A_∞ structures as a language for open Gromov-Witten theory

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Gromov-Witten theory $(g = 0)$

Setting: (X, ω, J) symplectic manifold with almost complex structure

$X = X^{2n}$, ω 2-form such that $\omega^{\wedge n}$ is a volume form

$J \in End(TX), J^2 = -\mathrm{Id}$, "$\omega$-tame"

Example: $(\mathbb{C}P^n, \omega_{FS}, J_0)$

***Problem*:** Count J-holomorphic maps from the sphere

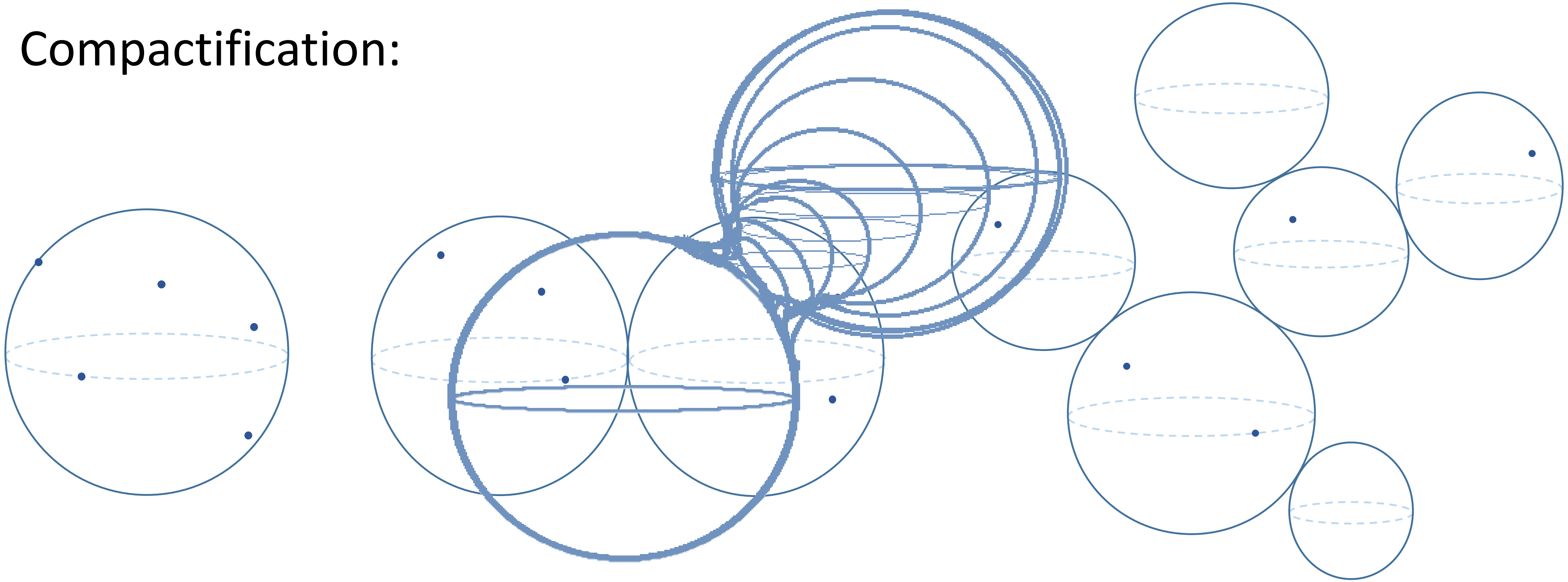
$$u: S^2 \to X$$

that satisfy various constraints

The moduli space of sphere maps

$$\overline{\mathcal{M}}_l(\beta) = \overline{\left\{ \left(u : S^2 \xrightarrow{J-hol.} X, w_1, \dots, w_l \right) : \begin{array}{c} [u] = \beta \in H_2(X;\mathbb{Z}) \\ w_j \in S^2, w_i \neq w_j \end{array} \right\} / \sim}$$

Compactification:



Rephrasing the problem

Count elements of $\overline{\mathcal{M}}_l(\beta)$ such that the marked points are mapped to given constraints.

