Nonlinear Brownian motion and nonlinear Feynman-Kac formula of path-functions

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Non-equilibrium Dynamics and Random Matrices
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Recall: Heat equation

A classical heat equation:

$$\partial_t u(t,x) + \frac{1}{2}\Delta u(t,x) = 0, \qquad t \in [0,T),$$
$$u(T,x) = \varphi(x), \quad x \in \mathbb{R}^d.$$

Consider u as a function of path $u(t,\omega(t))$, $\omega \in \Omega = C_0^d([0,T])$.

Heat equation on path space

- Solve the heat equation for solutions defined on the path space Ω of the form

$$u = u(t, \omega) = u(t, \omega(s)_{0\leq s\leq t}), \quad \text{given: } u(T, \omega) = \varphi(\omega(s)_{0\leq s\leq T}).$$

- Simple and typical situation: u is finite dimensional function of (t, ω)

$$D_t u(t,\omega) + \frac{1}{2} tr[D_x^2 u(t,\omega)] = 0, \qquad t \in [0, T), \tag{1}$$
$$u(T,\omega) = \varphi(\omega(t_1), \cdots, \omega(t_n)) \in C^\infty_{f.d.}(\Omega_T).$$

- Many stochastic processes has a form $u(t, \omega(s)_{s\in[0,t]})$, typically: Itô's process.

Derivatives of path process

- Define a 'good' $D_t u(t,\omega)$, $D_x u(t,\omega)$, $D_x^2 u(t,\omega)$ for

$$u = u(t,\omega) = u(t, \omega(s)_{0\le s\le t})$$

- Begin from a very simple case, for $0 = t_0 < t_1 < \cdots < t_n = T$,

$$u(t,\omega) = \begin{cases} u_{\omega(t_1),\cdots,\omega(t_{n-1})}(t,\omega(t)), & t \in [t_{n-1}, t_n] \\ \vdots & \\ u_{\omega(t_1),\cdots,\omega(t_{k-1})}(t,\omega(t)), & t \in [t_{k-1}, t_k] \\ \vdots & \\ u(t,\omega(t)), & t \in [0, t_1] \end{cases}$$

with $u_{\omega(t_1),\cdots,\omega(t_{k-1})}(\omega(t_k), t_k) = u_{\omega(t_1),\cdots,\omega(t_k)}(\omega(t_k), t_k)$

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with $u_{\omega(t_1),\cdots,\omega(t_{k-1})}(\omega(t_k),t_k) = u_{\omega(t_1),\cdots,\omega(t_k)}(\omega(t_k),t_k)$

Solving path PDE

To find $u \in C^{1,2}_{f.d}(0,T)$ ($u_{\omega(t_1),\cdots,\omega(t_{k-1})}(t,x)$, $C^{1,2}$-function of (t,x)), solution of (1).

$$
\begin{aligned}
D_t u(t,\omega) &:= \partial_t u_{\omega(t_1),\cdots,\omega(t_{k-1})}(t,\omega(t)),\\
D_x u(t,\omega) &=: \partial_x u_{\omega(t_1),\cdots,\omega(t_{k-1})}(t,\omega(t)),\\
D_x^2 u(t,\omega) &=: \partial_x^2 u_{\omega(t_1),\cdots,\omega(t_{k-1})}(t,\omega(t)), \qquad t \in [t_{k-1}, t_k)
\end{aligned}
$$

Expectation induced by through path PDE

- $$D_t u(t,\omega) + \frac{1}{2} D_x^2 u(t,\omega) = 0,$$
  $$u(T,\omega) = \varphi(\omega(t_1), \cdots, \omega(t_n)) \in C^\infty_{f.d.}(\Omega_T)$$
- We define $\mathbb{E}[\varphi(\omega)] := u(0, \omega(0))$. $\mathbb{E}[\cdot] : C^\infty_{f.d.}(\Omega_T) \mapsto \mathbb{R}$ is a linear functional s.t.

