# Black holes and random matrices

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Natifest

# Finite black hole entropy from the bulk

- What accounts for the finiteness of the black hole entropy–from the bulk point of view?
- For an observer hovering outside the horizon, what are the signatures of the Planckian "graininess" of the horizon?

# A diagnostic

- A simple diagnostic [Maldacena]. Let $O$ be a bulk (smeared boundary) operator.

$$\begin{aligned}\langle O(t)O(0)\rangle &= \mathrm{tr}\left(e^{-\beta H}O(t)O(0)\right)/\mathrm{tr}e^{-\beta H} \\ &= \sum_{m,n} e^{-\beta E_m}|\langle m|O|n\rangle|^2 e^{i(E_m-E_n)t}/\sum_n e^{-\beta E_n}\end{aligned}$$

- At short times can treat the spectrum as continuous. $\langle O(t)O(0)\rangle$ generically decays exponentially.

- Perturbative quantum gravity–quasinormal modes. [Horowitz-Hubeny]

# A diagnostic, contd.

$$\langle O(t)O(0)\rangle = \sum_{m,n} e^{-\beta E_m} |\langle m|O|n\rangle|^2 e^{i(E_m-E_n)t} / \sum_n e^{-\beta E_n}$$

- But we expect the black hole energy levels to be discrete (finite entropy) and generically nondegenerate (chaos).
- Then at long times $\langle O(t)O(0)\rangle$ oscillates in an erratic way. It is exponentially small and no longer decreasing.
- A nonperturbative effect in quantum gravity.
(See also [Dyson-Kleban-Lindesay-Susskind; Barbon-Rabinovici])

# Another diagnostic, $Z(t)Z^*(t)$

- To focus on the oscillating phases remove the matrix elements. Use a related diagnostic: [Papadodimas-Raju]

$$\sum_{m,n} e^{-\beta(E_m+E_n)} e^{i(E_m-E_n)t} = Z(\beta+it)Z(\beta-it) = Z(t)Z^*(t)$$

- The "spectral form factor"

# Properties of $Z(t)Z^*(t)$

$$Z(t)Z^*(t) = \sum_{m,n} e^{-\beta(E_m+E_n)} e^{i(E_m-E_n)} t$$

- $Z(\beta,0)Z^*(\beta,0) = Z(\beta)^2$
- Assume the levels are discrete (finite entropy) and non-degenerate (generic, implied by chaos)
- In the long time average
  $\langle Z(\beta,t)Z^*(\beta,t)\rangle_{T_0} = \frac{1}{2T_0}\int_{-T_0}^{T_0} dt\ Z(\beta,t)Z^*(\beta,t)$
  the oscillating phases go to zero and only the $n = m$ terms contribute.
- In the limit $T_0 \to \infty, \langle Z(\beta,t)Z^*(\beta,t)\rangle_{T_0} \to Z(2\beta)$
- $Z(\beta)^2 \to Z(2\beta)$. Roughly $e^{2S} \to e^S$. An exponential change.
- We will focus on the nature of this transition.

# A model system

- The Sachdev-Ye-Kitaev model is a promising system in which to investigate these questions

  [Jordan Cotler, Guy Gur-Ari, Masanori Hanada, Joe Polchinski, Phil Saad, Stephen Shenker, Douglas Stanford, Alex Streicher, Masaki Tezuka]
  [in progress]

# SYK Model

Sachdev-Ye-Kitaev model: Quantum mechanics of $N$ Majorana fermions with random couplings

$$H = \sum_{a,b,c,d} J_{abcd} \chi_a \chi_b \chi_c \chi_d$$

$$\{\chi_a, \chi_b\} = \delta_{ab}$$

$$\langle J_{abcd}^2 \rangle = \frac{1}{N^3} J^2$$

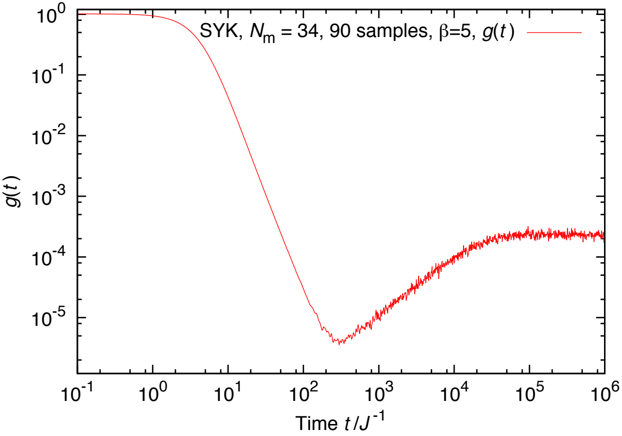
$$dim\ \mathcal{H} = 2^{N/2} = L$$

(We will often set $J = 1$.)