Can be expressed as an integral:

$$GW_\beta(\gamma_1, \dots, \gamma_l) = \int_{\overline{\mathcal{M}}_l(\beta)} ev_1^*\gamma_1 \wedge \cdots \wedge ev_l^*\gamma_l \, .$$

Some facts

- GW invariants are defined by the above integral if the space $\overline{\mathcal{M}}_l(\beta)$ is "nice"
- GW are generally hard to compute
- In some cases, can compute GW invariants by the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equation

Kontsevich (1994)

degree = d	No. of degree-d curves in $\mathbb{C}P^2$ through 3d-1 points
1	1
2	1
3	12
4	620
5	87,304
6	26,312,976
7	14,616,808,192

Open Gromov-Witten theory $(g = 0)$

Setting: (X, ω, J) symplectic manifold with almost complex structure

$L \subset X$ a Lagrangian submanifold $\left(\dim L = \frac{1}{2}\dim X\,, \omega|_L = 0\right)$

Example: $(X, L, \omega, J) = (\mathbb{C}P^n, \mathbb{R}P^n, \omega_{FS}, J_0)$

***Problem*:** Count J-holomorphic maps from the disk

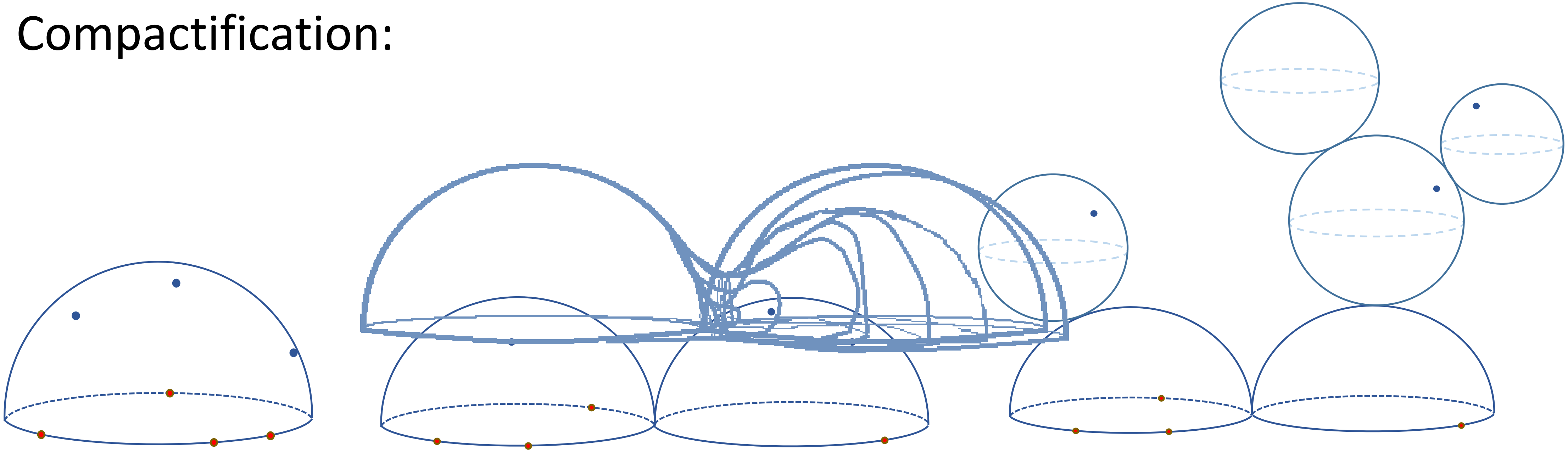
$$u: (D, \partial D) \to (X, L)$$

that satisfy various constraints.

The moduli space of disk maps

$$\overline{\mathcal{M}}_{k,l}(\beta) = \overline{\left\{ \begin{pmatrix} u:(D,\partial D) \xrightarrow{J-hol.} (X,L), \\ z_1,\dots,z_k,w_1,\dots,w_l \end{pmatrix} : \begin{matrix} [u]=\beta \in H_2(X,L;\mathbb{Z}) \\ z_i \in \partial D, w_j \in int(D) \end{matrix} \right\} / \sim}$$

Compactification:



Rephrasing the problem

Count elements of $\overline{\mathcal{M}}_{k,l}(\beta)$ such that the marked points are mapped to given constraints

Can be expressed as an integral:

$$OGW_\beta(\alpha_1, \dots, \alpha_k; \gamma_1, \dots \gamma_l) =$$
$$= \int_{\overline{\mathcal{M}}_{k,l}(\beta)} evb_1^*\alpha_1 \wedge \cdots \wedge evb_k^*\alpha_k \wedge evi_1^*\gamma_1 \wedge \cdots \wedge evi_l^*\gamma_l \, .$$

<u>**Issue**</u>**:** $\partial\overline{\mathcal{M}}_{k,l}(\beta) \neq \emptyset$.

