

some energy travels with speed of light

$$\int_{|x|\geq r+|t|} |\nabla_{x,t} u|^2(x,t)\, dx \gtrsim \int_{|x|\geq r} |\nabla u_0|^2 + |u_1|^2\, dx, \text{ for all } t \geq 0 \text{ or all } t \leq 0. \tag{2}$$

Channel of energy inequalities

When (2) is true, the proof is often (not always) quite simple. But these inequalities turn out to be much more useful than they might firstly appear.
A crucial point of the channel of energy inequality is that it implies that a fixed portion of the energy moves out with speed exactly equal to 1, which is a sharp counterpart of the finite speed of propagation.

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This allows one to obtain crucial information on the general (perhaps large) solutions by looking at what happens in the exterior of the lightcone, where dynamics are easier.

Channel of energy

Let us briefly review some of the channel of energy inequalities: for all $t \geq 0$ or all $t \leq 0$,

$$\int_{|x|\geq|t|} |\nabla_{x,t} u|^2(x,t)\,dx \gtrsim \int_{R^d} |\nabla u_0|^2 + |u_1|^2\,dx; \tag{3}$$
$$\text{for odd d, DKM, 2012}$$

$$\int_{r\geq r_0+|t|} |\nabla_{r,t}(ru)|^2(r,t)\,dr \gtrsim \int_{r\geq r_0} |\partial_r(ru_0)|^2 + |ru_1|^2\,dr; \tag{4}$$
$$\text{for radial } u \text{ and } d=3,\ \text{DKM, 2012};$$

$$\int_{r\geq r_0+|t|} |\nabla_{r,t} u|^2(r,t)\,dx \gtrsim \|\pi^{\perp}_{P(r_0)}(u_0,u_1)\|^2_{\dot H^1\times L^2(|x|\geq r_0)}, \tag{5}$$
$$\text{for odd d. Kenig – Lawrie – Liu – Schlag, 2015}$$

where

$$P(r_0) := \{(r^{2k_1-d},0),(0,r^{2k_2-d}) : k_1 = 1,\dots,[\frac{d+2}{4}];\ k_2 = 1,\dots,[\frac{d}{4}];\ r\geq r_0\}.$$

Channel of energy inequalities

The above channel of energy inequalities apply in radial case and are sensitive to dimensions.

Recently, a new channel of energy type inequality was found that applies in all dimensions and in the non-radial case.

Fix $\beta \in (0,1)$, for "special" initial data $(u_0,\, u_1)$ such that

$$\|\partial_\theta u_0\|_{L^2} + \|(u_0, u_1)\|_{\dot{H}^1 \times L^2(B_{1+\delta} \setminus B_{1-\delta})} + \|\partial_r u_0 + u_1\|_{L^2} \leq \delta \|(u_0, u_1)\|_{\dot{H}^1 \times L^2},$$

with a sufficiently small $\delta > 0$, then

$$\int_{|x| \geq \beta + t} |\nabla_{x,t} u|^2(x,t)\, dx \gtrsim \|(u_0, u_1)\|^2_{\dot{H}^1 \times L^2}. \tag{6}$$

DJKM 2016

Initial data of the above type is "outgoing", and appears naturally in many situations. For instance, a linear wave at large times is of such type after an appropriate scaling. This channel of energy inequality has played an essential role in the proof of soliton resolution along a sequence of times for the energy critical nonlinear wave equation.

Channel of energy inequality

We will showcase some interesting applications.

Consider the defocusing energy critical wave equation with a potential

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The potential term Vu does not affect the regularity properties much. However, the long time behavior can be very different from the energy critical defocusing wave equations.

Channel of energy and Dynamics of defocusing energy critical wave equation with trapping potential

Equation (7) is globally wellposed in the energy space: for any initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$, there exists a unique solution $\overrightarrow{u} \in C([0,\infty), \dot{H}^1 \times L^2)$, with $u \in L^5_t L^{10}_x(R^3 \times [0, T))$ for any $T < \infty$.

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For appropriate choice of the trapping potential V, equation (7) admits many steady states, including Q, $-Q$–the ground states, and excited states which change sign.