# SYK as a good model system

- Maximally chaotic [Kitaev]
- Gravitational sector with horizon (with enhanced amplitude), but stringy states in bulk [Maldacena-Stanford]
- $J$ average plays role of time average.
- $\langle Z(t)Z^*(t)\rangle_J$ is a smooth function of $t$.
- $Z = \int dG(t,t')d\Sigma(t,t') \ \exp(-N \ I[G,\Sigma])$
- Proxy for a bulk theory. $1/N \sim G_N$ .
- Finite dimensional $\mathcal{H}$. $N = 34 \to L = 2^{17} = 128K$.
- Numerics feasible....

# SYK $Z(t)Z^*(t)$

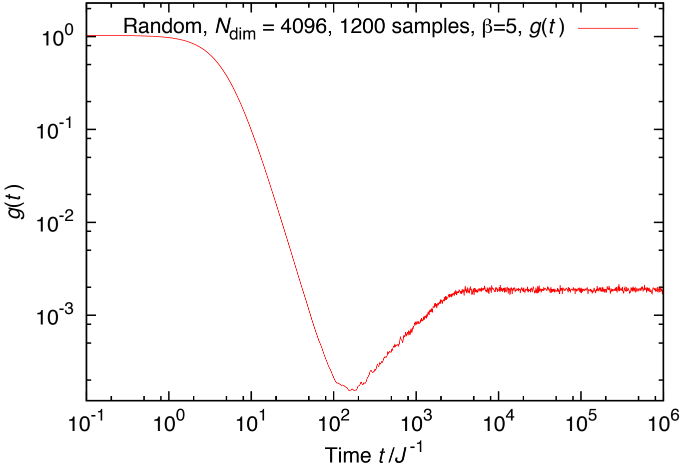



- The Slope
- The Dip
- The Ramp
- The Plateau
- What do they mean?

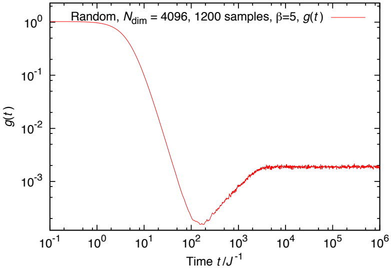
# Random Matrix Theory

- Chaotic quantum systems typically have fine grained energy level statistics described by Random Matrix Theory (RMT) [Wigner; Dyson; Bohigas-Giannoni-Schmit...]
- Consider a simple model where $H \to M$, an $L \times L$ (hermitian) random matrix
- $Z \sim \int dM_{ij} \exp(-\frac{L}{2} trM^2)$
- GUE ensemble
- Compute $\langle Z(t) Z^*(t) \rangle_M$
- The spectral form factor of RMT $(\beta = 0)$
- RMT connection and N mod 8 ensemble variation for near neighber eigenvalue statistics [You-Ludwig-Xu]. We will focus on longer range correlations, in fourier transform.

# Random Matrix Theory $Z(t)Z^*(t)$



# RMT spectral form factor, contd.




$$Z(t)Z^*(t) = \sum_{m,n} e^{-\beta(E_m+E_n)} e^{i(E_m-E_n)t}$$

- The plateau: $n = m$ only nonzero contribution
- What about the rest of the features: the ramp, and the slope?

# Lightning review of RMT theory

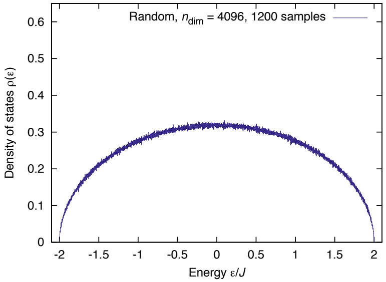
$$Z = \int dM_{ij} \exp(-\frac{L}{2} trM^2)$$
$$= \int dE_j \prod_{ij} (E_i - E_j)^2 \exp(-\frac{L}{2} \sum_I E_i^2)$$
$$= \int D\rho(E) \exp(-I[\rho(E)])$$
$$I[\rho(E)] = -L^2 (\int dEdE' \rho(E)\rho(E') \log((E - E')^2) + \int dE \rho(E) \frac{E^2}{2})$$

- $\rho(E)$ is the density of eigenvalues.
- The Dyson gas.