Some previous results

OGW are defined when

- S^1 acts on (X, L) *(Liu, 2004)*
- (X, L, ω, J) is a real symplectic manifold with $\dim_{\mathbb{C}} X = 2,3$, real interior constraints, point boundary constraints *(Cho, Solomon, 2006)*
- (X, L, ω, J) is a real symplectic manifold with $\dim_{\mathbb{C}} X$ odd, no boundary constraints *(Georgieva, 2013)*

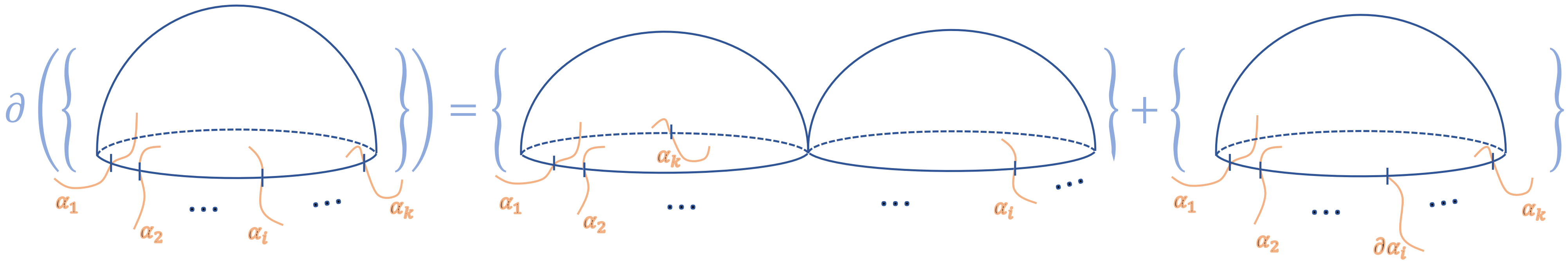
OGW are computable via a WDVV-like equation when

- (X, L, ω, J) is a real symplectic manifold with $\dim_{\mathbb{C}} X = 2$, real interior constraints, point boundary constraints *(Horev-Solomon, 2012)*
- (X, L, ω, J) is a real symplectic manifold with $\dim_{\mathbb{C}} X$ odd, no boundary constraints *(Georgieva-Zinger, 2013)*

A_∞ structure

= Algebraic language to describe boundary behavior

- A_∞ operators describe disks with prescribed boundary constraints
- A_∞ relations describe disk bubbling



- Special kind of boundary constraint: “bounding chain”

More results (joint with Jake Solomon)

- OGW can be defined using bounding chains when $\dim_{\mathbb{C}} X$ is odd, under cohomological conditions. E.g., $H^*(L;\mathbb{R}) = H^*(S^n;\mathbb{R})$.
- The boundary constraints can be interpreted as points.
- Whenever defined, OGW satisfy open WDVV equations.
- For $(\mathbb{C}P^n, \mathbb{R}P^n)$, all invariants are determined by the open WDVV.

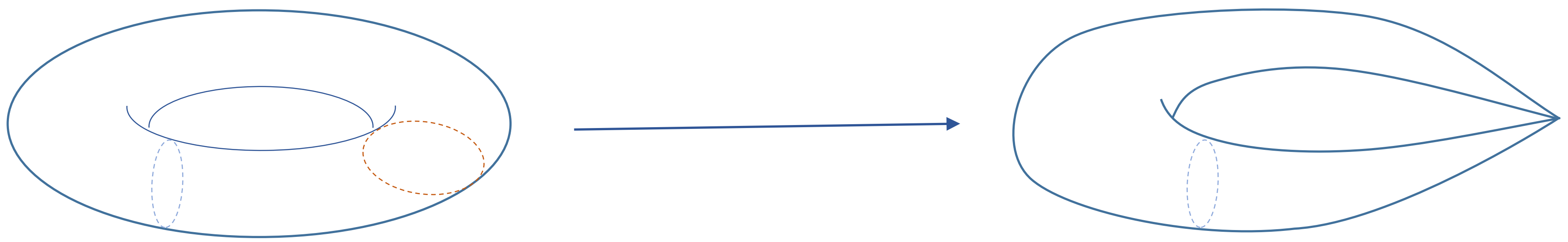
$(X, L) = (\mathbb{C}P^n, \mathbb{R}P^n)$

Initial condition: $\boldsymbol{OGW^n_{1,2} = 2}$

dim = n	degree = d	No. of boundary points = k	Resulting invariant $OGW^n_{d,k}$
3	3	6	-2
	5	10	90
	7	14	-29,178
	9	18	35,513,586
5	5	8	-2
	9	14	1974
	13	20	-42,781,410
	17	26	7,024,726,794,150
7	7	10	-2
	13	18	35,498
	19	26	-40,083,246,650
	25	34	680,022,893,749,060,370
9	9	12	-2
	17	22	587,334
	25	32	-31,424,766,229,890
	33	42	49,920,592,599,715,322,910,150
15	29	34	2,247,512,778

More questions

- Reduce cohomological assumptions
- Find structure suitable for $g > 0$



- Explore relative quantum cohomology

open WDVV $\leftrightarrow$ associativity of relative quantum product

Thank you