$$\begin{array}{ll} \mathbb{E}[\varphi(\omega)] \geq 0, & \text{if } \varphi \geq 0 \\ \mathbb{E}[c] = c, & c \text{ is constant} \\ \mathbb{E}[\varphi_i(\omega)] \downarrow 0 & \text{if } \varphi_i(\omega) \downarrow 0, \end{array}$$

Wiener expectation and Wiener probability measure

By Daniell-Stone theorem, there exists a unique probability measure P on $(\Omega, \mathcal{B}(\Omega)) = (\Omega, \mathcal{F})$ such that

$$\mathbb{E}[\varphi(\omega)] = \int_\Omega \varphi(\omega) dP.$$

$B_t(\omega) = \omega(t)$ $(\sim B_{h+t} - B_h$ indep. $(B_{h_1}, \cdots, B_{h_n}))$ is a Brownian motion under P, P is a Wiener process.

Nonlinear path PDE

Generalization to obtain sharper and more powerful tool: Consider HJB equation.

$$\partial_t u(t,x) + G(D^2 u(t,x)) = 0, \quad t \in [0,T), \qquad \text{(G-equ.)}$$
$$u(T,x) = \varphi(x), \quad x \in \mathbb{R}.$$

with $G(\alpha) = \frac{\overline{\sigma}^2}{2}\alpha^+ - \frac{\underline{\sigma}^2}{2}\alpha^-$, $\overline{\sigma} \geq 1 \geq \underline{\sigma} > 0$. Solve path G-equation permits us to introduce a new type of Brownian motion under probability model uncertainty.

Nonlinear PDE and nonlinear expectation

Solve $u \in C^{1,2}_{f,d}(0,T)$, solution of,

$$
\begin{aligned}
D_t u(t,\omega) + G(D_x^2 u(t,\omega)) &= 0, \qquad t \in [0,T), \\
u(T,\omega) &= \varphi(\omega(t_1),\cdots,\omega(t_n)) \in C^{\infty}_{f.d.}(\Omega_T).
\end{aligned}
$$

Then we define $\hat{\mathbb{E}}[\varphi(\omega)] = u(0,\omega(0))$. $\mathbb{E}[\cdot] : C^{\infty}_{f.d.}(\Omega_T) \mapsto \mathbb{R}$ is a sublinear functional s.t.

$$
\begin{aligned}
&\hat{\mathbb{E}}[\varphi(\omega)] \geq \hat{\mathbb{E}}[\psi(\omega)], && \text{if } \varphi \geq \psi \\
&\hat{\mathbb{E}}[c] = c, && c \text{ is constant} \\
&\mathbb{E}[\varphi_i(\omega)] \downarrow 0 && \text{if } \varphi_i(\omega) \downarrow 0.
\end{aligned}
$$

(P. 2005-2010)

Representation theorem

By Daniell-Stone and Hahn-Banach theorems, there exists a family of probability measures $\{P_\theta\}_{\theta\in\Theta}$ on $(\Omega, \mathcal{F})$ such that

$$\hat{\mathbb{E}}[\varphi(\omega)] = \sup_{\theta\in\Theta} \int_\Omega \varphi(\omega) dP_\theta.$$

$B_t(\omega) = \omega(t)$ ($\sim B_{h+t} - B_h$ indep. $(B_{h_1}, \cdots, B_{h_n})$), namely, $B_t(\omega)$ is a Brownian motion under the sublinear expectation $\hat{\mathbb{E}}$.

Sobolev Space induced through path PDE

Moreover $\|\cdot\|_{L_G^p} := \hat{\mathbb{E}}[|\cdot|^p]^{1/p}$ forms a norm on $C^\infty_{f.d.}(\Omega_T)$.

$$\|u\|_{S_G^p} := \left\| \sup_{0\le t\le T} |u(t,\omega)| \right\|_{L_G^p},$$

$$\|u\|_{H_G^p} := \left\| (\int_0^T |u(t,\omega)|^p ds)^{1/p} \right\|_{L_G^1},$$

$$\|u\|_{M_G^p} := \left\| \int_0^T |u(t,\omega)|^p ds \right\|_{L_G^1}^{1/p}$$

$$\|u\|_{W_G^{1,2;p}(0,T)} := \|u\|_{S_G^p} + \|D_x u\|_{H_G^p} + \left\| |D_t u| + |D_x^2 u| \right\|_{M_G^p}$$

$\|u\|_{W_G^{1,2;p}(0,T)}$ forms a "Sobolev norm" on $C^\infty_{f.d.}(0,T)$.