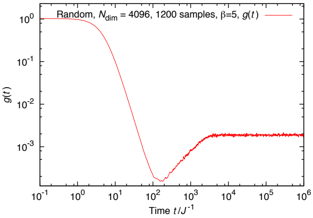
# The Wigner semicircle

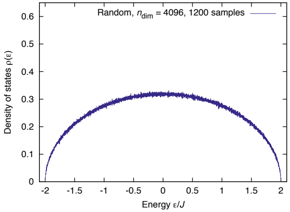
$$I[\rho(E)] = -L^2\left(\int dE dE' \rho(E)\rho(E') \log((E-E')^2) + \int dE \rho(E) \frac{E^2}{2}\right)$$

- The Dyson gas.
- Saddle point at large $L$ gives $\rho_W(E) \sim \sqrt{1-4E^2}$
- The Wigner semicircle law.
- Smallest energy spacing $\delta E \sim 1/L$.



# The slope in RMT




- At early times we can approximate $\langle Z(t)Z^*(t)\rangle \sim \langle Z(t)\rangle\langle Z^*(t)\rangle$
- $\langle Z(t)\rangle \sim L\int dE\ \rho_W(E)e^{-(\beta+it)E}$
- Long time behavior dominated by sharp edge of semicircle
- $\langle Z(t)\rangle \sim L\int dE(E-E_0)^{\frac{1}{2}}e^{-(\beta+it)E}$
- At long times, $\langle Z(t)\rangle \sim L/t^{3/2}\quad(\beta=0)$
- $\langle Z(t)Z^*(t)\rangle \sim L^2/t^3$

# The ramp

At later times fluctuations become important.

$$\langle Z(t)Z^*(t)\rangle = \langle\sum_{n,m} e^{-\beta(E_n+E_m)+it(E_n-E_m)}\rangle$$
$$= Z(2\beta) + \langle\sum_{n\neq m} e^{-\beta(E_n+E_m)+it(E_n-E_m)}\rangle$$
$$= Z(2\beta) + L^2\int dEdE'\rho^{(2)}(E,E')e^{-\beta(E+E')+it(E-E')}$$

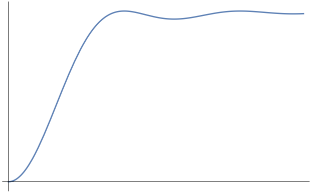
$\rho^{(2)}(E,E')$ is the eigenvalue pair correlation function.

$\langle Z(t)Z^*(t)\rangle$ is essentially the fourier transform of $\rho^{(2)}$, the spectral form factor.

# Spectral form factor in RMT

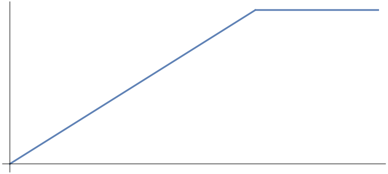
- (take $\beta = 0$ for convenience)
- Near the center of the semicircle $\rho^{(2)}(E, E')$ has a simple universal form given by the Sine kernel (GUE) [Dyson; Gaudin; Mehta]

$$\rho^{(2)}(E, E') \sim 1 - \frac{\sin^2(L(E-E'))}{(L(E-E'))^2}$$



# Fourier transform of $\rho^{(2)}$

Its fourier transform (accounting for the $n = m$ terms) gives $\langle Z(t)Z^*(t)\rangle$ in GUE ($\beta = 0$)



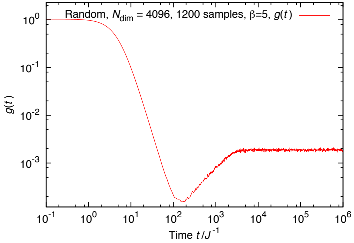
Plateau height is $\sim L$, $t_p \sim L$, ramp is $\sim t$.