The ground state is stable; small excited states are unstable; there are large stable excited states for some open set of V. Hence equation (7) admits *multiple stable regions*

Energy radiation

The channel of energy inequality has the following important consequence.

Theorem

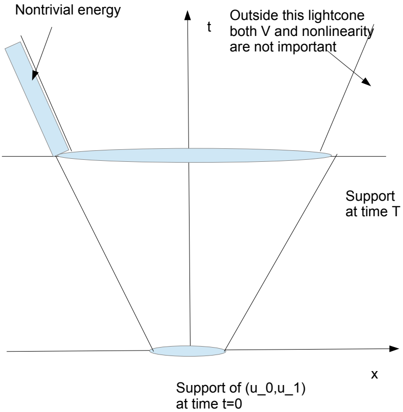
Suppose radial finite energy $(u_0, u_1) \not\equiv (u_c, 0)$ *for any steady state solution of equation (7). Let* $u \in C(R, \dot{H}^1) \cap L_t^5 L_x^{10}((-T, T) \times R^3)$ *for any* $T \in (0, \infty)$ *be the unique solution to equation (7) with* $\overrightarrow{u}(0) = (u_0, u_1)$*. Then there exists* $R > 0$ *such that*

$$\int_{|x| \geq R + |t|} |\nabla u|^2 + (\partial_t u)^2(t, x)\, dx \geq \delta > 0, \tag{8}$$

for all $t \geq 0$ *or all* $t \leq 0$*.*

Energy radiation

Compactly supported solutions emit energy to spatial infinity



Dynamics of solutions in the radial case I: scattering to steady states

Using this important fact, we obtained (J., B.P. Liu, G.X.Xu)

Theorem

Let u be the unique solution to equation (7) with initial data $(u(0), \partial_t u(0)) = (u_0, u_1)$. Then for some solution $(u^L, \partial_t u^L)$ to linear wave equation without potential and steady state $(\phi, 0)$ we have that $\overrightarrow{u}(t)$ scatters to $(\phi, 0)$, i.e.,

$$\lim_{t\to\infty} \|(u(t), \partial_t u(t)) - (\phi, 0) - (u^L(t), \partial_t u^L(t))\|_{\dot{H}^1\times L^2} = 0. \tag{9}$$

Heuristically the reason why solutions settle down to a steady state is clear: If the solution is not a steady state, then it will emit some amount of energy which then propagates to the "far field", and consequently the energy of the solution in the bounded region is reduced. This process repeats until the solution settles down to a steady state.

The actual proof uses a contradiction argument with the help of profile decompositions...

Dynamics of solutions in the radial case II: generic and non-generic behavior

Moreover, one can obtain much finer description of global dynamics.

Theorem (J.-Liu-Schlag-Xu, Generic and non-generic behavior)

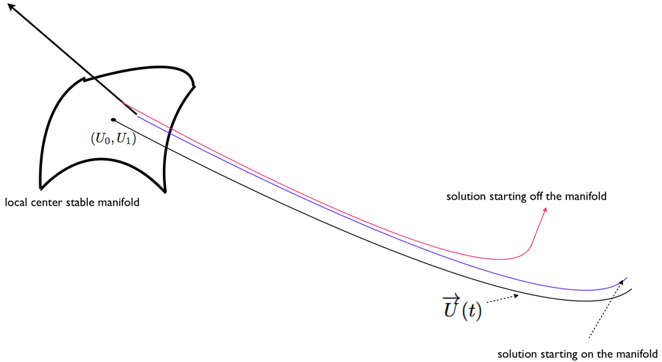
Suppose ϕ is a steady state solution. Denote $\overrightarrow{S}(t)$ as the solution operator to equation (7), and

$$\mathcal{M}_\phi := \{(u_0, u_1) \in \dot{H}^1_{\mathrm{rad}} \times L^2_{\mathrm{rad}} : \ \overrightarrow{S}(t)(u_0, u_1) \text{ scatters to } (\phi, 0)\}.$$

If ϕ is spectrally stable (i.e., no negative eigenvalues) then $\mathcal{M}_\phi$ is open.