Itô's calculus in $W_G^{1,2;p}(0, T)$ space

Itô's integrals are well defined:

- $\int_0^T \beta(s,\omega)dB_s, \quad \beta \in H_G^p(0,T)$

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- $\int_0^t \gamma(s,\omega)\, \langle B \rangle_s, \quad \gamma \in M_G^p(0,T)$

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- $\int_0^t \gamma(s,\omega) \langle B \rangle_s, \quad \gamma \in M_G^p(0,T)$
- $\langle B \rangle_t := B_t^2 - 2\int_0^t B_s dB_s = \lim_{\Delta_N \to 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2$.

Distinguishability of Itô process by G-expectation

Itô's process

$$u(t,\omega) = u(0,0) + \int_0^t \alpha(s,\omega)ds + \int_0^t \beta(s,\omega)dB_s + \frac{1}{2}\int_0^t \gamma(s,\omega)\,\langle B\rangle_s$$
$$\alpha,\gamma \in M_G^p(0,T), \quad \beta \in H_G^p(0,T).$$

is well defined (P. 2006, 2010).

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$u \equiv 0 \iff$

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$u \equiv 0 \iff u(0,0) = 0$ and $\alpha, \beta, \gamma \equiv 0$ □

Itô's process $\iff$ Itô's formula

For each $u \in C^{1,2}([t_k, t_{k+1}] \times \mathbb{R})$, $t \in [t_k, t_{k+1})$

$$u(t, B_t) - u(t_k, B_{t_k}) = \int_{t_k}^{t} \partial_s u(s, B_s) ds + \int_{t_k}^{t} \partial_x u(s, B_s) dB_s + \frac{1}{2} \int_{t_k}^{t} \partial_{xx}^2 u(s, B_s) d \langle B \rangle_s$$

Consequently, for each $u \in C^{\infty}_{f.d}(0, T)$, we have

$$u(t, B_{\cdot}) - u(0, B_0) = \int_0^t D_s u(s, B_{\cdot}) ds + \int_0^t D_x u(s, B_{\cdot}) dB_s + \frac{1}{2} \int_0^t D_{xx}^2 u(s, B_{\cdot}) d \langle B \rangle_s .$$

Closabilility of D_t, D_x and D_x^2

Proposition(P. &Song2013).

The norm $\|\cdot\|_{W_G^{1,2;p}}$ is closable in the space $S_G^p(0,T)$: Let $u_n \in C_{f.d}^\infty(0,T)$ be a Cauchy sequence w.r.t. the norm $\|\cdot\|_{W_G^{1,2;p}}$, if $\|u_n\|_{S_G^p} \to 0$, then $\|u_n\|_{W_G^{1,2;p}} \to 0$. □

Path functions and Itô's formula of path functions

Proposition(P. &Song2013) .

Assume that $u \in S_G^p(0, T)$. Then the following two conditions are equivalent:
(i) $u \in W_G^{1,2;p}(0, T)$;
(ii) u is a Itô process: there are $\alpha, \gamma \in M_G^p(0, T)$, $\beta \in H_G^p(0, T)$, such that

$$u(t,\omega) = u(0,0) + \int_0^t \alpha(s,\omega)ds + \int_0^t \beta(s,\omega)dB_s + \frac{1}{2}\int_0^t \gamma(s,\omega)\, \langle B \rangle_s .$$

Moreover, we have

$$D_t u(t,\omega) = \alpha(t,\omega), \quad D_x u(t,\omega) = \beta(t,\omega), \quad D_x^2 u(t,\omega) = \gamma(t,\omega).$$