# Physical meaning of ramp

- For times shorter than $t_p \sim L$, can average over the rapidly oscillating $\sin^2(L(E - E'))$ factor.
- $\rho^{(2)}(E - E') \sim 1 - 1/(L(E - E'))^2$. This gives the ramp.
- The ramp is lower than the plateau because of long range anticorrelation.
- Long wavelength fluctuations in the Dyson gas are strongly suppressed by the long distance log interaction.
- For example, near neighbor eigenvalue couplings would yield $\rho \sim |E - E'|$
- Spectral rigidity
- Analog of incompressibility in FQHE fluid

# The dip time in RMT

- ($\beta = 0$)
- The slope in RMT is $\sim L^2/t^3$
- The ramp is $\sim t$
- They meet at the dip time $t_d \sim L^{1/2}$
- The plateau time $t_p \sim L$
- $t_d/t_p \sim L^{-1/2} \sim e^{-S/2}$
- An exponentially long period dominated by spectral rigidity



# SYK analysis

- We now try to repeat this analysis for the SYK model.
- we assume the ramp and plateau structure are given by RMT, with an appropriate number of states.
- We then need to analyze the early time slope behavior

# SYK free energy

At low temperature and large $N$ the SYK free energy is given by the dominant saddle and is that of the near extremal black hole ([Sachdev-Ye; Parcollet-Georges; Kitaev; Maldacena-Stanford; Maldacena-Stanford-Yang; Jensen]):

$$-\beta F(\beta) = c_1 N/\beta + N s_0 - \frac{3}{2} \log \beta$$
$$Z(\beta) \sim \frac{1}{\beta^{3/2}} \exp\left(\frac{c_1 N}{\beta} + N s_0\right)$$

Here $N s_0$ is the zero temperature extremal entropy.

# The early time slope

At early times $\langle Z(t)Z^*(t)\rangle = \langle Z(t)\rangle\langle Z^*(t)\rangle$. (self averaging)
Compute $Z(\beta + it)$ by analytically continuing the large $N$ saddle.

$$Z(\beta + it) \sim \frac{1}{(\beta + it)^{3/2}} \exp\left(\frac{c_1 N}{\beta + it} + N s_0\right)$$

$$Z(\beta + it)Z^*(\beta + it)) \sim \frac{1}{(\beta^2 + t^2)^{3/2}} \exp\left(\frac{2 c_1 N \beta}{\beta^2 + t^2} + 2 N s_0\right)$$

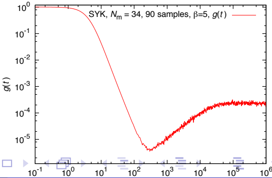
So in this approximation the slope is given by $e^{2Ns_0}/t^3$.
Large $t$ is a bit like large $\beta$.

# Corrections

- The very low temperature dynamics is governed by the reparametrization mode. The dynamics of this mode is described by the Schwarzian action ([Kitaev]).
- This produces $\beta/N$ perturbative corrections. At large time these should grow like $|\beta + it|/N$.
- This makes the slope difficult to analyze analytically for times $t > N$, the entropy time.
- Numerics are only marginally useful in the asymptotic regime because of large coefficients.

# The dip time in SYK

- At what time does new nonperturbative physics dominate over the slope? At what time does the gravitational description of horizon fluctuations break down?
- We can make a heuristic argument bounding the dip time $t_d$ at low enough $T$ by assuming the slope is governed by the low energy reparametrization mode. Then the slope would depend weakly on $T$. But the plateau depends exponentially on $T$.
- This argument gives an exponential separation between the dip and plateau, $t_d/t_p < e^{-\frac{cN}{\beta}}$.
- There would be an exponentially long period where long range spectral rigidity describes the physics of the model.
- $t_d$ could be as early as the entropy time $N$....