If the linearized operator $-\Delta - V + 5\phi^4$ has n negative eigenvalues when restricted to radial functions, then $\mathcal{M}_\phi$ is a connected, C^1 manifold of co-dimension n in $\dot{H}^1_{\mathrm{rad}} \times L^2_{\mathrm{rad}}$.

Illustration of the idea of proof: local center stable manifold



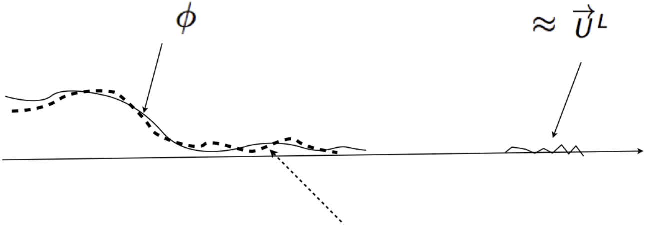
Main difficulty in the construction of the manifold: $\overrightarrow{U}(t)$ is not a steady state; presence of a large radiation term. Solution: Use endpoint Strichartz estimate, and dispersive estimate for the linearized operator.

Illustration of the idea of proof: second radiation

Suppose

$$\vec{U}(t) = (\phi, 0) + \vec{U}^L(t) + o_{\dot{H}^1_{\mathrm{rad}} \times L^2_{\mathrm{rad}}}(1) \quad \text{as } t \to \infty.$$

Decomposition of $\vec{u}(t)$ for large time T_1



a deformation of ϕ of at least a fixed amount of energy

the deformation leads to a second emission of energy, well separated from the first emission.

Soliton resolution for focusing energy critical wave and wave map equations

In the non-radial case, the situation is more complicated, and one has to combine the channel of energy inequality with additional monotonicity formula.
We will consider two important examples: the focusing energy critical wave equation

$$\partial_{tt} u - \Delta u = u^5, \tag{10}$$

in $R^3 \times [0, \infty)$,

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and the energy critical wave map from $R^{2+1} \to S^2$:

$$\partial_{tt} u - \Delta u = (|\nabla u|^2 - |\partial_t u|^2) u.$$

Soliton resolution for focusing energy critical wave and wave map equation

These two equations share many similarities: both admit many traveling wave solutions:

$$Q_\ell(x,t) = Q\left(x - \frac{x\cdot\ell}{|\ell|}\frac{\ell}{|\ell|} + \frac{\frac{x\cdot\ell}{|\ell|}\frac{\ell}{|\ell|} - \ell t}{\sqrt{1-|\ell|^2}}\right),$$

where Q is a steady state (or harmonic maps) and $|\ell| < 1$. Q_ℓ travels in the direction of ℓ with speed $|\ell|$.

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Both have conserved energy:

$$\int \frac{|\nabla_{x,t}u|^2}{2} - \frac{u^6}{6}(x,t)dx$$

for focusing energy critical wave equation and

$$\int \frac{|\nabla_{x,t}u|^2}{2}(x,t)dx$$

for wave maps. Both equations are locally wellposed in the energy space.

Soliton resolution conjecture

For focusing energy critical wave equation, the energy is not coercive. We focus on the so called Type II solutions, i.e, solution $\vec{u}$ with

$$\sup_{t\in[0,T_+)} \|\vec{u}(t)\|_{\dot{H}^1\times L^2} < \infty.$$

Solutions are called type I, if they are not type II.

It turns out that type II solutions model the dynamics of energy critical wave maps extremely well. So we consider this case first.