□

Examples

Example

$u(t,\omega) = \int_0^t \alpha(s,\omega)ds$: $D_t u(t,\omega) = \alpha(t,\omega)$, $D_x u(t,\omega) \equiv 0$, $D_x^2 u(t,\omega) \equiv 0$,

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$u(t,\omega) = \int_0^t \beta(s,\omega)dB_s$: $D_t u(t,\omega) = 0$, $D_x u(t,\omega) \equiv \beta(t,\omega)$, $D_x^2 u(t,\omega) \equiv 0$,

$$\text{if } \beta(s,\omega) = B(s,\omega), \text{ then } D_x\beta(t,\omega) = 1.$$

In general $D_x(D_x u(t,\omega)) \neq D_{xx}^2 u(t,\omega)$!

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Example

$u(t,\omega)=\int_0^t \beta(s,\omega)dB_s$: $D_t u(t,\omega)=0$, $D_x u(t,\omega)\equiv \beta(t,\omega)$, $D_x^2 u(t,\omega)\equiv 0$,

$$\text{if } \beta(s,\omega)=B(s,\omega), \text{ then } D_x\beta(t,\omega)=1.$$

In general $D_x(D_x u(t,\omega))\neq D_{xx}^2 u(t,\omega)$!

Example

$u(t,\omega)=\langle B\rangle_t(\omega)$: $D_t u(t,\omega)=0$, $D_x u(t,\omega)\equiv 0$, $D_x^2 u(t,\omega)\equiv 2$.

SDE (stochastic differential equation) and related path PDE

$$X_t(\omega) = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s + \int_0^t \kappa(X_s)d\langle B\rangle_s . \quad \text{(SDE)}$$

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Theorem

We assume that $b(x)$, $\sigma(x)$ and $\kappa(x)$ are all Lipschitz functions of x. Then there exists a unique solution $X \in S_G^p(0,T)$ ([Peng2006,2008]).

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Theorem

We assume that $b(x)$, $\sigma(x)$ and $\kappa(x)$ are all Lipschitz functions of x. Then there exists a unique solution $X \in S_G^p(0,T)$ ([Peng2006,2008]). Consequently $u(t,\omega) := X_t(\omega) \in W_G^{1,2;p}(0,T)$, and it is the unique solution of the PDE

$$D_t u(t,\omega) = b(u(t,\omega)), \quad D_x u(t,\omega) = \sigma(u(t,\omega)),$$
$$D_{xx}^2 u(t,\omega) = 2\kappa(u(t,\omega))$$

with initial condition $u(0,0) = X_0 \in \mathbb{R}$.

Backward SDE:

$$Y_t(\omega) = \xi(\omega) + \int_t^T f(Y_s, Z_s, \eta_s)ds - \int_t^T Z_s dB_s - (K_T - K_t), \tag{BSDE}$$

$$K_t = \frac{1}{2}\int_0^t \eta_s d\langle B\rangle_s - \int_0^t G(\eta_s)ds.$$

Problem: Given $\xi \in L_G^p(\Omega_T)$ and Lipschitz function $f(y, z, \eta)$, to find a unique triple

$$Y \in S_G^p(0, T), \; Z \in H_G^p(0, T) \text{ and } \eta \in M_G^p(0, T)$$

which solve the (BSDE).

BSDE $\Longleftrightarrow$ Path PDE

Theorem (P.&Song2013)

Let (Y, Z, η) *be a solution of the above BSDE. Then* $u(t,\omega) := Y_t(\omega) \in W_G^{1,2;p}$ *is a solution of*

$$D_t u(t,\omega) + G(D^2_{xx} u(t,\omega)) + f(u, D_x u, D^2_x u)(t,\omega) = 0,$$
$$u(T,\omega) = \xi(\omega).$$

with $D_x u(t,\omega) = Z(t,\omega)$ *and* $D^2 u(t,\omega) = \eta(t,\omega)$*. Conversely, if* $u(t,\omega) \in W_G^{1,2;p}(0,T)$ *is a solution of this PDE. Then* $(Y, Z, \eta) = (u, D_x u, D^2 u)(t,\omega)$ *is a solution of (BSDE).*