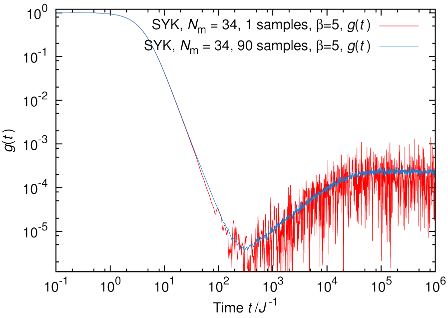


# Comments on SYM

- We can now try to apply these ideas to the canonical AdS/CFT example of N=4 SYM.
- Black holes in this system are known to saturate the chaos bound (at large $\lambda$). ([Kitaev; SS-Stanford; Maldacena-SS-Stanford])
- So it is natural to expect the fine grained structure of SYM energy eigenvalues to have RMT statistics (break T symmetry with $\theta$ to consider GUE ensemble).
- We do not average over Hamiltonians here. Expect large irregular fluctuations for one $H$. ([Barbon-Rabinovici])

# SYK one sample

- Order one fractional variance in one sample of $J$
- "The spectral form factor is not self averaging" ([Prange])
- Autocorrelation time $\sim 1/(\Delta E)$, $\Delta E \sim \sqrt{N}T$ is energy spread in system.
- Parametrically many independent intervals along the ramp. Time averaging makes a smooth signal.



# SYM ramp

- At high $T$ (small $\beta$) SYM has parametrically large entropy, $S(\beta) \sim N^2/\beta^3$, so the plateau is parametrically high.
- Entropy of SYM varies rapidly with energy so the level spacing does also.
- Need to "unfold," consider many small energy intervals with roughly constant level spacing.
- Each interval has its own ramp and plateau. Sum them up.

$$r(t) = \exp[-c\beta N^{-2/3}(\log t)^{4/3}]\ t\ . \tag{1}$$

# SYM at early time

- To compute the slope we compute $Z(\beta + it)$.
- Follow the dominant euclidean large black hole saddle as $\beta \to \beta + it$

$$Z(\beta + it) \sim \exp(N^2/(\beta + it)^3)$$

- $Z(t)Z^*(t)$ drops extremely rapidly.
- At $t \sim \beta, Z(t)Z^*(t) < 1$ and the thermal AdS saddle becomes dominant. Like going to low temperature.
- $ZZ^* \sim 1$.
- But, at $t \sim R_{AdS}$ the black hole saddle dominates again ($1/N^2$ corrections large).
- $ZZ^* \sim e^{cN^2}$.
- Other saddles could be involved, like the small 10D black hole.
- A complicated pattern that is a challenge to unravel.
- $D = 3$ is a promising arena ([Dyer–Gur-Ari])

# SYM dip time

To estimate the dip time we ask when these estimates for the slope intersect the ramp discussed above.

- If we assume the slope has $ZZ^* \sim 1$ (from the thermal AdS saddle) then $t_d \sim e^{\beta N^2}$
- If the slope has $ZZ^* \sim e^{cN^2}$ (from the large black hole saddle) then $t_d \sim e^{cN^2}$.
- In both cases $t_p \sim e^S \sim e^{N^2/\beta^3}$ and so $t_d/t_p \sim e^{-N^2/\beta^3}$, an enormous hierarchy.
- Again, there would be an exponentially long time between the gravitational regime and the plateau where spectral rigidity controls the dynamics of this system.

# Implications for bulk quantum gravity

- It is plausible that the late time dynamics of horizon fluctuations are governed by random matrix dynamics.
- What are the implications of the ramp and plateau for nonperturbative bulk quantum gravity?

# A research program

- A research program:
- In SYK we have an exact nonperturbative expression in terms of singlet bilocal fields: a proxy for a bulk theory.
- $Z = \int dG(t,t')d\Sigma(t,t')\ \exp(-N\ I[G,\Sigma])$
- What part of the $G, \Sigma$ functional integral accounts for this behavior?

# Nonperturbative effects

- $\rho^{(2)}(E, E') \sim 1 - \frac{\sin^2(L(E-E'))}{(L(E-E'))^2}$
- $t \ll t_p, \;\; \rho^{(2)}(E, E') \sim 1 - \frac{1}{L^2(E-E')^2}$
- $1/L^2$ perturbative in RMT, $\sim e^{-cN}$ nonperturbative in $1/N$, SYK
- A single saddle?, more likely a sum over saddles
- $\sin^2(L(E - E')) \rightarrow \exp(-2L(E - E')) \sim \exp(-e^{cN})$
- !

# Seiberg duality?

- Or, a Seiberg style duality at late (real) time? $1/N$ expansion would break down at late time. A new weakly coupled theory would take over, with $1/L$ as a coupling and RMT type degrees of freedom. $e^{-L}$ effects would be analogous to the Andreev-Altshuler instanton.
- We hope to have things to say about this by...

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