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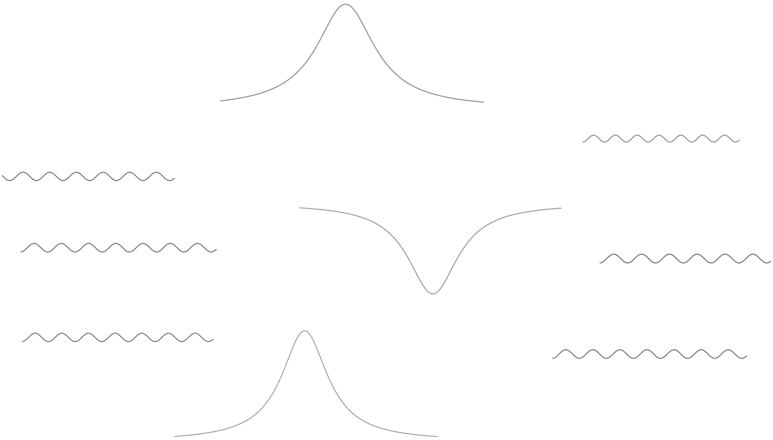
In the direction of characterizing type II behavior, an ambitious goal is to prove the soliton resolution conjecture:
Any type II solution $\vec{u}(t)$ can be decomposed as a finite sum of modulated solitons, a linear wave, plus a term which vanishes asymptotically in the energy space as $t \to T_+$.

Soliton resolution conjecture



Soliton resolution conjecture

More preciely, the conjecture predicts that

$$\vec{u}(t) = \sum_{j=1}^{J} \left(\lambda_j(t)^{-\frac{1}{2}} Q_{\ell_j} \left(\frac{x - x_j(t)}{\lambda_j(t)} \right), \lambda_j(t)^{-\frac{3}{2}} \partial_t Q_{\ell_j} \left(\frac{x - x_j(t)}{\lambda_j(t)} \right) \right) + \vec{u}^L(t) + \vec{\epsilon}(t),$$

where u^L is a linear wave, and $(\epsilon(t), \partial_t \epsilon(t)) = o_{\dot{H}^1 \times L^2}(1)$ as $t \to T_+$.

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The soliton resolution conjecture was settled in a remarkable work of Duyckaerts, Kenig and Merle, in the radial case for $d = 3$. The nonradial case is still wide open. However we have the following partial result.

Soliton resolution along a sequence of times, nonradial case

Theorem (Duyckaerts, J., Kenig and Merle 2016)

Let $\vec{u}$ be a Type II blow up solution. Define the singular set

$$\mathcal{S} := \left\{ x_* \in R^d : \|u\|_{L_t^{\frac{d+2}{d-2}} L_x^{2\frac{d+2}{d-2}} (B_\epsilon(x_*)\times[T_+-\epsilon,\, T_+))} = \infty, \quad \text{for any } \epsilon > 0 \right\}. \tag{11}$$

Then $\mathcal{S}$ is a set of finitely many points only. Then near a singular point, we have

$$\vec{u}(t_n) = \vec{v} + \sum_{j=1}^{J_*} \left((\lambda_n^j)^{-\frac{d}{2}+1} Q_{\ell_j}\left(\frac{x-c_n^j}{\lambda_n^j},\, 0\right),\, (\lambda_n^j)^{-\frac{d}{2}} \partial_t Q_{\ell_j}\left(\frac{x-c_n^j}{\lambda_n^j},\, 0\right)\right) + o_{\dot{H}^1\times L^2}(1), \tag{12}$$

as $n \to \infty$.

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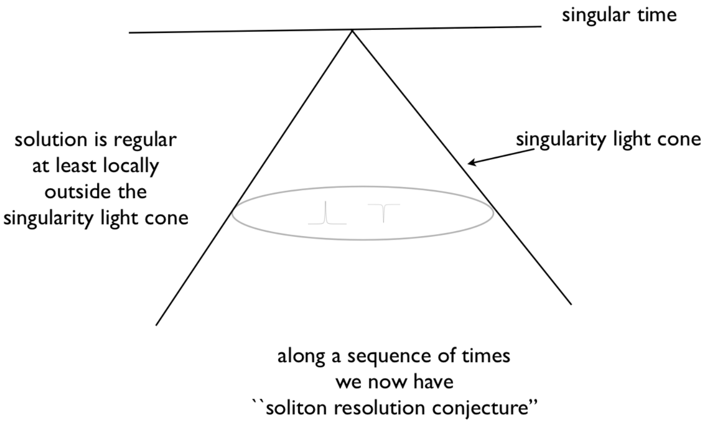
as $n \to \infty$.

There is a corresponding version for global Type II solutions.