Space $W_G^{\frac{1}{2},1;p}(0,T)$, $(W_{\mathcal{A}_G}^{\frac{1}{2},1;p}(0,T))$

Weaker solution in $u \in W_G^{\frac{1}{2},1;p}(0,T)$ write in the form

$$u(t,\omega) = u_0 + \int_0^t \mathcal{A}_G u(s,\omega)ds + \int_0^t D_x u(s,\omega)dB_s + K_t(\omega)$$

$$\mathcal{A}_G u := D_s u + G(D_x^2 u(s,\omega)),$$

$$K_t(\omega) := \frac{1}{2}\int_0^t D_x^2 u(s,\omega)d\langle B\rangle_s - \int_0^t G(D_x^2 u(s,\omega))ds.$$

$$d_{W_G^{\frac{1}{2},1;p}}(u,v) = \|u-v\|_{S_G^p} + \|\mathcal{A}_G u - \mathcal{A}_G v\|_{M_G^p} + \|D_x(u-v)\|_{H_G^p}$$

The path PDE:

$$D_t u + G(D_x^2 u) + f(u,Du) = \mathcal{A}_G u(t,\omega) + f(u,D_x u)(t,\omega) = 0.$$

Feynman-Kac formula of fully nonlinear path PDE

Theorem (P.&Song2014)

(i) If (Y, Z, K) is the solution of backward SDE

$$Y_t(\omega) = \xi(\omega) + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t)$$

Then $u(t, \omega) := Y_t(\omega) \in W_G^{\frac{1}{2},1;p}(0, T)$ is the sol. of the path PDE

$$D_t u(t, \omega) + G(D_x^2 u(t, \omega)) + f(u, D_x u)(t, \omega) = 0, \quad u(T, \omega) = \xi(\omega).$$

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(ii) Conversely, if $u \in W_G^{\frac{1}{2},1;p}(0, T)$ solves the above PDE, then $(Y, Z, K) = (u, D_x u, K)$ solves the BSDE, where

$$K_t = u(t, \omega) + \int_0^t f(u, D_x u)ds - \int_0^t D_x u(s, \omega)dB_s$$

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Existence and uniqueness of BSDE: obtained in [Hu-Ji-P.&Song2012].

Case without probability measure uncertainty: $G(a) = \frac{a}{2}$

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Case without probability measure uncertainty: $G(a) = \frac{a}{2}$

- SDE: $dX = b(X)dt + \sigma(X)dB_t + \kappa(X)d\langle B\rangle_t$,
- But we are unable to distinguish dt part and $d\langle B\rangle$ part
- $dX = (b(X) + \kappa(X))dt + \sigma(X)dB_t$, P-a.s.
- The corresponding PDE: $u(t,\omega) = X_t(\omega)$:

$$D_t u(t,\omega) + D^2_{xx}u(t,\omega) = b(u(t,\omega)) + \kappa(u(t,\omega)),$$
$$D_x u(t,\omega) = \sigma(u(t,\omega)), \quad u(0,\omega(0)) = X_0.$$

- This PDE is unique under the Wiener measure P, provided the initial condition $X_0 \in \mathbb{R}$.

BSDE and path PDE under Wiener measure P

3) For BSDE

$$Y_t(\omega) = \xi(\omega) + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t)$$

Since $K_t = \frac{1}{2}\int_0^t D_x^2 u(s,\omega)d\langle B\rangle_s - \int_0^t G(D_x^2 u(s,\omega))ds \equiv 0$, thus the term K_t disappear under the Wiener measure P. The equation becomes the classical BSDE: to solve (Y, Z), the solution of

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BSDE and Feynman-Kac formula

- The corresponding path-PDE becomes

$$\partial_t u(t,\omega) + \frac{1}{2}\Delta u(t,\omega) + f(u, D_x u)(t,\omega) = 0, \qquad u(T,\omega) = \xi(\omega).$$

- The solution is defined under the norm

$$\|u\|_{W_P^{\frac{1}{2},1;p}(0,T)} = \|u\|_{S_P^p(0,T)} + \left\|(D_t + \frac{1}{2}\Delta)u\right\|_{M_P^p(0,T)} + \|D_x u\|_{H_P^p(0,T)}$$

which is closable in $S_P^p(0,T)$.

- The smooth solution is given in (P&Wang2011) using a very different (BSDE) method

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Framework of Nonlinear (G) Expectation: $G(a) = \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$

Probability Space	Nonlinear Expectation Space
$(\Omega, \mathcal{F}, P)$	$(\Omega, \mathcal{H}, \mathbb{E})$: (sublinear is basic)
Distributions: $X \overset{d}{=} Y$	
Independence: Y indep. of X	
LLN and CLT	
Normal distributions	
Brownian motion $B_t(\omega) = \omega_t$	
Quadratic variable. $\langle B \rangle_t = t$	
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Framework of Nonlinear Expectation: nonlinear expectation

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Martingales	nonlinear Martingales
$E[X\|\mathcal{F}_t] = E[X] + \int_0^T z_s dB_s$	$\mathbb{E}[X\|\mathcal{F}_t] = \mathbb{E}[X] + \int_0^t z_s dB_s + K_t$ $K_t = \int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s)ds$

Probability Space	Nonlinear Expectation Space
P-almost surely analysis	
$X(\omega)$: P-quasi continuous $\iff$ X is $\mathcal{B}(\Omega)$-meas.	

Probability Space	Nonlinear Expectation Space
P-almost surely analysis	$\hat{c}$-quasi surely analysis
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Probability Space	Nonlinear Expectation Space
P-almost surely analysis	$\hat{c}$-quasi surely analysis
	$\hat{c}(A) = \sup_\theta E_{P_\theta}[\mathbf{1}_A]$
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$X(\omega)$: P-quasi continuous $\iff$ X is $\mathcal{B}(\Omega)$-meas.	$X(\omega)$: $\hat{c}$-quasi surely continuous $\implies$ X is $\mathcal{B}(\Omega)$-meas.

Central Limit Theorem (CLT) under Knightian Uncertainty

Theorem

Let $\{X_i\}_{i=1}^{\infty}$ *in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *be i.i.d.:* $X_i \sim X_1$ *and* X_{i+1} *Indep.* $(X_1, \cdots, X_i)$. *Assume:*

$$\hat{\mathbb{E}}[|X_1|^{2+\alpha}] < \infty \quad , \hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[-X_1] = 0.$$

Set: $\overline{\sigma}^2 = \hat{\mathbb{E}}[X_1^2]$, $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-X_1^2]$. *Then:*

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\varphi(\frac{X_1 + \cdots + X_n}{\sqrt{n}})] = \hat{\mathbb{E}}[\varphi(B_1)], \ \forall \varphi \in C_b(\mathbb{R}),$$

with $B_1 \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$.

Thank you

We consider situations of uncertainty
Probability method, distribution approach
People first think of
Probability space $(\Omega, \mathcal{F}, P)$,
Random variable $X = X(\omega)$
Distribution of $X(\omega)$: $F(A) = P(X \in A)$
Typical models of distributions $N(\mu, \sigma^2)$,
Stochastic processs: $X_t(\omega)$
It's finite dimensional distribution:
$F_{t_1,\cdots,t_n}(A_1 \times \cdots \times A_n) = P(X_{t_1} \in A_1, \cdots, X_{t_n} \in A_n)$
Tyoical models of stochastic processes
Brownian motion, Poisson process, Levy process, martingales

Important problem: what can we do if we cannot determinate the probability, the distrubution, the stochastic models we face and we still have to make decision?
In fact this is the situation we meet everyday, all the time, in any circumstances;
In enginnering, scientific research activity, humain activities, economic,

business (finance)
The more data we have the less we can be sure about the certainty of our probability and statistic models

This problem closely links In this talk we provide a useful PDE method to treat continuous time probability uncertainty
For fix our idea we concretely treat the following situation:
We can continuously observe a d-dim continuous process $\omega(t)$, $t \in [0, T]$,
To study a