Soliton resolution along a sequence of times, singular case



proof

The proof consists of several ingredients:
profile decompositions+Morawetz estimates+virial identites+channel of energy inequality.
The Morawetz estimate implies that

$$\int_{t_1}^{t_2}\int_{|x|<T_+-t}\left(\partial_t u+\frac{x}{T_+ - t}\cdot\nabla u+\left(\frac{d}{2}-1\right)\frac{u}{T_+ - t}\right)^2 dx\,\frac{dt}{T_+ - t}\leq C\left(\log\frac{T_+ - t_1}{T_+ - t_2}\right)^{\frac{d}{d+1}}, \qquad (13)$$

for some C independent of $T_+ > t_2 > t_1 > 0$. Such estimates have played a decisive role in "ground state conjecture" for wave maps, see Tao, Sterbernz-Tataru, and earlier work of Grillakis.

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Roughly speaking, the Morawetz estimates+virial identity+perturbation theory already allows to conclude:

1. $\overrightarrow{u}(t_n)$ = a regular part+ travelling waves + a residue term $\overrightarrow{\epsilon}_n = (\epsilon_{0n}, \epsilon_{1n})$.

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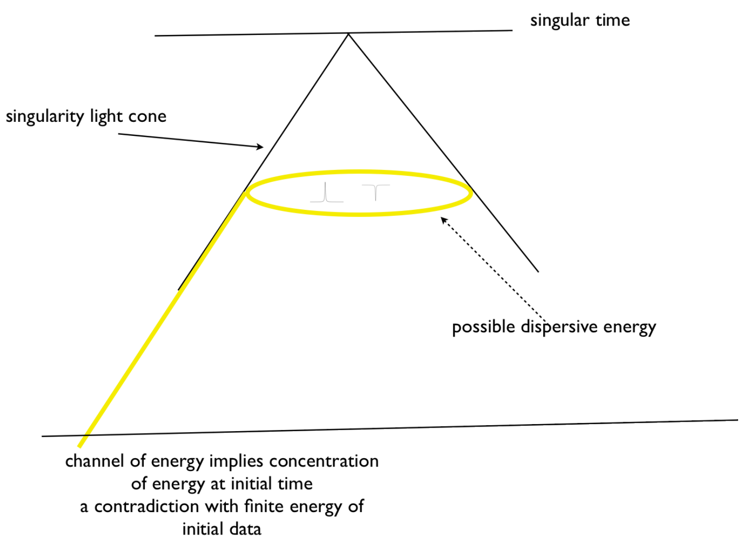
1. $\overrightarrow{u}(t_n)=$ a regular part+ travelling waves + a residue term $\overrightarrow{\epsilon}_n=(\epsilon_{0n},\epsilon_{1n})$.
2. The residue term satisfies "incoming" condition.

Elimination of dispersive energy, illustrated



Wave maps

Similar ideas have recently been applied to energy critical wave map equations from $R^{2+1} \to S^2$:

$$\partial_{tt} u - \Delta u = (|\nabla u|^2 - |\partial_t u|^2) u,$$

in the restricted setting when the energy of the map is only slightly greater than that of co-rotational harmonic maps.

For wave maps, the energy

$$\int |\nabla_{x,t} u|^2 (x, t) dx$$

is conserved and coercive, hence there is no need to restrict to type II solutions.

Wave maps

There is one complication in comparison to the semilinear wave equation we have considered so far. That is, even at small energy, the nonlinearity is not perturbative–hence the solution does not really behave linearly. Consequently the scattering statement is quite a bit more complicated.

A natural formulation for soliton resolution conjecture for the energy critical wave maps, in the global existence case, seems to be that as time becomes large, the energy splits itself into traveling waves, and a part de-coupling from the traveling waves and concentrating at $|x| = t + O(1)$.

The channel of energy for outgoing wave maps also play an essential role. It was proved by Duyckaerts-J-Kenig-Merle, that the channel of energy inequality for small outgoing wave maps holds, but the case of large energy is still open.

Wave maps

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This is in sharp contrast to the semilinear case

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where the outgoing condition on the initial data immediately implies that u^L (the free evolution) satisfies

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Wave maps

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There is no such an easy way to gain smallness for wave maps, at this moment.

Thank you!